Subdiffusion Through Switching

AN ABSTRACT
SUBMITTED ON THE THIRD DAY OF MAY, 2019
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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
BY

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Abstract

An ongoing effort in the study of microparticle movement in biofluids is the proper characterization of subdiffusive processes i.e. processes whose mean-squared displacement scales as a sublinear power law. In order to describe phenomena that lead to subdiffusive behavior, a few models have been developed: fractional Brownian motion, the generalized Langevin equation, and random walks with dependent increments. We will present perhaps a simpler model that leads to subdiffusion and is designed to characterize systems where a regularly diffusive particle intermittently becomes trapped for long periods of time.

By combining ideas from Hybrid Switching Diffusion and queuing systems literature we will describe the law of our process. The major obstacle is the introduction of heavy tail immobilization times and we will overcome it by representing the power law as an infinite mixture of exponentials. The description of the law allows us also to solve the First Passage Problem.

Modeling subdiffusion is a very active field of research both in mathematics and physics. Physicists often use a continuous model that originates in the theory of random walks - Brownian motion inversely subordinated to an α-stable process. In a similar way we will describe our process. With this description we will show that our process under rescaling is equivalent to the inverse subordinated Brownian motion, i.e., we will present the functional limit theorem for Switching Diffusion.

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Chapter 1

Introduction

An ongoing effort in the study of microparticle movement in biofluids is the proper characterization of subdiffusive processes, i.e., processes whose mean-squared displacement scales as a sublinear power law [1]. In order to describe phenomena that lead to subdiffusive behavior, a few models have been developed: Fractional Brownian motion, the generalized Langevin equation, and random walks with infinite waiting times [2],[3]. We will present perhaps a simpler model that leads to subdiffusion and is designed to characterize systems where a regularly diffusive particle intermittently becomes trapped for long periods of time.

The need to investigate such a model comes from a viral infectivity literature. Experimentalists such as Sam Lai [4] observed that HIV virions in the presence of virion specific antibodies becomes trapped in the cervicovaginal mucus. Mucus, which have a polymer-like structure, is not dense enough to effectively trap virions. Although, the exact binding interactions between antibodies and mucus are unknown, biologists have found evidence that antibodies accumulate on the virion sites and then bind to the mucus fibers - effectively becoming trapped for a long periods of time. Several experimental approaches have shown that virions in such environments have long times between switches, which could then result in long-term subdiffusivity.

Arising from queueing networks, financial engineering and biological systems many complex problems contain both continuous dynamics and discrete events. That is why our mathematical model of choice, presented in Chapter 2, will be hybrid switching diffusion: two component process with continuous-valued component that evolves according to some SDE and a discrete-valued component is a continuous-time Markov chain (CTMC) that encodes the current state of the process. In order to obtain subdiffusivity, the process has to spend “much more time” being immobilized than diffusing. In fact, the expected time that a particle spends trapped needs to be infinite! Such distributions are called heavy-tailed and they were extensively studied in the queueing theory literature [5],[6].

There is no unique definition for what makes a distribution heavy-tailed but what we usually talk about when we consider heavy-tailed phenomena is some kind of deviation from the “normal” behavior. In heavy-tailed analysis, typically the asymptotic behavior of variables is determined by the large values or merely a single large value. This is in contrast to many systems whose behavior is determined largely by an averaging effect. The most commonly studied class of heavy-tailed distributions are the distributions with regularly varying tails [7], i.e., with tails that behave like a power law. This class has many desirable properties that will manifest themselves as very “natural” in context of sums and extremes of independent and identically distributed random variables.

Modeling subdiffusion is a very active field of research both in mathematics and physics [1],[8],[9]. In order to capture the phenomenology leading to subdiffusion, Physicists often use a model that originates in the theory of random walks. Three types of models are often invoked, namely random walks in complex geometries, random walks with nonindependent increments (resulting in antipersistence effects), and walks displaying memory effects (aging). These are pathwise constructions for subdiffusive processes. Sometimes though, modelers take a population scale approach and propose the use of fractional diffusion equations. The stochastic formulation of transport phenomena in terms of random walk, as well as the description through the deterministic diffusion equation are two fundamental concepts in the theory of both normal and anomalous diffusion. Meerschaert and Scheffler [2] show the connection between the limits of CTRW and the fractional diffusion equation; we will fully explore their ideas in Chapter 3. The anomalous diffusion behavior manifested in the limiting process is intimately connected with the breakdown of the central limit theorem, caused by either broad distribution or long-range correlations. Here, anomalous diffusion rests on the validity of generalized central limit theorem - perfect for such situations where not all moments of the underlying elementary transport events exist.

In Chapter 4, we investigate the first passage time problem. For a stochastic process, the first passage time (FPT) is defined as the time when the process reaches a predetermined level for the first time. FPT is a random variable and one usually studies its tail distribution. Another question that usually arises in this context is the scaling of the mean first passage time with the interval width. The question is really interesting here because for many subdiffusive models the mean first passage time is infinite. We focus on presenting three different type of models: Brownian motion, Time-fractional diffusion and our own Switching diffusion. Also, we present simulations for quantile function of Switching Diffusion. We will demonstrate that it exhibits an interesting behavior of "switchover" between diffusive and subdiffusive regimes. We will present our results for two simulation procedures: a "Full" and a "Toy" model. The Full model simulates the exact behavior of the Switching Diffusion whereas in the Toy model we make some simplifications that significantly reduce running times.

Chapter 2

Subdiffusive Switching

Our model for a subdiffusive particle, which we call Switching Diffusion, is based on a very simple idea. It switches between two states: diffusive and immobilized. The times when the particle switches between diffusing and being immobilized is denoted by τ_i and immobilized to diffusing is denoted by σ_i, as illustrated in Figure 2.1. Further, we assume that they are independent and identically distributed (iid) with $\tau_i \sim \text{Exp}(\lambda)$ and $\sigma_i \sim F_\sigma$, where F_σ is a cumulative distribution function.

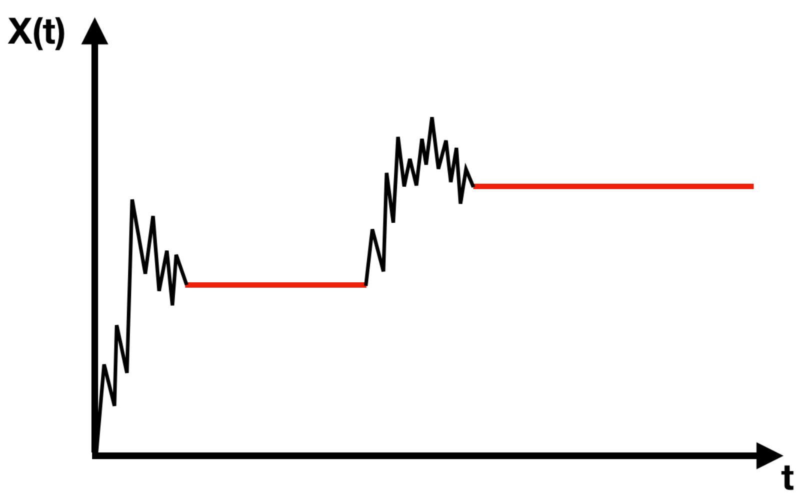


Figure 2.1: Sample path of Switching Diffusion.

In Section 2.1 we will start with a Stochastic Differential Equation (SDE) representation of our model. Also, we will introduce a notion of regular variation which is necessary to present sufficient conditions for the Switching Diffusion to be subdiffusive. The importance of Section 2.2 will be revealed in our further discussion where we use the idea coming from the queueing systems literature and represent the power law as an infinite mixture of exponentials. In Section 2.3 we will introduce the background on so called hybrid switching diffusions, a setting in which we describe the law of our process. Section 2.4 will present our model as a hybrid switching diffusion, i.e., in a case when the distribution of σ is a mixture of exponential distributions. In the final section of this chapter we will comment on going beyond the mixture of exponentials case and propose the largest class of distributions for F_σ that can be "successfully" approximated by the exponential mixtures.

2.1 Condition for Subdiffusivity

The model for the position of a switching particle at a given time t, $X = \{X(t)\}_{t \geq 0}$, which we mention in the introduction, can be represented in the language of stochastic differential equations. First, we introduce the sequence of times

$$0 = R_0 < S_1 < R_1 < ..., \tag{2.1}$$

where

R_i : Switch from being immobilized to diffusing,

S_i : Switch from diffusing to being immobilized.

For simplicity we assume that $R_0 = 0$, i.e., at time zero particle enters diffusive state. We let $\{\tau_i\}_{i\geq 1}$ be an iid sequence of random variables, such that

$$\tau_i \sim Exp(\lambda), \tag{2.2}$$

which represent the length of time intervals in which the process X is diffusing. In terms of switching times, for $i \geq 1$

$$\tau_i := S_i - R_{i-1}. \tag{2.3}$$

Similarly, let $\{\sigma_i\}_{i\geq 1}$ be an iid sequence of random variables

$$\sigma_i \sim F_\sigma \tag{2.4}$$

which represent the length of time intervals in which the process X is immobilized. Similarly, in terms of switching times, for $i \geq 1$

$$\sigma_i := R_i - S_i. \tag{2.5}$$

Finally, let us introduce a discrete valued process $\phi = \{\phi(t)\}_{t\geq 0}$, illustrated in Figure 2.2, which is defined as

$$\phi(t) := \begin{cases} 1 & R_{i-1} \leq t < S_i, \quad i \geq 1 \\ 0 & S_i \leq t < R_i, \quad i \geq 1. \end{cases} \tag{2.6}$$

The process ϕ is an ON-OFF process, a well studied object in the renewal theory [10]. In our model ϕ plays a role of a switch; it controls whether process is diffusing (ON) or immobilized (OFF). The following equation is the SDE for the pair

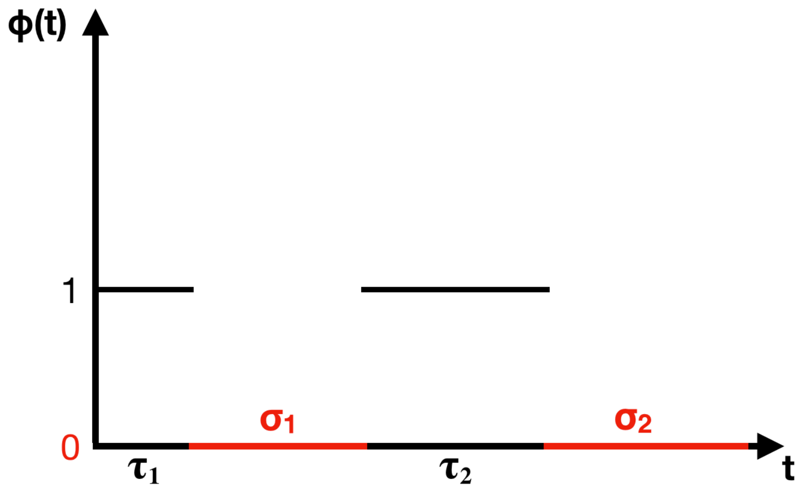


Figure 2.2: A sample path of ϕ.

$\big(X(t), \phi(t)\big)$:

$$\begin{cases} dX(t) = \sqrt{2D} \cdot \phi(t) dB(t) \\ \big(X(0), \phi(0)\big) = (0, 1) \end{cases} \tag{2.7}$$

where D is a diffusivity parameter and $B = \{B(t)\}_{t \geq 0}$ is a standard Brownian motion. We assume that $\phi(0) = 1$, i.e., our process starts in the diffusive state. From now on we refer to ϕ as a switch and X as a Switching Diffusion.

The main goal of this section is to present the conditions under which a Switching Diffusion is a subdiffusive process. In contrast to a typical diffusion process, in which the mean squared displacement (MSD:$= \mathbb{E}[X^2(t)]$) of a particle is a linear function of time, the MSD of anomalous diffusion is described by a power law. For the purposes of this work, we will define a process to be an anomalous diffusion if

$$\mathbb{E}[X^2(t)] \sim ct^{\alpha}, \quad t \to \infty. \tag{2.8}$$

Note that here and elsewhere we use the notation

$$f(t) \sim g(t), \quad t \to \infty \tag{2.9}$$

as a shorthand for

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1. \tag{2.10}$$

We call a process asymptotically diffusive if $\alpha = 1$. If $\alpha > 1$ the phenomenon is called superdiffusion. If $\alpha < 1$ the particle undergoes subdiffusion. The Switching Diffusion process X that we presented earlier can be used as model for subdiffusion but one have to carefully choose the distribution of σ. For example, we can directly calculate the MSD if distribution of σ is exponential.

Proposition 2.1.1. *If $\sigma \sim Exp(\mu)$ then MSD of X is*

$$\mathbb{E}[X^2(t)] = \frac{2D\mu}{\mu+\lambda}t - \frac{2D\lambda}{(\mu+\lambda)^2}(1 - e^{-(\mu+\lambda)t}) \sim \frac{2D\mu}{\mu+\lambda}t, \quad t \to \infty \tag{2.11}$$

and therefore X is asymptotically diffusive.

Proof. Using the SDE representation for $X(t)$ (2.7) and Ito's Isometry we have

$$\begin{aligned}
\mathbb{E}[X^2(t)] &= \mathbb{E}\left[\left(\sqrt{2D}\int_0^t \phi(s)dB_s\right)^2\right] \\
&= 2D\int_0^t \mathbb{E}[\phi^2(s)]ds \\
&= 2D\int_0^t \mathbb{E}[\phi(s)]ds \\
&= 2D\int_0^t \mathbb{P}\{\phi(s) = 1\}ds
\end{aligned} \tag{2.12}$$

Now, we need to investigate the behavior of the function

$$P(t) := \mathbb{P}\{\phi(t) = 1\}. \tag{2.13}$$

Let $T_1 = \tau_1 + \sigma_1$ be a time of the first renewal. Conditioning on T_1 yields

$$\begin{aligned} P(t) &= \mathbb{P}\{\phi(t) = 1, T_1 > t\} + \mathbb{P}\{\phi(t) = 1, T_1 \leq t\} \\ &= \mathbb{P}\{\tau > t\} + \int_0^t P(t-s) f_{T_1}(s) ds \\ &= (1 - F_\tau(t)) + \int_0^t P(t-s) f_{T_1}(s) ds. \end{aligned} \tag{2.14}$$

Now applying Laplace transform $\left(\hat{f}(s) = \int_0^\infty e^{-st} f(t) dt\right)$ on the both sides of the equation we get

$$\begin{aligned} \hat{P}(s) &= \frac{1}{s+\lambda} + \hat{P}(s)\hat{f}_{T_1}(s) \\ &= \frac{1}{s+\lambda} + \hat{P}(s)\hat{f}_\sigma(s)\hat{f}_\tau(s) \\ &= \frac{1}{s+\lambda} + \frac{\lambda \hat{P}(s)\hat{f}_\sigma(s)}{s+\lambda}. \end{aligned} \tag{2.15}$$

Reorganizing above equation gives

$$\hat{P}(s) = \frac{1}{s + \lambda(1 - \hat{f}_\sigma(s))}. \tag{2.16}$$

Since we assume that $\sigma \sim \mathrm{Exp}(\mu)$ we have

$$\hat{P}(s) = \frac{s+\mu}{s^2 + (\mu+\lambda)s} \tag{2.17}$$

and by inverting Laplace transform we get

$$P(t) = e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}(1 - e^{-(\mu+\lambda)t}). \tag{2.18}$$

Finally,

$$\begin{aligned}\mathbb{E}[X^2(t)] &= 2D\int_0^t \left(e^{-(\mu+\lambda)s} + \frac{\mu}{\mu+\lambda}(1-e^{-(\mu+\lambda)s})\right)ds \\ &= \frac{2D\mu}{\mu+\lambda}t - \frac{2D\lambda}{(\mu+\lambda)^2}(1-e^{-(\mu+\lambda)t}),\end{aligned} \tag{2.19}$$

which implies

$$\mathbb{E}[X^2(t)] \sim \frac{2D\mu}{\mu+\lambda}t, \quad t\to\infty. \tag{2.20}$$

$\square$

Remark 2.1.1. What is more, any choice of σ such that $\mathbb{E}[\sigma] < \infty$ would lead to a similar result. We will prove this fact in the next chapter (see 3.2.3). Therefore, we have to consider distributions that have infinite mean; heavy-tailed distributions.

Roughly speaking, a random variable Y has a heavy tail if there exists a positive parameter $\alpha > 0$ such that

$$\bar{F}(y) := \mathbb{P}\{Y > y\} \sim y^{-\alpha}, \quad y \to \infty. \tag{2.21}$$

Examples of such random variables are Cauchy, Pareto, t-student or stable distributions. The most successful and very well studied class of heavy-tailed distributions have been distributions with regularly varying tails [7],[11]. They admit very nice analytical properties that are summed up in Appendix A.1. Here we provide the definition of regular variation.

Definition 2.1.2. A positive, measurable function L on $(0,\infty)$ is called slowly varying at infinity ($L \in RV_0$) if

$$\lim_{x\to\infty}\frac{L(tx)}{L(x)} = 1, \text{ for all } t>0. \tag{2.22}$$

Definition 2.1.3. A positive, measurable function U is called regularly varying at infinity with index $\alpha \in \mathbb{R}$ ($U \in RV_\alpha$) if

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^\alpha, \text{ for all } t > 0. \tag{2.23}$$

Note that if $U \in RV_\alpha$ then $U(x)/x^\alpha \in RV_0$, and setting $L(x) = U(x)/x^\alpha$, we can see that it is always possible to represent a regularly varying function as $x^\alpha L(x)$.

Theorem 2.1.4. *If $\bar{F}_\sigma(t) \in RV_{-\alpha}$ for $\alpha \in (0,1)$ with absolutely continuous, eventually monotone density f, then the associated Switching Diffusion X is subdiffusive.*

Proof. Using the SDE representation of X (2.7) we have that

$$\mathbb{E}[X^2(t)] = \mathbb{E}\left[\left(\sqrt{2D}\int_0^t \phi(s)dB_s\right)^2\right]. \tag{2.24}$$

Now performing similar calculations as in Proposition 2.1.1 we arrive at

$$\hat{P}(s) = \frac{1}{s + \lambda(1 - \hat{f}_\sigma(s))}. \tag{2.25}$$

Since $\bar{F}_\sigma \in RV_{-\alpha}$, using Theorem A.1.8 we have

$$1 - \hat{f}_\sigma(s) \sim \Gamma(1-\alpha)s^\alpha L(1/s), \quad s \downarrow 0. \tag{2.26}$$

Therefore,

$$\hat{P}(s) \sim \frac{s^{-\alpha}}{\lambda\Gamma(1-\alpha)L(1/s)}, \quad s \downarrow 0. \tag{2.27}$$

Now, by Karamata's Tauberian Theorem A.1.6

$$P(t) \sim \frac{\alpha}{\lambda\Gamma(1-\alpha)\Gamma(1+\alpha)} \frac{t^{\alpha-1}}{L(t)}, \quad t \to \infty. \tag{2.28}$$

Finally, using Theorem A.1.4 we obtain

$$\mathbb{E}[X^2(t)] = 2D\int_0^t P(s)ds \sim \frac{2D}{\lambda\Gamma(1-\alpha)\Gamma(1+\alpha)}\frac{t^\alpha}{L(t)}, \quad t \to \infty, \tag{2.29}$$

which ends the proof. $\square$

2.2 Exponential-Series Representation of Power Laws

Here we want to present the idea that the distribution with a regularly varying tail can be obtained by mixing the exponential distributions. In other words, we obtain a cumulative distribution function $F(t)$ so its tail

$$\bar{F}(t) := 1 - F(t) \tag{2.30}$$

is in the following form

$$\bar{F}(t) \sim At^{-\alpha} \text{ as } t \to \infty, \tag{2.31}$$

where $A > 0$ and $\alpha \in (0,1)$ are constants. Of course, any distribution with a tail as seen in (2.31) is regularly varying. Let us write

$$\frac{\bar{F}(tx)}{\bar{F}(x)} = \frac{\bar{F}(tx)}{A(tx)^{-\alpha}} \cdot \frac{Ax^{-\alpha}}{\bar{F}(x)} \cdot t^{-\alpha} \to t^{-\alpha} \text{ as } t \to \infty, \tag{2.32}$$

and therefore $\bar{F} \in RV_{-\alpha}$.

Abate and Whitt [12] developed such representations and we briefly present their methods. Let

$$\bar{F}(t) = \sum_{n=1}^{\infty} p_n \bar{F}_n(t) = \sum_{n=1}^{\infty} p_n e^{-\lambda_n t}, t \geq 0, \tag{2.33}$$

where $\{p_n\}_{n\geq 1}$ is a probability mass function and $\{\lambda_n\}_{n\geq 1}$ is the sequence of rates of the component exponential pdf's. Usually one assumes that $\lambda_n > \lambda_{n+1}$ and $\lambda_n \to 0$.

In simulations we will have to truncate the infinite series in (2.33). Unfortunately, by truncating the resulting mixture, which is in fact a hyperexponential distribution, has only exponential tail, hence this truncation cannot be a good approximation for all t at once. Although, it can yield a sufficient approximation for any given interval $[0, t]$.

We show the outline of the method used in [12] and then calculate it in our specific setting. The asymptotic form and the truncation point can be found by using Euler-Maclaurin summation formula to approximate sum by the integral and then use Laplace's method [13] to determine the asymptotic of the integral. Let us start with $\bar{F}(t)$ such that for $t \geq 0$

$$\bar{F}(t) = \sum_{n=1}^{\infty} p_n \bar{F}_n(t). \tag{2.34}$$

It can be rewriten as

$$\bar{F}(t) = \sum_{n=1}^{\infty} e^{-\phi(n,t)}, \text{ where } \phi(n,t) = -\log p_n - \log \bar{F}_n(t). \tag{2.35}$$

Now, we treat $\phi(x,t)$ as a continuous function of x and approximate the sum by the integral, i.e.,

$$\bar{F}(t) \approx \int_0^{\infty} e^{-\phi(x,t)} dx. \tag{2.36}$$

We assume there is an unique $x^*(t)$ that minimizes $\phi(x,t)$. If ϕ reaches its minimum at $(x^*(t), t)$ then for t large enough, the integral is dominated by the neighborhood of $x^*(t)$. Then using the Laplace's method in (2.36) yields

$$\int_0^{\infty} e^{-\phi(x,t)} dx \sim \sqrt{\frac{2\pi}{\phi''(x^*(t), t)}} e^{-\phi(x^*(t),t)} \text{ as } t \to \infty. \tag{2.37}$$

To sum up, the method gives the asymptotic form

$$\bar{F}(t) \sim \sqrt{\frac{2\pi}{\phi''(x^*(t),t)}} e^{-\phi(x^*(t),t)} \text{ as } t \to \infty \tag{2.38}$$

and if we are interested in time t_0 the truncation point should be at least $x^*(t_0)$. In the following proposition we will apply the above method in our specific setting.

Proposition 2.2.1. *Let $F(t)$ be a mixture of exponential cdfs, with*

$$\bar{F}(t) = \sum_{n=1}^{\infty} p_n e^{-\lambda_n t}, \tag{2.39}$$

where $p_n = kn^{-\beta}$ for k, such that $\sum p_n = 1$, $\beta > 1$ and $\lambda_n = n^{-1/\gamma}$ for $\gamma > 0$. Then

$$\bar{F}(t) \sim At^{-(\beta-1)/\gamma} \textit{ as } \; t \to \infty, \tag{2.40}$$

where

$$A = k\sqrt{2\pi/\beta\gamma}e^{-\beta/\gamma}(\gamma/\beta)^{-(\beta-1)/\gamma}. \tag{2.41}$$

Proof. By changing $\bar{F}(t)$ to the form given in (2.35) we get

$$\phi(x,t) = -\log k + \beta \log x + \frac{t}{x^\gamma}, \tag{2.42}$$

so that

$$x^*(t) = (\gamma t/\beta)^{1/\gamma}), \quad \phi''(x,t) = \frac{-\beta}{x^2} + \frac{\gamma(\gamma+1)t}{x^{\gamma+2}}, \tag{2.43}$$

$$\phi''(x^*(t),t) = \left(\frac{\beta}{\gamma t}\right)^{2/\gamma}(\beta\gamma). \tag{2.44}$$

Combining everything and substituting into (2.38) gives us

$$\bar{F}(t) \sim At^{-(\beta-1)/\gamma} \text{ as } \; t \to \infty, \tag{2.45}$$

where

$$A = k\sqrt{2\pi/\beta\gamma}e^{-\beta/\gamma}(\gamma/\beta)^{-(\beta-1)/\gamma}. \tag{2.46}$$

□

Remark 2.2.1. Although there is more flexibility in the method described above, for the remainder of the text we use the following representation. Let $k = 6/\pi^2, \beta = 2$ and $\gamma = 1/\alpha$, $\alpha \in (0, 1)$. Therefore

$$p_n = \frac{6}{\pi^2 n^2}, \quad \lambda_n = n^{-1/\alpha} \tag{2.47}$$

and we obtain

$$\bar{F}(t) = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-t/n^{1/\alpha}}}{n^2} \sim At^{-\alpha} \text{ as } t \to \infty, \tag{2.48}$$

where

$$A = \frac{6}{\pi^2}\sqrt{\pi\alpha}e^{-2\alpha}(2\alpha)^{-\alpha}. \tag{2.49}$$

2.3 Background: Hybrid Switching Diffusions

This section is based on the chapters in [14], a textbook on Hybrid Switching Diffusions, and [15], a paper that extends the theory from finite to countable switching. We will work with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A family of σ-algebras $\{\mathcal{F}_t\}$, $t \geq 0$ is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. We say $\mathcal{F}_t$ is complete if it contains all null sets and that the filtration $\{\mathcal{F}_t\}$ satisfies the usual condition if $\mathcal{F}_0$ is complete. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\{\mathcal{F}_t\}$ is said to be a filtered probability space, denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

Suppose that $Z = \{Z(i)\}_{i\in\mathcal{M}}$ is a stochastic process with right-continuous with left-hand limits sample paths and discrete state space $\mathcal{M}$, that could be finite or countable [15], and x-dependent generator $Q(x)$ so that for a suitable function $f(\cdot,\cdot)$,

$$Q(x)f(x,\cdot)(i) = \sum_{j\in\mathcal{M}} q_{ij}(x)f(x,j), \text{ for each } i \in \mathcal{M}. \tag{2.50}$$

Let $B = \{B(t)\}_{t\geq 0}$ be the $\mathbb{R}$-valued standard Brownian motion defined in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Suppose that $b(\cdot,\cdot) : \mathbb{R}\times\mathcal{M} \mapsto \mathbb{R}$ and that $\sigma(\cdot,\cdot) : \mathbb{R}\times\mathcal{M} \mapsto \mathbb{R}$. Then the two-component process $(X(\cdot), Z(\cdot))$, satisfying

$$\begin{aligned} &dX(t) = b(X(t), Z(t))dt + \sigma(X(t), Z(t))dB(t) \\ &(X(0), Z(0)) = (x, 0), \end{aligned} \tag{2.51}$$

and for $i \neq j$,

$$\mathbb{P}\{Z(t+\Delta) = j | Z(t) = i, X(s), Z(s), s \leq t\} = q_{ij}(X(t))\Delta + o(\Delta), \tag{2.52}$$

is called a hybrid switching diffusion. For two-component process $(X(t), Z(t))$, we call $X(t)$ the continuous component and $Z(t)$ the discrete component, in accordance with their sample path properties.

There is an associated operator defined as follows. For each $i \in \mathcal{M}$ and each $f(\cdot, i) \in C^2$, where C^2 denotes the class of functions whose partial derivatives with respect to variable x up to second-order are continuous, we have

$$\mathcal{L}f(x,i) = b(x,i)\frac{\partial}{\partial x}f(x,i) + \frac{1}{2}\sigma^2(x,i)\frac{\partial^2}{\partial x^2}f(x,i) + \sum_{j\in\mathcal{M}} q_{ij}(x)f(x,j). \tag{2.53}$$

Now, the probability density $p(x,t) = (p(x,t,1), p(x,t,2), ...)$ of the process $Y(\cdot)$, with

$$\int_\Gamma p(x,t,i)dx = \mathbb{P}\{X(t)\in\Gamma, Z(t) = i\}, \tag{2.54}$$

satisfies the adjoint equation, namely the system of equations

$$\frac{\partial}{\partial t}p(x,t,i) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x,i)p(x,t,i)) - \frac{\partial}{\partial x}(b(x,i)p(x,t,i)) + \sum_{j\in\mathcal{M}} p(x,t,j)q_{ij}(x)$$

$$p(x,0,i) = g_i(x)$$

for $i \in \mathcal{M}$ and $g(x) = (g_1(x), g_2(x), ...)$ is the initial distribution of $Y(t)$.

2.4 Law of Switching Diffusion: Mixture of Exponentials Case

Note that in Section 2.1 when we introduced the Switching Diffusion model, we only assumed that F_σ has a regularly varying tail. Now we impose further assumptions on distribution of σ that allow us to obtain the law of subdiffusive Switching Diffusion. We return to ideas from Section 2.2 and we start with F_σ that has a specific form: it is a mixture of exponential distributions, i.e., we can write its tail as

$$\bar{F}_\sigma(t) = \sum_{i=1}^{\infty} p_i e^{-\lambda_i t}, \tag{2.55}$$

where

$$\sum_{i=1}^{\infty} p_i = 1 \text{ and } \lambda_i > 0 \text{ for } i \in \mathbb{Z}_+, \tag{2.56}$$

with density

$$f_\sigma(t) = \sum_{i=1}^{\infty} p_i \lambda_i e^{-\lambda_i t}. \tag{2.57}$$

This assumption allows us to use the ideas coming from hybrid switching diffusion models. The choice for such a class of distributions for σ is a result of natural restrictions on discrete component of the hybrid switching process. The mixtures of exponential distributions is a subclass of a more general phase-type distributions.

A phase-type distribution is constructed by a convolution or mixture of exponential distributions. It results from a system of one or more inter-related processes occurring in sequence, or phases. Each of the states of the Markov process represents one of the said phases. In case of hybrid switching diffusions the phase process is Z. In our case the phase process is ϕ and it controls whether the particle is diffusing or immobilized.

In the language of hybrid switching diffusions our model can be written as a two component process

$$Y(t) = (X(t), Z(t)) \tag{2.58}$$

where the continuous component $X(t)$ evolves according to SDE which coefficients depend on $Z(t)$. The discrete component: $Z(t)$ is a continuous time Markov chain taking values in $\mathbb{N}$ that evolves according to a rate matrix Q:

$$Q = \begin{bmatrix} -\lambda & p_1\lambda & p_2\lambda & p_3\lambda & \dots \\ \lambda_1 & -\lambda_1 & 0 & 0 & \dots \\ \lambda_2 & 0 & -\lambda_2 & 0 & \dots \\ \lambda_3 & 0 & 0 & -\lambda_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.59}$$

In our case the pair $(X(t), Z(t))$ is a solution to

$$\begin{cases} dX(t) = \sqrt{2D} \cdot \mathbb{1}_{\{Z(t)=0\}} dB(t) \\ (X(0), Z(0)) = (0, 0) \end{cases} \tag{2.60}$$

where D is a diffusivity parameter and B is the standard Brownian Motion. Note that now Z plays the role of a switch. When Z is in state zero, process X is diffusing. The states $\{1, 2, 3, ...\}$ are auxiliary states where process X is immobilized for exponential amount of time and then comes back to state zero. There is a very simple connection between the ϕ in (2.7) and Markov chain Z in (2.60) for $t \geq 0$

$$\phi(t) = \mathbb{1}_{\{Z(t)=0\}}. \tag{2.61}$$

The density of Y satisfies:

$$\partial_t p(x,t,0) = D\partial_{xx} p(x,t,0) - \lambda p(x,t,0) + \sum_{i=1}^{\infty} \lambda_i p(x,t,i) \tag{2.62}$$

$$\partial_t p(x,t,i) = \lambda p_i p(x,t,0) - \lambda_i p(x,t,i) \text{ for } i = 1,2,3,... \tag{2.63}$$

with initial distribution $p(x,0) = (\delta_0(x), 0, 0...)$.

Theorem 2.4.1. *Let X be a Switching Diffusion with iid diffusion times*

$$\{\tau_i\}_{i\geq 1} \sim Exp(\lambda) \tag{2.64}$$

and iid immobilization times

$$\{\sigma_i\}_{i\geq 1} \sim F_\sigma, \tag{2.65}$$

where F_σ is a mixture of exponentials as in Equation 2.55, with density f_σ. Then for $t > 0$ and $\Gamma \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}\{X(t) \in \Gamma\} = \int_\Gamma p(x,t)dx \tag{2.66}$$

where

$$p(x,t) = p(x,t,0) + \lambda \int_0^t f_\sigma(t-s)p(x,s,0)ds \tag{2.67}$$

and $p(x,t,0)$ is a solution to

$$\partial_t p(x,t,0) = D\partial_{xx} p(x,t,0) - \lambda p(x,t,0) + \lambda \int_0^t f_\sigma(t-s)p(x,s,0)ds. \tag{2.68}$$

Proof. Notice that for every i we can solve (2.63) in terms of $p(x,t,0)$, i.e.,

$$p(x,t,i) = \lambda \int_0^t p_i e^{-\lambda_i(t-s)} p(x,s,0)ds \tag{2.69}$$

Now, by plugging into (2.62) and changing sum with the integral we obtain autonomous equation for $p(x,t,0)$:

$$\partial_t p(x,t,0) = D\partial_{xx}p(x,t,0) - \lambda p(x,t,0) + \lambda \int_0^t \sum_{i=1}^{\infty} p_i \lambda_i e^{-\lambda_i(t-s)} p(x,s,0)ds \tag{2.70}$$

Combining above we can get a formula for the law of $X(t)$. Notice that we can write

$$\begin{aligned}
\mathbb{P}\{X(t) \in \Gamma\} &= \sum_{i=0}^{\infty} \mathbb{P}\{X(t) \in \Gamma, Z(t) = i\} \\
&= \int_\Gamma \sum_{i=0}^{\infty} p(x,t,i)dx \\
&= \int_\Gamma \left(p(x,t,0) + \sum_{i=1}^{\infty} p(x,t,i) \right) dx \\
&= \int_\Gamma \left(p(x,t,0) + \lambda \int_0^t \sum_{i=1}^{\infty} p_i e^{-\lambda_i(t-s)} p(x,s,0)ds \right) dx \\
&= \int_\Gamma \left(p(x,t,0) + \lambda \int_0^t f_\sigma(t-s)p(x,s,0)ds \right) dx,
\end{aligned} \tag{2.71}$$

where $p(x,t,0)$ is a solution to (2.70). ∎

2.5 Beyond the Exponential Mixtures

So far we have been able to describe the law of Switching Diffusion when σ is a countable mixture of exponentials. Our goal is to extend the work beyond this class. It turns out that it is possible to fully characterize the range of exponential mixtures. In order to do this we need the notion of a completely monotone function. A positive function defined on $(0,\infty)$ of the class C^∞, such that the sequence of its derivatives alternates signs at every point, is called completely monotone (CM). Bernstein [16] showed that a function on positive reals is CM if and only if it is a mixture of exponentials. Therefore, we would like to prove a stronger result along the lines of Theorem 2.4.1 with the assumption F_σ has a density which is CM. Unfortunately, tools that we have developed in this chapter are not enough, but we will explore a number of results in this direction. In order to do this we have to reach to the theory of subordinators which introduces a new perspective on Switching Diffusion.

Chapter 3

Subdiffusion by Subordination

Modeling subdiffusion is a very active field of research both in mathematics and physics [8],[1],[9]. Physicists often use a continuous model that originates in the theory of random walks - Brownian motion inversely subordinated to an α-stable process [2]. We noticed that we can use a similar language to describe the Switching Diffusion process introduced in Chapter 2.

In this chapter we start with some background information on Lévy processes and their special relation with infinitely divisible distributions. The most important examples are given with a focus on the Poisson and stable cases. Then we discuss non-decreasing Levy processes - subordinators, mostly used as a model of a random time evolution. Physicists observed that the first passage time process of a subordinator, known as a inverse subordinator, is a great tool in modeling subdiffusive phenomenons. Magdziarz [17] showed that inversely subordinated Brownian motion is a martingale. We use this fact to show that Brownian motion inversely subordinated to the compound Poisson process with a drift is a subdiffusive process and therefore we present another proof for subdiffusivity of Switching Diffusion.

3.1 Lévy Processes and Infinitely Divisible Distributions

Lévy processes, named after the French mathematician Paul Lévy, are, generally speaking, processes with stationary independent increments. The most common examples are Brownian motion and Poisson process. As we will see later this class is in fact very rich due to an intimate relationship with the infinitely divisible distributions.

Let us start with the definition of Lévy processes [18].

Definition 3.1.1. A process $X = \{X(t)\}_{t\geq 0}$ defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it has the following properties:

1. $\mathbb{P}\{X(0) = 0\} = 1$.

2. Independent and stationary increments: for each $n = 1, 2, ...$ and each $0 \leq t_1 \leq t_2 \leq \cdots < t_{n+1} < \infty$ the random variables $\{X(t_{j+1} - X(t_j)\}_{1\leq j\leq n}$ are independent, moreover each

$$X(t_{j+1}) - X(t_j) \overset{\mathrm{d}}{=} X(t_{j+1} - t_j) - X(0). \tag{3.1}$$

3. Continuity in probability: for all $\varepsilon > 0$ and for all $s \geq 0$

$$\lim_{t\to s} \mathbb{P}\{|X(t) - X(s)| > \varepsilon\} = 0. \tag{3.2}$$

If X is a Lévy process then one may construct a version of X that has cádlág paths (right continuous with left limits) [18], therefore throughout the remainder of the text we will refer the cádlág paths.

Simply looking at the definition it is not immediately obvious which processes belong to this class. However there is a certain perspective that can give us a good idea about the members. As we will see there is a natural connection between Lévy processes and infinitely divisible distributions.

Definition 3.1.2. A random variable Y has an infinitely divisible distribution if for each $n = 1, 2, ...$ there exists a sequence of i.i.d random variables $\{Y_{j,n}\}_{1 \leq j \leq n}$, such that

$$Y \stackrel{\mathrm{d}}{=} Y_{1,n} + ... + ..Y_{n,n}. \tag{3.3}$$

Alternatively, the law μ of a random variable is called infinitely divisible if for any $n \geq 1$, there exists a probability measure μ^n, such that μ can be expressed as the n-th convolution power of μ^n, i.e.

$$\mu = \mu^{*n}. \tag{3.4}$$

The standard way to establish whether the law of a random variable is infinitely divisible, is through the characteristic exponent. Let Y be a random variable with law μ. We calculate the characteristic function

$$\mathbb{E}\big[e^{i\theta Y}\big] = \int_{\mathbb{R}} e^{i\theta x}\mu(dx) \tag{3.5}$$

and then we obtain the characteristic exponent given by

$$\Psi(\theta) := -\log \mathbb{E}\big[e^{i\theta Y}\big], \quad \theta \in \mathbb{R}. \tag{3.6}$$

Now, Y has an infinitely divisible distribution if and only if for all $n \geq 1$ a characteristic exponent can be expressed as

$$\Psi(\theta) = n\Psi_n(\theta), \quad \theta \in \mathbb{R}, \tag{3.7}$$

where Ψ_n is characteristic exponent corresponding to a possibly different random variable. The following theorem is known as the Lévy-Khintchine formula. It characterizes all the infinitely divisible distributions by their characteristic exponents.

Theorem 3.1.3 (Lévy-Khintchine formula [Citation)**.** *] A function $\Psi : \mathbb{R} \to \mathbb{C}$ is the characteristic exponent of an infinitely divisible probability measure on $\mathbb{R}$ if and only if there are $a \in \mathbb{R}$, $\sigma \geq 0$, and a measure Π with $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$ such that*

$$\Psi(\theta) = ia\theta + \frac{\theta^2\sigma^2}{2} + \int_{\mathbb{R}} \big(1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x|<1\}}\big)\Pi(dx) \tag{3.8}$$

for every $\theta \in \mathbb{R}$.

Now we turn our attention to the connection between the Lévy processes and infinitely divisible distributions. From the definition of a Lévy process X we see that for any fixed $t > 0$, $X(t)$ is a random variable with an infinitely divisible distributions. To obtain that for $n = 1, 2...$, we write

$$X(t) \stackrel{\mathrm{d}}{=} X(t/n) + \big(X(2t/n) - X(t/n)\big) + ... + \big(X(t) - X((n-1)t/n)\big) \tag{3.9}$$

and use independence and stationarity of the increments. Now suppose that we define

$$\Psi_t(\theta) := -\log \mathbb{E}\big[e^{i\theta X(t)}\big], \tag{3.10}$$

for all $\theta \in \mathbb{R}$, $t \geq 0$.

By (3.9), for any two positive integers k, l we can show that

$$k\Psi_1(\theta) = \Psi_k(\theta) = l\Psi_{k/l}(\theta). \tag{3.11}$$

Therefore for any rational $t > 0$,

$$\Psi_t(\theta) = t\Psi_1(\theta). \tag{3.12}$$

If t is irrational number, then we can choose a decreasing sequence of rational numbers

t_n such that $t_n \downarrow t$ as $n \to \infty$. Right continuity of X implies right continuity of $\exp\big(-\Psi_t(\theta)\big)$ (by the Dominated Convergence Theorem) and thus (3.12) holds for all $t \geq 0$.

Now we can see that every Lévy process has the following property

$$\mathbb{E}[e^{i\theta X(t)}] = e^{-t\Psi(\theta)} \tag{3.13}$$

for $t \geq 0$, where $\Psi(\theta) := \Psi_1(\theta)$ is the characteristic exponent of $X(1)$, which has an infinitely divisible distribution. We will now refer to Ψ as the characteristic exponent of the Lévy process.

3.1.1 Examples of Lévy Processes

Poisson Process

Let us begin by recalling the definition of a Poisson process. A process valued on the non-negative integers $N = \{N(t)\}_{t\geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be a Poisson process with intensity $\lambda > 0$ if the following hold:

1. The paths of N are $\mathbb{P}$-almost surely right continuous with left limits.
2. $\mathbb{P}\{N(t) = 0\} = 1$.
3. N has independent and stationary increments.
4. For each $t > 0$, $N(t)$ is equal in distribution to a Poisson random variable with rate parameter λt.

First, consider the Poisson random variable N with a rate parameter λ. Let us observe that

$$\begin{aligned}\mathbb{E}\big[e^{i\theta N}\big] &= \sum_{k=0}^{\infty} e^{i\theta k}\frac{e^{-\lambda}\lambda^k}{k!} \\ &= e^{-\lambda(1-e^{i\theta})} \\ &= \Big[e^{-\frac{\lambda}{n}(1-e^{i\theta})}\Big]^n.\end{aligned} \tag{3.14}$$

The right-hand side of (3.14) is the characteristic function of the sum of n independent Poisson random variables, each with rate λ/n. In the Lévy-Khintchine decomposition we see that $a = \sigma = 0$ and $\Pi = \lambda\delta_1$, where δ_1 is the Dirac measure supported on 1.

Now, for the Poisson process $\{N(t)\}_{t\geq 0}$ we have

$$\mathbb{E}\big[e^{i\theta N(t)}\big] = e^{-\lambda t(1-e^{i\theta})} \tag{3.15}$$

and therefore its characteristic exponent is given by

$$\Psi(\theta) = \lambda(1-e^{i\theta}), \quad \theta \in \mathbb{R}. \tag{3.16}$$

Compound Poisson Process

Let N be a Poisson random variable with rate parameter $\lambda > 0$ and let $\{\xi_i\}_{i\geq 1}$ be a sequence of i.i.d random variables, independent of N, with a common cdf F having no atom at zero. By conditioning on N, we get for $\theta \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left[e^{i\theta\sum_{i=1}^{N}\xi_i}\right] &= \sum_{n=0}^{\infty}\mathbb{E}\left[e^{i\theta\sum_{i=1}^{n}\xi_i}\right]\frac{e^{-\lambda}\lambda^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\int_{\mathbb{R}} e^{i\theta x}F(dx)\right)^n\frac{e^{-\lambda}\lambda^n}{n!} \\
&= e^{-\lambda\int_{\mathbb{R}}1-e^{i\theta x}F(dx)} \\
&= \left[e^{-\frac{\lambda}{n}\int_{\mathbb{R}}1-e^{i\theta x}F(dx)}\right]^n .
\end{aligned} \tag{3.17}$$

From the above calculations we can conclude that distributions of the random variables of the form $\sum_{i=0}^{N}\xi_i$ are infinitely divisible with the Lévy-Khintchine decomposition with $a = -\lambda\int_{-1}^{1}xF(dx)$, $\sigma = 0$ and $\Pi(dx) = \lambda F(dx)$. Notice that when F has an atom of unit mass at 1 then the sum simply has a Poisson distribution.

Suppose that $N = \{N(t)\}_{t\geq 0}$ is a Poisson process with intensity λ. Let us consider a compound Poisson process $X = \{X(t)\}_{t\geq 0}$ which is defined as

$$X(t) := \sum_{i=1}^{N(t)}\xi_i, \quad t \geq 0. \tag{3.18}$$

Using the fact that N has stationary independent increments together with the mutual independence of the random variables $\{\xi\}_{i\geq 1}$, for $0 \leq s < t < \infty$ by writing

$$X(t) = X(s) + \sum_{i=N(s)+1}^{N(t)}\xi_i \tag{3.19}$$

it is clear that $X(t)$ is the sum of $X(s)$ and an independent copy of $X(t-s)$. The right continuity and left limits of the process N ensure the right continuity and left limits of X. Therefore the compound Poisson processes are Lévy processes. Finally

$$\mathbb{E}\left[e^{i\theta\sum_{i=1}^{N(t)}\xi_i}\right] = e^{-\lambda t\int_{\mathbb{R}}1-e^{i\theta x}F(dx)} \tag{3.20}$$

and therefore its characteristic exponent is given by

$$\Psi(\theta) = \lambda \int_{\mathbb{R}} \left(1 - e^{i\theta x}\right) F(dx), \quad \theta \in \mathbb{R}. \tag{3.21}$$

If a drift of rate $c \in \mathbb{R}$ is added to a compound Poisson process, i.e.,

$$X(t) = \sum_{i=1}^{N(t)} \xi_i + ct, \quad t \geq 0 \tag{3.22}$$

then it is straightforward to see that the resulting process is again a Lévy process. The associated infinitely divisible distribution is a shifted compound Poisson distribution with shift c. The Lévy-Khintchine exponent is given by

$$\Psi(\theta) = \lambda \int_{\mathbb{R}} \left(1 - e^{i\theta x}\right) F(dx) - ic\theta, \quad \theta \in \mathbb{R}. \tag{3.23}$$

Brownian motion

A real-valued process $B = \{B(t)\}_{t \geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be a standard Brownian motion if the following hold:

1. The paths of B are $\mathbb{P}$-almost surely continuous.

2. $\mathbb{P}\{B(0) = 0\} = 1$.

3. B has independent and stationary increments.

4. For each $t > 0$, $B(t)$ is equal in distribution to a normal random variable with mean 0 and variance t.

From the above definition we can see that the Brownian motion is a Lévy process.

Consider standard normal random variable Z. It is well known that

$$\begin{aligned}\mathbb{E}\Big[e^{i\theta Z}\Big] &= e^{-\frac{1}{2}\theta^2} \\ &= \left[e^{-\frac{1}{2}(\frac{1}{\sqrt{n}})^2\theta^2}\right]^n.\end{aligned} \tag{3.24}$$

From the above it follows that Z has an infinitely divisible distribution with $a = 0$, $\sigma = 1$ and $\Pi = 0$. Finally

$$\mathbb{E}\Big[e^{i\theta B(t)}\Big] = e^{-\frac{1}{2}t\theta^2} \tag{3.25}$$

and hence its characteristic exponent is given by

$$\Psi(\theta) = \frac{1}{2}\theta^2, \quad \theta \in \mathbb{R}. \tag{3.26}$$

Stable Lévy Processes

A comprehensive introduction to stable processes, can be found in Samorodnitsky and Taqqu [19]. A stable Lévy Process is a Lévy process $X = \{X(t)\}_{t\leq 0}$ in which each $X(t)$ is a stable random variable. A random variable X has a stable distribution if for all $n \geq 1$ there exist $a_n > 0$ and $b_n \in \mathbb{R}$ so we have

$$X_1 + \cdots + X_n \stackrel{\mathrm{d}}{=} a_n X + b_n, \tag{3.27}$$

where $X_1, ..., X_n$ are independent copies of X. A random variable is said to have a strictly stable distribution if (3.27) is satisfied but with $b_n = 0$. By subtracting b_n/n from each of the terms on the left-hand side of (3.27) one sees that the definition implies that any stable random variable is infinitely divisible.

In general the stable distributions do not have closed-form densities and the most common characterization is through the characteristic functions [20].

Theorem 3.1.4. *A random variable X is stable if and only if there exists $\sigma > 0$,*

$\beta \in [-1,1]$ *and* $\mu \in \mathbb{R}$ *such that for all* $\theta \in \mathbb{R}$*:*

- *when* $\alpha = 2$*,*

$$\Psi(\theta) = \frac{1}{2}\sigma^2\theta^2 - i\mu\theta \tag{3.28}$$

- *when* $\alpha \neq 1, 2$*,*

$$\Psi(\theta) = \sigma^\alpha|\theta|^\alpha\left[1 - i\beta sgn(\theta)\tan\left(\frac{\pi\alpha}{2}\right)\right] - i\mu\theta \tag{3.29}$$

- *when* $\alpha = 1$*,*

$$\Psi(\theta) = \sigma|\theta|\left[1 + i\beta\frac{2}{\pi}sgn(\theta)\log(|\theta|)\right] - i\mu\theta \tag{3.30}$$

Note that stable laws cover two very familiar situations: for $\alpha = 2$, the normal distribution, $X \sim N(\mu, \sigma^2)$, and for $\alpha = 1, \beta = 0$, the Cauchy distribution.

Suppose that $\mathcal{S}_\alpha(\sigma, \beta, \mu)$ is the distribution of a stable random variable with parameters $c > 0, \alpha \in (0,2), \beta \in [-1,1]$ and $\mu \in \mathbb{R}$. From the definition of its characteristic exponent it is clear that at each fixed time t the stable Lévy process has distribution $\mathcal{S}_\alpha(\sigma t, \beta, \mu)$.

One of the reasons why stable laws are so important in applications is the decay properties of the tails. Whenever $\alpha \neq 2$ we have a distribution with regularly varying tail (with index $-\alpha$) [19]

$$\lim_{y\to\infty} y^\alpha \mathbb{P}\{X > y\} = C_\alpha \frac{1+\beta}{2}\sigma^\alpha, \tag{3.31}$$

$$\lim_{y\to\infty} y^\alpha \mathbb{P}\{X < -y\} = C_\alpha \frac{1-\beta}{2}\sigma^\alpha, \tag{3.32}$$

where $C_\alpha > 1$.

Moreover, stable processes display self-similarity. Stochastic process $X = \{X(t)\}_{t\geq 0}$ is self-similar with Hurst index $H > 0$ if the two processes $\{X(at)\}_{t\geq 0}$ and $\{a^H X(t)\}_{t\geq 0}$ have the same finite-dimensional distributions for all $a \geq 0$. For example, by examining the Theorem 3.1.4 it is easily verified that "rotationally invariant" Levy process X ($\beta = \mu = 0$) with the characteristic exponent

$$\Psi(\theta) = \sigma^{\alpha}|\theta|^{\alpha}, \tag{3.33}$$

with $\alpha \in (0, 2], \sigma > 0$, is self-similar with Hurst index $H = 1/\alpha$.

3.2 Subordinators

In this section we present some background and examples of subordinators. Relevant theory can be found in the book about Lévy processes by Applebaum [18].

A subordinator is an almost surely non-decreasing Lévy process. Such processes can be thought of as a model of a random time evolution, since if $T = \{T(t)\}_{t\geq 0}$ is a subordinator we have

$$T(t) \geq 0 \tag{3.34}$$

for each $t \geq 0$ and whenever $t_1 \leq t_2$

$$T(t_1) \leq T(t_2). \tag{3.35}$$

One can check whether a given Lévy process is a subordinator by the special form of its Levy triple (Theorem 1.2 in Bertoin [21]).

Theorem 3.2.1. *A Levy process T is a subordinator iff its Levy triple has the form $(b, 0, \Pi)$, i.e, its characteristic exponent takes the form*

$$\Psi(\theta) = \int_0^{\infty} \left(1 - e^{i\theta x}\right)\Pi(dx) - ib\theta \tag{3.36}$$

where $b \geq 0$ and the measure Π satisfies additional requirements

$$\Pi\{(-\infty, 0)\} = 0 \quad \textit{and} \quad \int_0^\infty (1 \wedge x)\Pi(dx) < \infty. \tag{3.37}$$

Conversely, any mapping from $\mathbb{R} \to \mathbb{C}$ of the form (3.36) is the characteristic exponent of a subordinator.

An important feature of the subordinators is that one can work with the Laplace transform, instead of characteristic function. Since T is nonnegative almost surely, for each $t \geq 0$ the map $\theta \to \mathbb{E}[e^{i\theta T(t)}]$ can be analytically continued to the region $\{i\theta : \theta > 0\}$. We obtain the following expression for the Laplace transform of the distribution

$$\mathbb{E}\left[e^{-\theta T(t)}\right] = e^{-t\psi(\theta)}, \tag{3.38}$$

where

$$\psi(\theta) = \Psi(i\theta) = b\theta + \int_0^\infty \left(1 - e^{-\theta x}\right)\Pi(dx), \tag{3.39}$$

for each $\theta > 0$. The function ψ is called the Laplace exponent of the subordinator.

There are two main examples of subordinators that we have already seen in Chapter 3.1.1. It is easy to see that the Poisson process is a subordinator. More generally compound Poisson process with a drift is a subordinator if and only if the random variables ξ_n and drift c in Equation 3.22 are nonnegative. The Laplace exponent is easily calculated and is equal to

$$\psi(\theta) = c\theta + \lambda \int_0^\infty \left(1 - e^{-\theta x}\right)F(dx), \quad \theta \geq 0. \tag{3.40}$$

Another important example is an α-stable subordinator. Using straightforward calculus we can show that for $0 < \alpha < 1$,

$$\theta^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\theta x})\frac{dx}{x^{1+\alpha}}, \quad \theta \geq 0. \tag{3.41}$$

Therefore by Theorem 3.2.1 and Equation 3.39 we see that for each $\alpha \in (0,1)$ there exists an α-stable subordinator T with Laplace exponent

$$\psi(\theta) = \theta^\alpha, \quad \theta \geq 0. \tag{3.42}$$

Note that this corresponds to the characteristic exponent in the Theorem 3.1.4 with $\mu = 0, \beta = 1$ and $\sigma^\alpha = \cos(\alpha\pi/2)$.

One of the most important probabilistic applications of the subordinators is the "time change". Let $X = \{X(t)\}_{t\geq 0}$ be a Lévy process and let $T = \{T(t)\}_{t\geq 0}$ be a subordinator independent of X. Let us define a new stochastic process $Z = \{Z(t)\}_{t\geq 0}$ by the composition

$$Z(t) := X \circ T(t) = X(T(t)), \quad t \geq 0. \tag{3.43}$$

It turns out that again $\{Z(t)\}_{t\geq 0}$ is a Lévy process (Theorem 1.3.25 [18]). Moreover, by conditioning we find that

$$\mathbb{E}\big[e^{i\theta Z(t)}\big] = \mathbb{E}\big[e^{-U(t)\Psi_X(\theta)}\big] = e^{\psi(\Psi(\theta))}, \quad t \geq 0, \theta \in \mathbb{R}, \tag{3.44}$$

so the Lévy exponent of process Z is given by

$$\Psi_Z(\theta) = \psi_T \circ \Psi_X(\theta), \quad \theta \in \mathbb{R}. \tag{3.45}$$

For example, let T be an α-stable subordinator, $\alpha \in (0,1)$, and let

$$X(t) := \sqrt{2}B(t), \quad t \geq 0. \tag{3.46}$$

Recall that Laplace exponent for T is $\psi_T(s) = s^\alpha$ and the Lévy exponent of X is $\Psi_X(\theta) = \theta^2$. Therefore

$$\psi_Z(\theta) = \theta^{2\alpha}, \tag{3.47}$$

and we recognize Z as a Cauchy process, so each $Z(t)$ has a symmetric Cauchy distribution with parameters $\mu = 0$ and $\sigma = 1$.

3.2.1 Inverse Subordinated Brownian Motion as a Subdiffusive Process

As observed by Revuz and Yor [22] "It is a natural idea to change the speed at which a process runs through its path". In this section we look closer at the technique of time change and we show how it can be used to obtain subdiffusive processes.

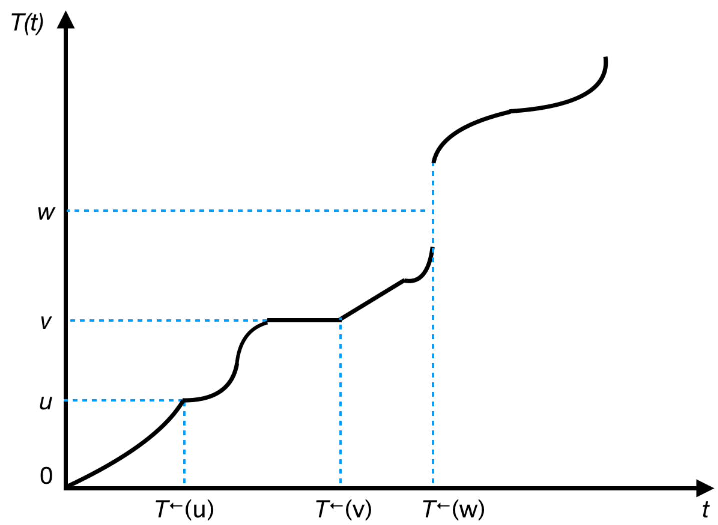


Figure 3.1: Path of increasing process T and the way to find it's inverse $T^{\leftarrow}$.

Given a subordinator $T = \{T(t)\}_{t\geq 0}$, the first-passage time process defined as

$$T^{\leftarrow}(t) := \inf\{s : T(s) > t\}, \quad t \geq 0, \tag{3.48}$$

is called an inverse subordinator. Here and below, it is understood that $\inf\{\emptyset\} = \infty$.

To better understand what follows, consider Figure 3.1 showing the path of T and the way to find its inverse $T^{\leftarrow}$. Let us observe few things. Note that T being increasing process implies its inverse is increasing as well. Similarly, it is always positive. Therefore the inverse subordinator also seems to be a good choice for a random time change. The key observation from the point of modeling subdfiffusive processes is that the jumps of subordinator T correspond to the flat periods of its inverse. Therefore "sufficiently large" jumps of the subordinator lead to "long" flat periods of its inverse, very characteristic for the subdiffusive dynamics, since they represent long immobilization periods. Finally let us note that the inverse subordinators are in general non-Markovian (they correspond to local times of some Markov processes, therefore they have memory [23]).

In what follows, we will consider the time changed process $X = \{X(t)\}_{t\geq 0}$ defined as

$$X(t) := B \circ T^{\leftarrow}(t) = B(T^{\leftarrow}(t)), \quad t \geq 0, \tag{3.49}$$

where $B = \{B(t)\}_{t\geq 0}$ is a standard Brownian Motion independent of an inverse subordinator $T^{\leftarrow} = \{T^{\leftarrow}(t)\}_{t\geq 0}$.

Let us introduce the filtration $\{\mathcal{F}_t\}$ proposed by Magdziarz in [17], where

$$\mathcal{F}_t = \bigcap_{u>t} \sigma\big(\{B(y) : 0 \leq y \leq u\}, \{T^{\leftarrow}(y) : y \geq 0\}\big). \tag{3.50}$$

Note that by definition $\{\mathcal{F}_t\}$ is right-continuous, B is a $\mathcal{F}_t$-martingale, and for every fixed $t_0 > 0$ the random variable $T^{\leftarrow}(t_0)$ is a stopping time with respect to $\{\mathcal{F}_t\}$. Therefore $\{\mathcal{G}_t\}$, where

$$\mathcal{G}_t = \mathcal{F}_{T^{\leftarrow}(t)} \tag{3.51}$$

is a well-defined filtration. Here, the filtration at the stopping time τ is

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, t \geq 0\}. \tag{3.52}$$

It turns out that inverse subordinated Brownian motion is a martingale with respect to that filtration (Theorem 1 in [17]), i.e, $X = B \circ T^{\leftarrow}$ is a $\mathcal{G}_t$ martingale.

The above theorem has some interesting consequences. Since X is a $\{\mathcal{G}_t\}$-martingale, we know that $X^2 - \langle X, X\rangle$ is a $\{\mathcal{G}_t\}$-martingale as well. Here $\langle X, X\rangle$ stands for quadratic variation of X. By Proposition 1.5 in [22] we have

$$\langle X, X\rangle(t) = \langle B, B\rangle(T^{\leftarrow}(t)) = T^{\leftarrow}(t), \tag{3.53}$$

implying that $X^2 - T^{\leftarrow}$ is a $\{\mathcal{G}_t\}$-martingale as well. Now we have

$$\mathbb{E}[X^2(t) - T^{\leftarrow}(t)] = \mathbb{E}[X^2(0) - T^{\leftarrow}(0)] = 0. \tag{3.54}$$

Therefore we arrive at

$$\mathbb{E}[X^2(t)] = \mathbb{E}[T^{\leftarrow}(t)]. \tag{3.55}$$

Now onto a specific example. First consider a well studied case [9],[24], where T is an α-stable subordinator T_α, $\alpha \in (0, 1)$, discussed in section 3.1.1. Using the fact that T_α is $1/\alpha$ similar, we have

$$\mathbb{P}\{T_\alpha^{\leftarrow}(t) \le \tau\} = \mathbb{P}\{T_\alpha(\tau) \ge t\} = \mathbb{P}\{(t/T_\alpha(1))^\alpha \le \tau\}. \tag{3.56}$$

That observation allows us to derive some properties of the process $T_\alpha^{\leftarrow}$ (Corollary 3.1 in [24]).

Theorem 3.2.2. *For any $t > 0$*

1. $T^{\leftarrow}(t) \stackrel{d}{=} (T_\alpha(1)/t)^{-\alpha}$,

2. *for any $\gamma > 0$ the γ-moment of $T_\alpha^{\leftarrow}(t)$ exists and there exists a positive finite constant $C(\alpha, \gamma)$ such that*

$$\mathbb{E}[(T_\alpha^{\leftarrow}(t))^\gamma] = C(\alpha, \gamma)t^{\alpha\gamma}, \tag{3.57}$$

3. the random variable $T_\alpha^{\leftarrow}(t)$ has density

$$f_t(\tau) = \frac{t}{\alpha}\tau^{-1-1/\alpha} g_\alpha(t\tau^{-1/\alpha}), \tag{3.58}$$

where g_α is the density of the α-stable random variable $T_\alpha(1)$.

Furthermore Bingham [25] showed that the Laplace transform of $T_\alpha^{\leftarrow}$ equals

$$\mathbb{E}\left[e^{-\theta T_\alpha^{\leftarrow}(t)}\right] = E_\alpha(-\theta t^\alpha), \tag{3.59}$$

where $E_\alpha(z)$ is the Mittag-Leffler function, discussed in the Appendix A.2.

Now knowing moments of $T_\alpha^{\leftarrow}$ we can write

$$\mathbb{E}[X^2(t)] = \mathbb{E}[T_\alpha^{\leftarrow}(t)] = C(\alpha, 1)t^\alpha, \tag{3.60}$$

showing that process X is subdiffusive.

Now let us consider a case when subordinator T is a compound Poisson process with a drift and its inverse $T^{\leftarrow}$, i.e,

$$T(t) = \sum_{i=1}^{N(t)} \sigma_i + t, \quad t \geq 0, \tag{3.61}$$

where, $N(t)$ is a Poisson process with intensity λ, $\{\sigma_i\}_{i\geq 1}$ is a sequence of i.i.d. nonnegative random variables, independent of $N(t)$, with distribution with regularly varying tail, i.e.,

$$\bar{F}_\sigma(t) \in RV_{-\alpha}, \quad \alpha \in (0, 1). \tag{3.62}$$

Recall that the Laplace exponent of compound Poisson process with a drift is given by

$$\psi(\theta) = \theta + \lambda \int_0^\infty \left(1 - e^{-\theta x}\right) F(dx), \quad \theta \geq 0. \tag{3.63}$$

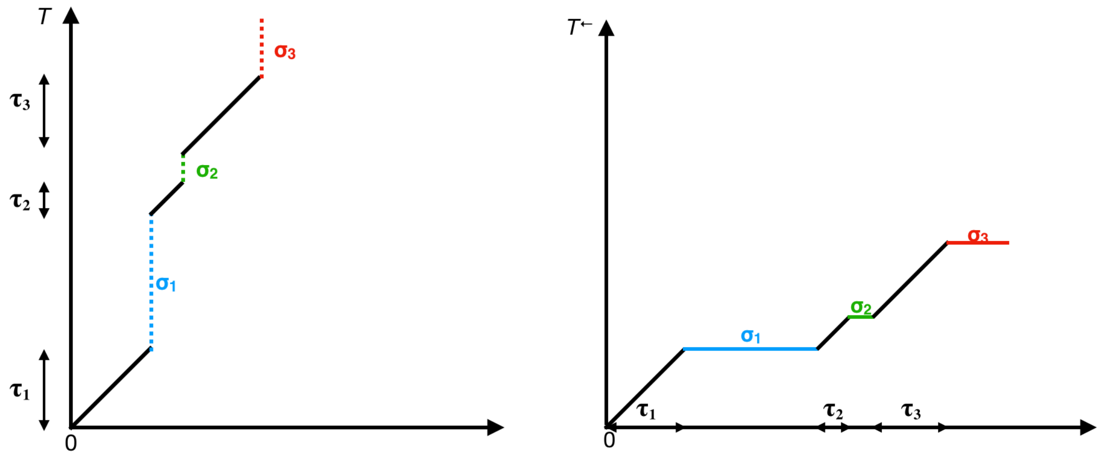


Figure 3.2: Comparison of a path of a compound Poisson process with a drift and it's inverse.

Note that we can rewrite above in terms of Laplace transform of F, i.e.,

$$\psi(\theta) = \theta + \lambda(1 - \hat{f}(\theta)), \quad \theta \geq 0. \tag{3.64}$$

Since $\bar{F}_\sigma \in RV_{-\alpha}$, using Theorem A.1.8 we have

$$1 - \hat{f}_\sigma(\theta) \sim \Gamma(1-\alpha)\theta^\alpha L(1/\theta), \quad \theta \downarrow 0. \tag{3.65}$$

Therefore

$$\psi(\theta) \sim \lambda\Gamma(1-\alpha)\theta^\alpha L(1/\theta), \quad \theta \downarrow 0. \tag{3.66}$$

Now let us present a connection to the renewal theory. Notice that subordinator is a transient Markov process and we can define its potential measure

$$\mathcal{U}(A) := \mathbb{E}\bigg(\int_0^\infty \mathbb{1}_{\{T(t)\in A\}} dt\bigg). \tag{3.67}$$

The distribution function $U(x)$ of the potential measure $\mathcal{U}$

$$\mathcal{U}([0,x]) := U(x) = \mathbb{E}\bigg(\int_0^\infty \mathbb{1}_{\{T(t)\leq x\}} dt \bigg), \quad x \geq 0, \tag{3.68}$$

is known as the renewal function. Notice that since the Laplace transform of the renewal measure is

$$\hat{U}(\theta) = \int_{[0,\infty)} e^{-\theta x} U(dx) = \frac{1}{\psi(\theta)}, \quad \theta > 0 \tag{3.69}$$

the renewal measure characterizes the law of the subordinator. The most important observation is that 3.68 implies that

$$U(x) = \mathbb{E}[T^{\leftarrow}(x)]. \tag{3.70}$$

Now combining 3.55 with 3.66 and 3.70 we get

$$\mathbb{E}[X^2(t)] = \mathbb{E}[T^{\leftarrow}(t)] = U(t) \sim \frac{1}{\lambda\Gamma(1+\alpha)\Gamma(1+\alpha)} \frac{t^\alpha}{L(t)}, \quad t \to \infty. \tag{3.71}$$

showing that process X is subdiffusive.

Remark 3.2.3. With the above we can prove the statement we made in Remark 2.1.1. It has been shown in [26] that if $\mathbb{E}[T(1)] < \infty$ then as $t \to \infty$

$$U(t) \sim \frac{t}{\mathbb{E}[T(1)]}. \tag{3.72}$$

Note that

$$\begin{aligned}\mathbb{E}[T(1)] &= -\frac{d}{d\theta}\mathbb{E}[e^{-\theta T(1)}]_{\theta=0} \\ &= -\frac{d}{d\theta}[e^{-\psi(\theta)}]_{\theta=0} \\ &= \psi'(0).\end{aligned} \tag{3.73}$$

Therefore if T is a compound Poisson process with a drift we obtain

$$\mathbb{E}[T(1)] = 1 + \lambda\mathbb{E}[\sigma]. \tag{3.74}$$

From the above it is pretty visible that the assumption that σ has a finite mean implies that the process X is diffusive, i.e, as $t \to \infty$

$$\mathbb{E}[X^2(t)] = U(t) \sim \frac{t}{\mathbb{E}[T(1)]} = \frac{t}{1 + \lambda\mathbb{E}[\sigma]}. \tag{3.75}$$

3.2.2 Local Times and Subordinators

The purpose of this chapter is to reveal interesting connection between the excursion intervals of a Markov process Z and inverse subordinator $T^{\leftarrow}$, called the local time, which stays constant on the excursion intervals. This topic is studied in detail in chapter IV of Bertoin [27], we will use this perspective to show the connection between the inverse subordinated Brownian motion and the subdiffusive switching model presented in Chapter 1.

Let $Z = \{Z(t)\}_{t\geq 0}$ be a Markov process, introduced in Chapter 1. Recall the sequence of switching times

$$0 = R_0 < S_1 < R_1 < ... \tag{3.76}$$

In terms of Z, R_0 denotes the first return of Z to zero and S_1 is the first exit time from 0,

$$S_1 = \inf\{t \geq 0, Z(t) \neq 0\}. \tag{3.77}$$

The assumption, that at time zero particle just entered the diffusive state is equivalent with assuming that 0 is a holding point of Z, that is $\mathbb{P}\{R_0 = 0\} = 1$ and $\mathbb{P}\{S_1 = 0\} = 0$. Now $R_0 < S_1 < R_1 < ...$, where $R_0 = 0$, $R_n = \inf\{t > S_n, Z(t) = 0\}$, $S_{n+1} = \inf\{t > R_n, Z(t) \neq 0\}$, denotes the sequence of successive exits from/returns to 0 of Z.

Recall that we assumed that the diffusion times $\{\tau_i\}_{i\geq 1}$ are exponentially distributed

$$\tau_i \sim \mathrm{Exp}(\lambda), \tag{3.78}$$

and immobilization times $\{\sigma_i\}_{i\geq i}$

$$\sigma_i \sim F_\sigma, \tag{3.79}$$

where F_σ is a mixture of exponentials.

It has been shown in [27] that for all $t \geq 0$ the process

$$\int_0^t \mathbb{1}_{\{Z(s)=0\}} ds, \tag{3.80}$$

known as a local time of Z at zero, coincides with the inverse subordinator, i.e,

$$T^{\leftarrow}(t) = \int_0^t \mathbb{1}_{\{Z(s)=0\}}, \tag{3.81}$$

where T is a compound Poisson process with a drift, with Laplace exponent

$$\psi(\theta) = \theta + \lambda \int_0^\infty \left(1 - e^{-\theta x}\right) F_\sigma(dx), \quad \theta \geq 0. \tag{3.82}$$

Let us recall the SDE representation of Switching Diffusion presented in Chapter 1.

$$\begin{cases} dX(t) = \sqrt{2D} \cdot \phi(t) dB(t) \\ \big(X(0), \phi(0)\big) = (0, 1) \end{cases} \tag{3.83}$$

where

$$\phi(t) := \begin{cases} 1 & R_{i-1} \le t < S_i, \quad i \ge 1 \\ 0 & S_i \le t < R_i, \quad i \ge 1. \end{cases} \tag{3.84}$$

Let $n(t)$ be defined as

$$n(t) = \sup\{i : S_i \le t\}. \tag{3.85}$$

Notice that we can write

$$\begin{aligned} X(t) &= \int_0^t \phi(s) dB(s) \\ &= \sum_{i=1}^{n(t)} \int_{R_{i-i}}^{S_i} dB(s) + \int_{R_{n(t)}}^t \mathbb{1}_{\{R_{n(t)}<t\}} dB(s). \end{aligned} \tag{3.86}$$

Now, let $\{B\}_{i \ge 0}$ be a sequence of independent Brownian motions. We have equality in distribution

$$\begin{aligned} X(t) &= \sum_{i=1}^{n(t)} B_i(\tau_i) + \mathbb{1}_{\{R_{n(t)}<t\}} B(t - R_{n(t)}) \\ &= B\bigg(\sum_{i=1}^{n(t)} \tau_i + \mathbb{1}_{\{R_{n(t)}<t\}} (t - R_{n(t)}) \bigg) \\ &= B\bigg(\int_0^t \mathbb{1}_{\{Z(s)=0\}} ds \bigg) \\ &= B(T^{\leftarrow}(t)). \end{aligned} \tag{3.87}$$

Therefore we conclude that Switching Diffusion X and Brownian motion inverse subordinated to compound Poisson process with a drift $B \circ T^{\leftarrow}$ have the same one dimensional distributions.

3.2.3 Generalized Diffusion Equation

In the literature there is a formulation for the probability density function of the process $X(t) = B(T^{\leftarrow}(t))$. It was proposed by Gajda and Magdziarz in [28] that $p(x,t)$ satisfies following integro-differential equation,

$$\partial_t p(x,t) = \frac{1}{2}\Phi_t \partial_{xx} p(x,t), \tag{3.88}$$

where Φ_t is an integro-differential operator given by

$$\Phi_t f(t) = \frac{d}{dt}\int_0^t M(t-s)f(s)ds \tag{3.89}$$

for sufficiently smooth function f. The memory kernel $M(t)$ is defined via its Laplace transform

$$\hat{M}(\theta) = \frac{1}{\psi(\theta)}, \tag{3.90}$$

where ψ is a Laplace exponent of the subordinator T.

The case when T is an α-stable subordinator (and hence infinite activity process) is well studied [2],[17]. Recall that the Laplace exponent for T_α is given by

$$\psi(\theta) = \theta^\alpha. \tag{3.91}$$

In this case the operator Φ_t introduced in 3.89 is the Riemann-Liouville fractional derivative $D_t^{1-\alpha}$ A.20,

$$\Phi_t f(t) = D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds. \tag{3.92}$$

Therefore the density $p(x,t)$ of process X satisfies the following equation

$$\partial_t p(x,t) = \frac{1}{2}D_t^{1-\alpha}\partial_{xx} p(x,t). \tag{3.93}$$

We would like to show that the Generalized Diffusion Equation is not true for all inverse subordinated Brownian motions. As an counterexample let us consider the case where T is a compound Poisson process with a drift, with Laplace exponent

$$\psi(\theta) = \theta + \lambda(1 - \hat{f}_\sigma(t)). \tag{3.94}$$

Suppose that σ is exponentially distributed with rate μ. Now we have

$$\hat{M}(\theta) = \frac{\theta + \mu}{\theta^2 + (\mu + \lambda)\theta}, \tag{3.95}$$

and by inverting Laplace transform we obtain

$$M(t) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\mu+\lambda)t}. \tag{3.96}$$

In this case operator Φ_t becomes

$$\Phi_t f(t) = f(t) - \lambda \int_0^t e^{-(\mu+\lambda)(t-s)} f(s) ds. \tag{3.97}$$

Therefore $p(x,t)$ satisfies

$$\partial_t p(x,t) = \partial_{xx} p(x,t) - \lambda \int_0^t e^{-(\mu+\lambda)(t-s)} \partial_{xx} p(x,s) ds. \tag{3.98}$$

The Fourier-Laplace transform can be evaluated and is equal to

$$\bar{p}(u,\theta) = \tilde{p}(u,0) \left(\frac{\theta + \mu + \lambda}{(\theta + \mu + \lambda)\theta + (\theta + \mu) u^2} \right). \tag{3.99}$$

On the other hand recall the equations for the $p(x,t)$ for Switching Diffusion

$$p(x,t) = p(x,t,0) + \lambda \int_0^t f_\sigma(t-s) p(x,s,0) ds \tag{3.100}$$

where $p(x,t,0)$ is a solution to

$$\partial_t p(x,t,0) = \partial_{xx} p(x,t,0) - \lambda p(x,t,0) + \lambda \int_0^t f_\sigma(t-s) p(x,s,0) ds. \tag{3.101}$$

On the Fourier-Laplace side 3.101 becomes

$$\bar{p}(u,\theta,0) = \frac{\tilde{p}(u,0,0)}{s + \lambda(1 - \hat{f}_\sigma(\theta)) + u^2} \tag{3.102}$$

therefore

$$\bar{p}(u,\theta) = \tilde{p}(u,0,0)\left[\frac{1 + \lambda \hat{f}_\sigma(\theta)}{s + \lambda(1 - \hat{f}_\sigma(\theta)) + u^2}\right]. \tag{3.103}$$

Now since $\sigma \sim \text{Exp}(\mu)$ we get

$$\bar{p}(u,\theta) = \tilde{p}(u,0,0)\left[\frac{\theta + \mu + \mu\lambda}{(\theta + \mu + \lambda)\theta + (\theta + \mu)u^2}\right]. \tag{3.104}$$

Comparing the 3.99 and 3.104 we conclude that the one dimensional probability densities for inverse subordinated Brownian Motion and Switching diffusion are different which is contradicting the equality in distribution between the processes shown in section 3.2.2.

3.3 Stochastic-Process Limits

This section will be devoted to the topic of stochastic-process limits, i.e., limits in which a sequence of stochastic processes converges to another stochastic process. The idea of such limits is to rescale the time and space of a process, so the limiting process is stripped away from unessential details and only key features remain. The most famous result of that type is Donsker's theorem (or functional central limit theorem (FCLT)) which shows how random walk converges to a Brownian motion.

Let $\mathbb{S}$ be a complete, separable metric space with metric d and let $\mathcal{S}$ be the Borel

σ-algebra of subsets of $\mathbb{S}$ generated by open sets. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A random element X in $\mathbb{S}$ is a measurable map from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{S}, \mathcal{S})$. Given a sequence of $\{X_n\}_{n\geq 0}$ of random elements of $\mathbb{S}$, there is a corresponding sequence of distributions on $\mathcal{S}$

$$P_n = \mathbb{P}\{X_n \in \cdot\}, \quad n \geq 0. \tag{3.105}$$

P_n is called the distribution of X_n. Then X_n converges weakly to X_0 (written $X_n \implies X_0$) if whenever $f \in C(\mathbb{S})$, the class of bounded, continuous, real-valued functions on $\mathbb{S}$, we have

$$\mathbb{E}[f(X_n)] = \int_{\mathbb{S}} f(x)P_n(dx) \to \mathbb{E}[f(X_0)] = \int_{\mathbb{S}} f(x)P_0(dx). \tag{3.106}$$

The definition of weak convergence of random variables in $\mathbb{R}$ is given in terms of one-dimensional distribution functions, which does not generalize nicely to higher dimensions. Often to prove weak convergence, for separable and complete spaces $\mathbb{S}$, one have to show that the family of distributions $\{P_n\}$ is tight, i.e., for any ϵ, there exist a compact $K_\epsilon \in \mathcal{S}$ such that

$$P(K_\epsilon) > 1 - \epsilon \quad \text{for } P \in \{P_n\}. \tag{3.107}$$

Tightness can be checked, however it is rarely easy.

The power of weak convergence theory comes from the fact that once a basic convergence result has been proved, many corollaries emerge with little effort. There are many useful techniques but one that will be the most fruitful for our needs is the continuous mapping theorem.

Theorem 3.3.1 (Continuous mapping theorem)**.** *Let* $(\mathbb{S}_i, d_i)$*,* $i = 1, 2$*, be two metric spaces, and suppose* $\{X_n\}_{n\geq 0}$ *are random elements of* $(\mathbb{S}_1, \mathcal{S}_1)$ *and* $X_n \implies X_0$*. If*

$h : \mathbb{S}_1 \to \mathbb{S}_2$ satisfies

$$\mathbb{P}\Big\{X_0 \in \{s_1 \in \mathbb{S}_1 : h \textit{ is discontinuous at } s_1\}\Big\} = 0, \tag{3.108}$$

then in $\mathbb{S}_2$

$$h(X_n) \implies h(X_0). \tag{3.109}$$

3.3.1 Central Limit Theorems on $\mathbb{R}$

In our research we are mainly interested in convergence in functional spaces. Most of the results however are based on the one dimensional theorems. Let us start our discussion with the most fundamental limit theorem that is used when constructing Brownian Motion from a random walk - Central Limit Theorem (CLT). Classical version of CLT states: given iid random variables $X_1, X_2, ..$ with $\mathbb{E}[X_1] = \mu$ and $\mathbb{E}[X_1^2] = \sigma^2 < \infty$ define

$$S_n := \sum_{i=1}^{n} X_i \tag{3.110}$$

and then as $n \to \infty$, we have

$$\frac{S_n - n\mu}{\sqrt{n}} \implies S, \tag{3.111}$$

where $S \sim \mathcal{N}(0, \sigma^2)$ (here $\implies$ means convergence in distribution). An essential assumption for classical CLT to work is that the second moment of X_i is finite. In the world of "heavy tailed" phenomenon that is rarely a case. What would happen if the collection of random variables had infinite second or even first moment? Let start with introducing an idea of a domain of attraction.

Definition 3.3.2. We say that random variable X is in a domain of attraction of S if for n-iid copies of $X : X_1, X_2, ..., X_n$ there exist sequences a_n, b_n s.t

$$\frac{S_n - b_n}{a_n} \implies S. \tag{3.112}$$

The random variable S in the literature is called an attractor. The Central Limit theorem shows that normal random variables can be attractors, but it is very natural to ask what are the other possible attractors? The answer is surprisingly simple; the only other attractors beside normal are stable random variables. We already discussed the stable random variables in 3.1.1. Recall that stable random variables are characterized by four parameters

$$\alpha \in (0,2], \sigma \leq 0, \beta \in (-1,1), \mu \in \mathbb{R}, \tag{3.113}$$

and we will denote stable distributions by $S_\alpha(\sigma,\beta,\mu)$. Next theorem characterizes the domain of attraction of stable distributions [19].

Theorem 3.3.3 (stable CLT). *Let $X_1,..,X_n$ be i.i.d. with cumulative distribution function F. There exist $a_n > 0$, $b_n \in \mathbb{R}$, $n = 1,2,...,$ such that the distribution of $a_n^{-1}(S_n - b_n)$ converges as $n \to \infty$ to $S_\alpha(1,\beta,0)$ if and only if both*

1. *$x^\alpha[1 - F(x) + F(-x)] = L(x)$ is slowly varying at infinity,*

2. *$\frac{F(-x)}{1+F(-x)+F(-x)} \to \frac{1-\beta}{2}$ as $x \to \infty$.*

The a_n must satisfy

$$\lim_{n\to\infty} \frac{nL(a_n)}{a_n^\alpha} = \begin{cases} \Gamma(1-\alpha)\cos(\pi\alpha/2) & \text{if } 0<\alpha<1, \\ 2/\pi & \text{if } \alpha = 1 \\ \frac{\Gamma(2-\alpha)}{\alpha-1}|\cos(\frac{\pi\alpha}{2})| & \text{if } 1<\alpha<2. \end{cases} \tag{3.114}$$

The b_n can be chosen as follows:

$$b_n = \begin{cases} 0 & \text{for } 0<\alpha<1, \\ na_n\int_{-\infty}^{\infty}\sin(x/a_n)dF(x) & \text{for } \alpha = 1 \\ n\int_{-\infty}^{\infty} x dF(x) & \text{for } 1<\alpha<2. \end{cases} \tag{3.115}$$

In all cases, $a_n = n^{1/\alpha} L_0(n)$ *where* L_0 *is slowly varying.*

3.3.2 Functional Limit Theorems

In this section we will discuss important limit theorems that take place in in the cádlág space (right continuous functions with left limits). As mentioned before, proving weak convergence in higher dimensional spaces is not easy and therefore we establish some main results here and then we will re-use this results to prove new functional theorems. The following results are often invoked when proving theorems about convergence of CTRW (continuous times random walks).

We start with discussion about the underlying function space of possible sample paths for the stochastic processes. We want to consider stochastic processes with discontinuous sample paths. Therefore for $0 < T < \infty$ let $\mathcal{C}[0,T)$ be the space of all continuous real-valued functions defined on $[0,T)$, and, $\mathcal{D}[0,T)$ be the space of all right-continuous functions on $[0,T)$ with left hand limits on $(0,T]$. We define $\mathcal{C}[0,\infty)$ and $\mathcal{D}[0,\infty)$ in a similar way.

The metric on $\mathcal{C}[0,1]$ is the uniform metric

$$d(x(\cdot), y(\cdot)) = \sup_{0 \le t \le 1} |x(t) - y(t)| := ||x(\cdot) - y(\cdot)||. \tag{3.116}$$

Note that $\mathcal{C}[0,1]$ is a subspace of $\mathcal{D}[0,1]$. The uniform metric works really well on $\mathcal{C}$ but not on $\mathcal{D}$. Since elements of $\mathcal{D}$ are discontinuous, in order for the functions to be close, under uniform metric, the corresponding jumps had to occur at the same time. This can be avoided by introducing time deformations

$$\Lambda = \{\lambda : [0,1] \to [0,1] : \lambda(0) = 0, \lambda(1) = 1\}, \tag{3.117}$$

such that every $\lambda(\cdot) \in \Lambda$ is continuous and strictly increasing. Let $e(\cdot) \in \Lambda$ be the identity map on $[0, 1]$. Then the standard J_1 metric on $\mathcal{D}[0, 1]$ is

$$d_{J_1}(x(\cdot), y(\cdot)) = \inf_{\lambda \in \Lambda} \left\{ ||x \circ \lambda - y|| \vee ||\lambda(\cdot) - e(\cdot)|| \right\} \tag{3.118}$$

The idea behind going from uniform to this metric is to say that two functions are close if they are uniformly close over $[0, 1]$ after allowing small perturbations of time. Finally note that converge in the uniform metric implies convergence in the J_1 metric.

The most famous result in the theory of weak convergence is Donsker's theorem, which shows that a random walk with suitable time and space scaling looks roughly like a Brownian Motion. A classical proof of Donsker's theorem using convergence of the finite-dimensional distributions plus tightness can be found in [29].

Theorem 3.3.4 (Donsker's theorem)**.** *Suppose that* $\{X_i\}_{i \geq 1}$ *are iid random variables satisfying*

$$\mathbb{E}[X_i] = 0 \quad \textit{and} \quad \mathbb{E}[X_i^2] = 1. \tag{3.119}$$

Define $S_0 = 0$ *and for* $n \geq 1$

$$S_n := \sum_{i=1}^{n} X_j. \tag{3.120}$$

Then in $\mathcal{D}[0, \infty)$*, as* $n \to \infty$*,*

$$\frac{S_{\lfloor n \cdot \rfloor}}{\sqrt{n}} \implies B(\cdot), \tag{3.121}$$

where $B = \{B(t)\}_{t \geq 0}$ *is a standard Brownian motion.*

All central limit theorems have their functional central limit theorems (FCLT) counterparts. The next theorem shows that rescaled CTRW with stable "steps" look like standard stable Lévy motion.

Theorem 3.3.5 (stable FCLT)**.** *Under the conditions of Theorem 3.3.3, in addition* *to the stable CLT*

$$\frac{S_n - b_n}{a_n} \implies S_\alpha(1, \beta, 0), \tag{3.122}$$

there is convergence in $\mathcal{D}[0, \infty)$

$$\frac{S_{\lfloor n \cdot \rfloor} - b_n \cdot}{a_n} \implies S(\cdot), \tag{3.123}$$

where the limit S *is a standard* (α, β)*- stable Lévy motion, with*

$$S(t) = t^{1/\alpha} S_\alpha(1, \beta, 0) = S_\alpha(t^{1/\alpha}, \beta, 0). \tag{3.124}$$

3.3.3 Functional Convergence of Switching Diffusion

In this section we present the stochastic-process limit of our own. We show that Brownian motion inversely subordinated to compound Poisson with a drift under rescaling is equivalent to the inverse subordinated Brownian motion, i.e, we present the functional limit theorem for Switching Diffusion. Let $X = \{X(t)\}_{t \geq 0}$ be a Brownian motion inverse subordinated to compound Poisson process with a drift,

$$X(t) = B\big(T^{\leftarrow}(t)\big), \tag{3.125}$$

where T is a compound Poisson process with a drift. Let $X_\alpha = \{X_\alpha(t)\}_{t \geq 0}$ be a Brownian motion inverse subordinated $\alpha-$stable process,

$$X_\alpha(t) = B\big(T_\alpha^{\leftarrow}(t)\big), \tag{3.126}$$

where T_α is a $\alpha-$stable process with $\alpha \in (0, 1)$. The essential result to show the functional convergence of inverse subordinate processes is to show that there is a functional convergence of the clocks. Establishing the functional convergence in the space $\mathcal{D}[0,\infty)$ is not easy but fortunately functional convergence of the increasing processes reduces to finite dimensional convergence (Theorem 3 of [25]).

Theorem 3.3.6. *Let $\{X_n\}_{n\geq 1}$ be a sequence of stochastic processes whose path-functions lie in $\mathcal{D}[0,\infty)$. If*

1. *the finite-dimensional distributions of X_n converge as $n \to \infty$ to those of X;*

2. *the process X is continuous in probability;*

3. *the processes X_n have monotone path-functions.*

Then $X_n \implies X$ in $\mathcal{D}[0,\infty)$.

Notice that since T and T_α are strictly increasing their inverses $T^{\leftarrow}$ and $T_\alpha^{\leftarrow}$ are increasing as well.

Before we prove the main result let us prove well known functional theorem for Poisson process.

Theorem 3.3.7. *As $c \to \infty$,*

$$N(c\cdot)/c \implies \lambda\cdot \ \text{ in } \mathcal{D}[0,\infty) \tag{3.127}$$

Proof. Fix $t_1, ..., t_m$ such that $0 < t_1 < t_2 < ... < t_m$ and $x_1, ..., x_m \geq 0$. For $n \in \mathbb{N}$ let

$$S_n = \sum_{i=1}^{n} \tau_i. \tag{3.128}$$

Note that we have

$$\{N(t) \geq x\} = \{S_{\lceil x \rceil} \leq t\}, \tag{3.129}$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. Therefore we can write

$$\begin{aligned} \mathbb{P}\{N(ct_i)/c < x_i, \text{ for } i = 1, ..., m\} &= \mathbb{P}\{N(ct_i) < cx_i, \text{ for } i = 1, ..., m\} \\ &= \mathbb{P}\{S_{\lceil cx_i \rceil} > ct_i, \text{ for } i = 1, ..., m\} \\ &= \mathbb{P}\{S_{\lceil cx_i \rceil}/c > t_i, \text{ for } i = 1, ..., m\} \end{aligned} \tag{3.130}$$

Now as $c \to \infty$ by the law of large numbers

$$\mathbb{P}\{N(ct_i)/c < x_i, \text{ for } i = 1, ..., m\} = \mathbb{P}\{\lambda t_i < x_i, \text{ for } i = 1, ..., m\} \tag{3.131}$$

and we establish convergence in distribution of all finite-dimensional marginal distributions. Now, since the sample paths of $\{N_t\}_{t\geq 0}$ and $\{\lambda t\}_{t\geq 0}$ are increasing and $\{\lambda t\}_{t\geq 0}$ is continuous, Theorem 3 of [25] and above calculation give us as $c \to \infty$

$$N(c\cdot)/c \implies \lambda \cdot \text{ in } \mathcal{D}[0, \infty). \tag{3.132}$$

$\square$

With the above we are ready to prove the main result of the section.

Theorem 3.3.8. *Let T be a compound Poisson process with a drift, i.e,*

$$T(t) = \sum_{i=1}^{N(t)} \sigma_i + t, \quad t \geq 0, \tag{3.133}$$

where N is a Poisson process with intensity λ and σ is a random variable with cdf F, where $\bar{F} \in RV_{-\alpha}$, $\alpha \in (0, 1)$. Let B be a standard Brownian motion. Then as $s \to \infty$,

$$B(\bar{F}(s)T^{\leftarrow}(s\cdot)) \implies B(T_\alpha^{\leftarrow}(\cdot)) \tag{3.134}$$

in $\mathcal{D}[0, \infty)$, where T_α is a α-stable subordinator.

Proof. Let us define the quantile function of σ

$$b(s) := (1/\bar{F})^{\leftarrow}(s). \tag{3.135}$$

Note that $\bar{F} \in RV_{-\alpha}$, implies that

$$1/\bar{F} \in RV_{\alpha}. \tag{3.136}$$

Finally by part (v) of Proposition A.1.1

$$b = (1/\bar{F})^{\leftarrow} \in RV_{1/\alpha}. \tag{3.137}$$

From above we can conclude that as $s \to \infty$

$$\frac{s}{b(s)} \to 0. \tag{3.138}$$

So we can write

$$(b(s))^{-1} T(s\cdot) = (b(s))^{-1} \sum_{i=1}^{N(s\cdot)} \sigma_i + o(1), \tag{3.139}$$

which holds for the finite-dimensional distributions. Now regular variation of $\bar{F}$ implies that in $\mathcal{D}[0,\infty)$

$$(b(s))^{-1} \sum_{i=1}^{[s\cdot]} \sigma_i \implies T_{\alpha}(\cdot), \tag{3.140}$$

where T_α is an α-stable subordinator. Therefore

$$\left(\frac{N(s\cdot)}{s}, (b(s))^{-1} \sum_{i=1}^{[s\cdot]} \sigma_i \right) \implies (\lambda\cdot, T_{\alpha}(\cdot)) \tag{3.141}$$

in $\mathcal{D}([0,\infty), \mathbb{R}^2)$. By the Continuous Mapping Theorem we conclude that

$$(b(s))^{-1} T(s\cdot) \implies \lambda^{1/\alpha} T_{\alpha}(\cdot), \tag{3.142}$$

where $\Longrightarrow$ refers to the convergence of the finite-dimensional distributions. Since $(b(s))^{-1}T(s\cdot)$ is non-decreasing and continuous in probability, Theorem 3.3.6 gives us convergence in $\mathcal{D}[0,\infty)$. Now since both $(b(s))^{-1}T(s\cdot)$ and T_α are nondecreasing, the inverse processes also converge in $\mathcal{D}[0,\infty)$:

$$\big((b(s))^{-1}T(s\cdot)\big)^{\leftarrow} \implies \big(\lambda^{1/\alpha}T_\alpha(\cdot)\big)^{\leftarrow}. \tag{3.143}$$

Unwinding the last result, we get in $\mathcal{D}[0,\infty)$

$$\frac{T^{\leftarrow}(b(s)\cdot)}{s} \implies \lambda T_\alpha^{\leftarrow}(\cdot), \tag{3.144}$$

or, simply,

$$\bar{F}(s)T^{\leftarrow}(s\cdot) \implies T_\alpha^{\leftarrow}(\cdot). \tag{3.145}$$

Finally the Continuous Mapping Theorem yields that in $\mathcal{D}[0,\infty)$

$$B\big(\bar{F}(s)T^{\leftarrow}(s\cdot)\big) \implies B\big(T_\alpha^{\leftarrow}(\cdot)\big), \tag{3.146}$$

which ends the proof. $\square$

3.4 Return to Switching Diffusion

In Chapter 2 we proposed a model of Switching Diffusion with immobilization times which have a distribution that is a countable mixture of exponentials. This natural restriction comes from the limitations in a hybrid switching diffusion model. Its discrete component has to be a continuous time Markov chain therefore its state space can only be countable. However, the countable mixtures of exponentials are fairly good approximation class. So the following question emerges: what is the largest class of densities that can be approximated by the countable mixtures of exponentials? And consequently, what is the largest class of Switching Diffusion processes that its law can be represented as in Theorem 2.4.1. Let us start the discussion with the theory of exponential mixtures and its connection to the completely monotone functions.

3.4.1 Mixtures of Exponentials and Completely Monotone Functions

When we talk about convergence of probability distributions on the real line we use the notion of weak convergence. We already discussed this matter in Section 3.3 so here we only make extra few comments. On the real line weak convergence of random variables is given in terms of cumulative distribution functions

$$F_n(t) \to F(t) \text{ as } n \to \infty \tag{3.147}$$

for all t that are continuity points of the limiting cdf F, which we denote by

$$F_n \implies F. \tag{3.148}$$

The exponential distribution is a probability distribution on $[0, \infty)$ with density f and its Laplace transform $\hat{f}$ given by

$$f(t) = \lambda e^{-\lambda t}, \quad \hat{f}(s) = \frac{\lambda}{\lambda + s}. \tag{3.149}$$

Mixing with respect to λ leads to distribution on $(0, \infty)$ with density f and Laplace transform $\hat{f}$ of the form

$$f(t) = \int_{(0,\infty)} \lambda e^{-\lambda t} dG(\lambda), \quad \hat{f}(s) = \int_{(0,\infty)} \frac{\lambda}{\lambda + s} dG(\lambda), \tag{3.150}$$

with G a distribution on $(0, \infty)$. Observe that the resulting mixture can be viewed as a scale mixture by putting $\lambda = 1/\mu$

$$\hat{f}(s) = \int_{\mathbb{R}_+} \frac{1}{1 + \mu s} dH(\mu), \tag{3.151}$$

with H a distribution on $[0, \infty)$. This means that an additional mixing with degenerate distribution at zero is allowed. This representation has an advantage that the resulting class of distributions is closed under the weak convergence [30]. It turns out that there is a close relationship of exponential mixtures and completely monotone functions, which definition is provided below.

Definition 3.4.1. A function f defined on $(0, \infty)$ is completely monotone (CM) if it is of class C^∞ and

$$(-1)^n f^{(n)}(t) \geq 0, \quad t \in (0, \infty), n = 0, 1, 2, ... \tag{3.152}$$

With the above definition we can fully characterize the range of exponential mixtures. The following theorem is known as Bernstein theorem on monotone functions.

Theorem 3.4.2 (Bernstein)**.** *The function f is CM if and only if*

$$f(t) = \int_{[0,\infty)} e^{-\lambda t} d\mu(dt), \tag{3.153}$$

where $\mu(t)$ is a positive measure on Borel sets of $[0, \infty)$ and the integral converges for $0 < t < \infty$.

In other words, completely monotone functions are real one-side Laplace transforms of a positive measure on $[0, \infty)$. Note that in Bernstein theorem function f does not have to be a probability density function. However, if f is a probability density on $[0,\infty)$ after integration with respect to $t \in (0,\infty)$ we obtain

$$1 = \int_{[0,\infty)} \frac{1}{\lambda}\mu(dt). \tag{3.154}$$

Defining a new probability measure by $\nu(\Gamma) := \int_\Gamma \frac{1}{\lambda}\mu(d\lambda)$, we get

$$f(t) = \int_{[0,\infty)} \lambda e^{-\lambda t}\nu(d\lambda) = \int_0^\infty \lambda e^{-\lambda t} G(d\lambda), \tag{3.155}$$

where G is the distribution function for ν. Above shows that in the probabilistic setting Bernstein theorem says that a probability density function is a CM function if and only if it is a mixture of exponential densities. Now, it is pretty clear that mixtures of exponential densities are completely monotone. Moreover, using the fact that cdfs with finite support are dense in the family of all cdfs, we get

Corollary 3.4.1. *If F is a cdf with CM pdf f, then there are cdfs $\{F_n\}_{n\geq 0}$, with CM densities $\{f_n\}_{n\geq 0}$ of the form*

$$f_n(t) = \sum_{i=1}^{k_n} \lambda_{ni} p_{ni} e^{-\lambda_{ni} t}, \quad t \geq 0, \tag{3.156}$$

with $\lambda_{ni} \leq \infty$ and $\sum_{i=1}^{k_n} p_{ni} = 1$ such that for $t \in (0,\infty)$

$$f_n(t) \to f(t) \text{ as } n \to \infty. \tag{3.157}$$

Note that convergence of densities implies the convergence of cdfs so we also have $F_n \implies F$. Also it is worth mentioning that Corollary 3.4.1 can be strengthened. Since for CM functions the pointwise convergence, locally uniform convergence, and even convergence in space $C^\infty(0,\infty)$ coincide [30].

3.4.2 Switching Diffusion with CM Immobilization Times

Now, we revisit Section 2.5 and present a collection of results that goes partially towards the goal of establishing a law for a Switching Diffusion with immobilization times that have completely monotone densities. Recall that in Chapter 2 we showed that for Switching Diffusion with f_σ a countable mixture of exponentials we have autonomous equation for $p(x,t,0)$ which satisfies

$$\begin{aligned}\partial_t p(x,t,0) &= D\partial_{xx}p(x,t,0) - \lambda p(x,t,0) + \lambda\int_0^t \sum_{i=1}^{\infty} p_i\lambda_i e^{-\lambda_i(t-s)}p(x,s,0)ds \\ &= D\partial_{xx}p(x,t,0) - \lambda p(x,t,0) + \lambda\int_0^t f_\sigma(t-s)p(x,s,0)ds.\end{aligned} \tag{3.158}$$

We found a similar formulation for the probability density function in physics literature. Lomholt et al [31] presented a model for a particle which switches between diffusing and ballistic relocation. The authors claim that $p(x,t)$ is the probability density for the position x of the particle at time t (independent of which state the particle is in):

$$\partial_t p(x,t) = D\partial_{xx}p(x,t) - \lambda p(x,t) + \lambda\int_{-L}^{L}\int_0^\infty K^*(x-y,t-s)p(y,s)dsdy, \tag{3.159}$$

where $K^*(x,t)$ represents relocation kernel. Notice that if $K^*(x,t)$ has a special form:

$$K^*(x,t) = \delta_0(x)K(t) \tag{3.160}$$

then the equation (3.159) becomes:

$$\partial_t p(x,t) = D\partial_{xx}p(x,t) - \lambda p(x,t) + \lambda\int_0^\infty K(t-s)p(x,s)ds, \tag{3.161}$$

which is the equation (3.158) for the density of Switching diffusion in diffusing state, where $K(t)$ is a mixture of exponentials. In this paper only the case when $K(t)$ is an exponential or Levy density were considered. Unfortunately, the authors did not justify the equations for the probability density function therefore the validity of these equations for the choice of Levy density remains in question.

Nonetheless, this paper inspired our research on the maximum class of models we can present the law for. The perfect theorem would be in the following form.

Theorem 3.4.3. *(OPEN PROBLEM) Let $X(t)$ be a switching diffusion with iid diffusion times*

$$\{\tau_i\}_{i\geq 1} \sim Exp(\lambda) \tag{3.162}$$

and iid immobilization times

$$\{\sigma_i\}_{i\geq 1} \sim F_\sigma, \tag{3.163}$$

where F_σ is a cumulative distribution function with completely monotone density f_σ. Then for $t > 0$ and $\Gamma \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}\{X(t) \in \Gamma\} = \int_\Gamma p(x,t)dx \tag{3.164}$$

where

$$p(x,t) = p(x,t,0) + \lambda \int_0^t f_\sigma(t-s)p(x,s,0)ds \tag{3.165}$$

and $p(x,t,0)$ is a solution to

$$\partial_t p(x,t,0) = D\partial_{xx}p(x,t,0) - \lambda p(x,t,0) + \lambda \int_0^t f_\sigma(t-s)p(x,s,0)ds. \tag{3.166}$$

Unfortunately, we were not able to prove the above. Here we present the partial results and potential obstacles. Figure 3.3 summarizes the steps needed for the proof. First, we can use Corollary 3.4.1 to obtain sequence of CM densities f_{σ_n}, in the following form

$$f_n(t) = \sum_{i=1}^{k_n} \lambda_{ni} p_{ni} e^{-\lambda_{ni} t}, \quad t \geq 0, \tag{3.167}$$

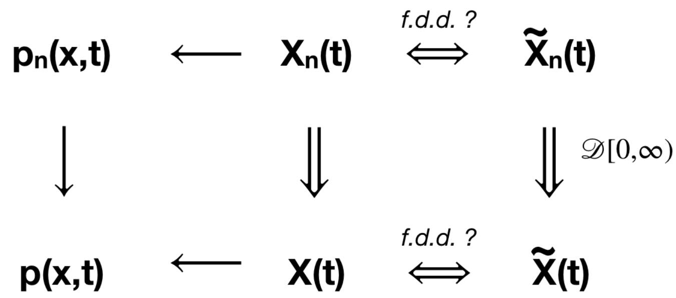


Figure 3.3: Sketch of the possible proof of Theorem 3.4.3.

such that $f_{\sigma_n}(t) \to f_\sigma(t)$. Now, let

$$\tilde{X}_n := B \circ T_n^{\leftarrow} \quad \text{and} \quad \tilde{X} := B \circ T^{\leftarrow} \tag{3.168}$$

be Brownian motions inversely subordinated to Compound Poisson processes with

$$\psi_n(\theta) = \theta + \lambda(1 - \hat{f}_{\sigma_n}(\theta)) \quad \text{and} \quad \psi(\theta) = \theta + \lambda(1 - \hat{f}_\sigma(\theta)). \tag{3.169}$$

Observe that T_n converges in finite dimensional distributions to subordinator T. To see that we write

$$|\psi_n(\theta) - \psi(\theta)| = |\hat{f}_n(\theta) - \hat{f}(\theta)|. \tag{3.170}$$

By Theorem 2 from [32] weak convergence implies convergence in Laplace transforms and therefore we obtain

$$T_n \implies T, \tag{3.171}$$

where the convergence is in the finite dimensional distributions. With the help of Theorem 3.3.6 we get the convergence in $\mathcal{D}[0,\infty)$. Consequently, we have

$$T_n^{\leftarrow} \implies T^{\leftarrow} \tag{3.172}$$

in $\mathcal{D}[0,\infty)$ and finally by the Continuous Mapping Theorem

$$\tilde{X}_n = B(T_n^{\leftarrow}) \implies B(T^{\leftarrow}) = \tilde{X} \tag{3.173}$$

in $\mathcal{D}[0,\infty)$. In Section 3.2.2 we found a link between inverse subordinated Brownian motion and switching diffusion processes. If we denote by X_n sequence of Switching diffusions with immobilization times σ_n and probability density functions $p_n(x,t)$ then X_n and $\tilde{X}_n$ have the same one dimensional distributions. By Equation 3.173 we can conclude that one dimensional distributions of X_n converge to that of X, since X and $\tilde{X}$ have the same one dimensional distributions. It seems plausible that under appropriate conditions one can show that there exists a function p such that sequence of probability densities p_n converge to p and p is the probability density function of X. However, this is not the same as showing that the laws of the associated processes X_n converge in distribution to X. This would require analysis of the finite dimensional distributions which remains unsolved.

Chapter 4

First Passage Problem

In this chapter we investigate the first passage time problem for Brownian motion, Time-Fractional Diffusion and subdiffusive Switching Diffusion. For a stochastic process, the first passage time (FPT) is defined as the time T when the process, which starts from a given point, reaches a predetermined level, say L, for the first time. T is a random variable and usually one tries to compute the survival function $S(t) := \mathbb{P}\{T > t\}$. $S(t)$ is called the survival function because it is just the probability that the particle has not been absorbed by the boundaries during the time interval $[0, t]$. Another question that usually arises in this context is the scaling of mean first passage time with L. The question is really interesting here because for many subdiffusive models the mean first passage time is infinite. In Section 4.1 we calculate the survival function for Brownian motion. In this setting the random variable T is well understood with explicit density and mean $E[T] = L^2/2D$. In Section 4.2 we present calculations for Time-Fractional Diffusion and comment on the mean first passage time. Section 4.3 is devoted to our subdiffusive Switching Diffusion where we calculate asymptotic form of the survival function and show that the mean first passage time is infinite. The next section contains a discussion about the behavior of a quantile function for Switching Diffusion model and suggests an alternative statistic that which can be used to study the relationship of T with L. Also, we present simulations for it in Section 4.5.

4.1 FPT for Brownian Motion

The solution to FPT problem for Brownian motion by way of the evolving law of the particle is a classical result [33]. Let $B = \{B(t)\}_{t\geq 0}$ denotes Brownian motion starting from the origin with diffusivity $2D$. We denote by T_{BM} the time process exits the interval $[-L, L]$, i.e.,

$$T_{BM} := \inf\{t > 0 : |B(t)| > L, \text{ where } B(0) = 0\}. \tag{4.1}$$

We demonstrate that $\mathbb{E}[T_{BM}] = \frac{L^2}{2D}$, while

$$S(t) = \mathbb{P}\{T_{BM} > t\} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left\{-D\left(\frac{(2k+1)\pi}{2L}\right)^2 t\right\}. \tag{4.2}$$

Let $p(x, t)$ denote the law of the particle in the interval $[-L, L]$ until it reaches either of the absorbing boundaries. That is

$$\partial_t p(x, t) = D\partial_{xx} p(x, t), \tag{4.3}$$

with $p(x, 0) = \delta(x)$ and $p(-L, t) = p(L, t) = 0$. The solution to (4.3) can be obtained through the method of separation of variables, i.e., we start looking for solutions which satisfy the boundary conditions, and have special form

$$p(x, t) = X(x)T(t), \tag{4.4}$$

where X and T are functions of variables x and t, respectively. By differentiating and separating variables from (4.4) we obtain

$$\frac{T_t}{DT} = \frac{X_{xx}}{X} = -\theta, \tag{4.5}$$

where θ is a separation constant. Equation 4.5 leads to the following system of ODEs:

$$X'' = -\theta X \quad -L < x < L, \tag{4.6}$$

$$T' = -\theta DT \quad\quad t > 0 \tag{4.7}$$

which are coupled only by the separation constant θ. First, we focus on (4.6). The function X should be a solution of the boundary value problem

$$X'' + \theta X = 0 \quad -L < x < L \tag{4.8}$$

$$X(-L) = X(L) = 0. \tag{4.9}$$

The solution to the above is

$$X_n(x) = \frac{1}{L}\sin\left[\frac{n\pi}{2}\right]\sin\left[\frac{n\pi(L+x)}{2L}\right], \quad \theta_n = \left(\frac{n\pi}{2L}\right)^2, \quad n = 1, 2, 3, ... \tag{4.10}$$

Where the solution to (4.7) is simply

$$T_n(t) = \exp\left\{ -D\left(\frac{n\pi}{2L}\right)^2 t\right\}, \quad n = 1, 2, 3, ... \tag{4.11}$$

The superposition principle implies that the solution to (4.3) is:

$$p(x,t) = \frac{1}{L}\sum_{n=1}^{\infty}\sin\left[\frac{n\pi}{2}\right]\sin\left[\frac{n\pi(L+x)}{2L}\right]\exp\left\{ -D\left(\frac{n\pi}{2L}\right)^2 t\right\}. \tag{4.12}$$

Then the survival function is given by

$$\begin{aligned} S(t) &= \int_{-L}^{L} p(x,t)dx \\ &= \frac{1}{L}\sum_{n=1}^{\infty} \sin\left[\frac{n\pi}{2}\right]\left[\frac{2L(1-\cos\pi n)}{\pi n}\right]\exp\left\{-D\left(\frac{n\pi}{2L}\right)^2 t\right\} \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left\{-D\left(\frac{(2k+1)\pi}{2L}\right)^2 t\right\}. \end{aligned} \tag{4.13}$$

Now, using the survival function we can compute the mean first passage time,

$$\mathbb{E}[T_{BM}] = \int_0^{\infty} \mathbb{P}\{T_{BM} > t\}dt = \frac{16L^2}{D\pi^3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3} = \frac{L^2}{2D}, \tag{4.14}$$

where we use that $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$.

4.2 FPT for Time-Fractional Diffusion

Now, we shift our focus to the FPT problem for Time-Fractional diffusion. As shown in Section 3.3.3 Time-Fractional diffusion is a stochastic-process limit of Switching Diffusions. Therefore we decide to investigate this problem first.

Let us present the calculation of survival function for Time-Fractional Diffusion and show that the mean first passage time is infinite. This result is well known, for example see [34]. Let $X = \{X(t)\}_{t\geq 0}$ denote Time-Fractional Diffusion with diffusivity parameter K_α, $\alpha \in (0,1)$, i.e.

$$X(t) = K_\alpha (B \circ T_\alpha^{\leftarrow})(t), \tag{4.15}$$

where $B = \{B(t)\}_{t\geq 0}$ is the standard Brownian motion and $T_\alpha = \{T_\alpha(t)\}_{t\geq 0}$ is an

α-stable subordinator. T_{TFD} is the first time the process X exits the interval $[-L, L]$,

$$T_{TFD} := \inf\{t > 0 : |X(t)| > L, \text{ where } X(0) = 0\}. \tag{4.16}$$

Let $p(x,t)$ denote the law of the particle in the interval $[-L, L]$ until it reaches either of the absorbing boundaries. Recall that $p(x,t)$ solves the following generalized diffusion equation

$$\partial_t p(x,t) = K_\alpha D_t^{1-\alpha} \frac{\partial^2}{\partial x^2} p(x,t), \tag{4.17}$$

with $p(x,0) = \delta(x)$ and $p(-L,t) = p(L,t) = 0$. Here the operator $D_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative (A.20). The solution of Equation 4.17 with the given boundaries and the initial condition can be found by the method of separation of variables and is provided below

$$p(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left[\frac{n\pi}{2}\right] \sin\left[\frac{n\pi(L+x)}{2L}\right] E_\alpha\left[-\frac{n^2\pi^2}{4L^2} K_\alpha t^\alpha\right], \tag{4.18}$$

where $E_\alpha(-z)$ is the Mittag-Leffler function, discussed in the Appendix A.2.

The survival function is given by

$$S(t) = \int_{-L}^{L} p(x,t)dx. \tag{4.19}$$

Substituting for $p(x,t)$ into this equation and using the following fact

$$\int_{-L}^{L} \sin\left[\frac{n\pi(L+x)}{2L}\right] dx = \begin{cases} \frac{4L}{n\pi} & n \text{ is odd,} \\ 0 & n \text{ is even} \end{cases} \tag{4.20}$$

we obtain

$$S(t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} E_\alpha\left[-\frac{(2n+1)^2\pi^2}{4L^2} K_\alpha t^\alpha\right]. \tag{4.21}$$

Notice that for $\alpha = 1$ the Mittag-Leffler function reduces to the exponential e^{-z} and thus yields the usual solution for the diffusive problem (4.2).

Now, we would like to show the behavior of survival function for large times and show that the mean first passage time for time-fractional diffusion is infinite. To address that we need to analyze the behavior of $S(t)$ as $t \to \infty$. For large z the Mittag-Leffler function behaves as

$$E_\alpha(-z) \sim \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\Gamma(1-\alpha m)} z^{-m}, \tag{4.22}$$

which also implies

$$E_\alpha(-z) \sim \frac{1}{\Gamma(1-\alpha)z}. \tag{4.23}$$

Consequently, for $t \to \infty$,

$$\begin{aligned} S(t) &\sim \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{(2L)^2}{\Gamma(1-\alpha) K_\alpha (2n+1)^2 \pi^2 t^\alpha} \\ &\sim \frac{1}{\Gamma(1-\alpha)} \frac{L^2}{2K_\alpha} t^{-\alpha}. \end{aligned} \tag{4.24}$$

Here we use a fact that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$. Then it follows that for $t \to \infty$

$$\int_0^t S(s)ds \sim \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \frac{L^2}{2K_\alpha} t^{1-\alpha}. \tag{4.25}$$

Since

$$\mathbb{E}[T_{FTD}] = \lim_{t\to\infty} \int_0^t S(s)ds, \tag{4.26}$$

we thus conclude that for any $\alpha \in (0,1)$ the mean first passage time for a time-fractional diffusion is infinite.

4.3 FPT for Switching Diffusion

Here we present two approaches to solve the FPT problem for Switching Diffusion model introduced in Chapter 1. In the first approach we use the equations for the law of SD (Section 2.4) and similar techniques to these used in two previous sections. The second approach uses the analysis of trapping events and the properties of the regularly varying functions.

4.3.1 FPT for SD using the Law

Let $X = \{X(t)\}_{t\geq 0}$ denote Switching Diffusion defined as in (2.7). Here we assume that σ has a distribution which is a mixture of exponentials, i.e., for $t \geq 0$

$$\bar{F}_\sigma(t) = \sum_{i=1}^{\infty} p_i e^{-\lambda_i t}, \tag{4.27}$$

with the density

$$f_\sigma(t) = \sum_{i=1}^{\infty} p_i \lambda_i e^{-\lambda_i t}, \tag{4.28}$$

where the coefficients $\{p_i\}_{i\geq 1}$ and rates $\{\lambda_i\}_{i\geq 1}$ are chosen as in (2.2.1) so that as $t \to \infty$

$$\bar{F}_\sigma(t) \sim At^{-\alpha}. \tag{4.29}$$

We denote by T_{SD} the first time process X exits the interval $[-L, L]$,

$$T_{SD} := \inf\{t > 0 : |X(t)| > L\}. \tag{4.30}$$

We are interested in obtaining survival function

$$S(t) = \mathbb{P}\{T_{SD} > t\} = \int_{-L}^{L} p(x,t)dx, \tag{4.31}$$

where $p(x,t) = \sum_{i=0}^{\infty} p(x,t,i)$. Therefore we need to solve

$$\partial_t p(x,t,0) = D\partial_{xx} p(x,t,0) - \lambda p(x,t,0) + \sum_{i=1}^{\infty} \lambda_i p(x,t,i) \tag{4.32}$$

$$\partial_t p(x,t,i) = \lambda p_i p(x,t,0) - \lambda_i p(x,t,i) \text{ for } i = 1,2,3,... \tag{4.33}$$

for $x \in (-L, L)$ and $t > 0$ with boundary conditions $p(L,t,i) = p(-L,t,i) = 0$ where $i = 0,1,....$ Recall that for every i we can solve (4.33) in terms of $p(x,t,0)$, i.e.,

$$p(x,t,i) = \lambda \int_0^t p_i e^{-\lambda_i (t-s)} p(x,s,0) ds \tag{4.34}$$

Now by plugging into (4.32) and changing sum with the integral we obtain autonomous equation for $p(x,t,0)$:

$$\partial_t p(x,t,0) = D\partial_{xx} p(x,t,0) - \lambda p(x,t,0) + \lambda \int_0^t f_\sigma(t-s) p(x,s,0) ds. \tag{4.35}$$

Moreover with (4.34) we can rewrite (4.31) in terms of $p(x,t,0)$, i.e,

$$\begin{aligned} S(t) &= \int_{-L}^{L} p(x,t) dx \\ &= \int_{-L}^{L} \big(p(x,t,0) + \sum_{i=1}^{\infty} p(x,t,i)\big) dx \\ &= \int_{-L}^{L} \left(p(x,t,0) + \lambda \int_0^t \sum_{i=1}^{\infty} p_i e^{-\lambda_i (t-u)} p(x,u,0) du\right) dx \\ &= \int_{-L}^{L} \left(p(x,t,0) + \lambda \int_0^t \bar{F}_\sigma(t-u) p(x,u,0) du\right) dx. \end{aligned} \tag{4.36}$$

Now in order to solve (4.35), we use the method of separation of variables, i.e., we start by looking for solutions that satisfy the boundary conditions and have the special form

$$p(x,t,0) = X(x)T(t). \tag{4.37}$$

We obtain

$$X(x)T'(t) = DX''(x)T(t) - \lambda X(x)T(t) + \lambda X(x)(f_\sigma * T(t)), \tag{4.38}$$

which leads to the following system of ODE's

$$X'' = -\theta X \quad -L < x < L, \tag{4.39}$$

$$T' + \lambda T - \lambda f_\sigma * T = -\theta T \quad t > 0 \tag{4.40}$$

which are coupled only by the separation constant θ. First, we focus on (4.39). The function X should be a solution of the boundary value problem

$$X'' + \theta X = 0 \quad -L < x < L \tag{4.41}$$

$$X(-L) = X(L) = 0. \tag{4.42}$$

The solution to above is

$$X_n(x) = \frac{1}{L} \sin\left[\frac{n\pi}{2}\right] \sin\left[\frac{n\pi(L+x)}{2L}\right], \quad \theta_n = \left(\frac{n\pi}{2L}\right)^2 D, \quad n = 1, 2, 3, ... \tag{4.43}$$

To solve (4.40) we apply the Laplace transform to obtain

$$s\hat{T}(s) - 1 + \lambda\hat{T}(s) - \lambda\hat{f}_\sigma(s)\hat{T}(s) = -\theta\hat{T}(s). \tag{4.44}$$

Substituting θ_n, we get the following solution on the Laplace side

$$\hat{T}_n(s) = \frac{1}{s + \lambda(1 - \hat{f}_\sigma(s)) + \theta_n}, \quad n = 1, 2, 3, ... \tag{4.45}$$

Combining (4.43) and (4.45) we obtain Laplace transform of $p(x,t,0)$:

$$\hat{p}(x,s,0) = \frac{1}{L}\sum_{n=0}^{\infty}(-1)^n \sin\left[\frac{(2n+1)\pi(L+x)}{2L}\right]\frac{1}{s+\lambda(1-\hat{f}_\sigma(s))+\theta_{2n+1}}, \tag{4.46}$$

where we use that for $k = 0,1,2,....$

$$\sin\left[\frac{n\pi}{2}\right] = \begin{cases} 0 & n = 2k, \\ (-1)^k & n = 2k+1. \end{cases} \tag{4.47}$$

With the above we can calculate the Laplace transform of the survival function.

$$\begin{aligned} \hat{S}(s) &= \mathcal{L}\left\{\int_{-L}^{L} p(x,t)dx\right\}(s) \\ &= \mathcal{L}\left\{\int_{-L}^{L}\left(p(x,t,0)+\lambda\int_0^t \bar{F}_\sigma(t-u)p(x,u,0)du\right)dx\right\}(s). \end{aligned} \tag{4.48}$$

Now, by observing that

$$\widehat{1-F_\sigma}(s) = \frac{1}{s}(1-\hat{f}_\sigma(s)), \tag{4.49}$$

we obtain

$$\begin{aligned} \hat{S}(s) &= \int_{-L}^{L}\big(\hat{p}(x,s,0)+\frac{\lambda}{s}(1-\hat{f}_\sigma(s))\hat{p}(x,s,0)\big)dx \\ &= \int_{-L}^{L}\hat{p}(x,s,0)\frac{1}{s}\big(s+\lambda(1-\hat{f}_\sigma(s))\big)dx. \end{aligned} \tag{4.50}$$

By plugging in (4.46) and integrating with respect to x we finally have

$$\hat{S}(s) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{s+\lambda(1-\hat{f}_\sigma(s))}{s(s+\lambda(1-\hat{f}_\sigma(s))+\theta_{2n+1})}. \tag{4.51}$$

Our goal is to show that as $s\downarrow 0$

$$\hat{S}(s) \sim \frac{\lambda L^2}{2D}\Gamma(1-\alpha)As^{\alpha-1}. \tag{4.52}$$

By Theorem A.1.8 the assumption (4.29) implies that as $s \downarrow 0$

$$1 - \hat{f}_\sigma(s) \sim \Gamma(1-\alpha) A s^\alpha. \tag{4.53}$$

Fix $\varepsilon > 0$. Choose $\delta > 0$, s.t. for all $s < \delta$

$$\left| \frac{1 - \hat{f}_\sigma(s)}{s^\alpha} - \Gamma(1-\alpha) A \right| < \varepsilon. \tag{4.54}$$

We have that

$$\frac{\hat{S}(s)}{s^{\alpha-1}} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{s^{1-\alpha} + \lambda(1 - \hat{f}_\sigma(s))/s^\alpha}{s + \lambda(1 - \hat{f}_\sigma(s)) + \theta_{2n+1}} := \frac{4}{\pi} \sum_{n=0}^{\infty} a_n f_n(s), \tag{4.55}$$

where $a_n = \frac{(-1)^n}{2n+1}$. The strategy is to show that series $\sum_{n=0}^{\infty} a_n f_n(s)$ converges uniformly. Let us start with noticing that

$$\sum_{n=0}^{\infty} a_n = \frac{\pi}{4} < \infty. \tag{4.56}$$

Also, since θ_n's are decreasing $\{f_n\}_{n \geq 0}$ is a monotonic decreasing sequence. What is more, $\{f_n\}_{n \geq 0}$ is uniformly bounded. To see that we write for $n \geq 0$

$$0 \leq \frac{s^{1-\alpha} + \lambda(1 - \hat{f}_\sigma(s))/s^\alpha}{s + \lambda(1 - \hat{f}_\sigma(s)) + \theta_{2n+1}} \leq \frac{s^{1-\alpha} + \lambda(1 - \hat{f}_\sigma(s))/s^\alpha}{\theta_{2n+1}}. \tag{4.57}$$

Now, by (4.54) for $s \in [0, \delta/2]$

$$0 \leq f_n(s) \leq \frac{(\delta/2)^{1-\alpha} + \lambda(\Gamma(1-\alpha)A + \epsilon)}{(\pi/2L)^2 D}. \tag{4.58}$$

Therefore by Abel's Uniform Convergence Test we conclude that for $s \in [0, \delta/2]$

$$\sum_{n=0}^{\infty} a_n f_n(s) \tag{4.59}$$

is uniformly convergent series. This allows us to write

$$\begin{aligned}\lim_{s\downarrow 0}\frac{\hat{S}(s)}{s^{\alpha-1}} &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\lim_{s\downarrow 0}\frac{s^{1-\alpha}+\lambda(1-\hat{f}_\sigma(s))/s^\alpha}{s+\lambda(1-\hat{f}_\sigma(s))+\theta_{2n+1}} \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{\lambda}{\theta_{2n+1}}\Gamma(1-\alpha)A \\ &= \frac{4}{\pi}\lambda\Gamma(1-\alpha)A\frac{4L^2}{\pi^2 D}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} \\ &= \frac{\lambda L^2}{2D}\Gamma(1-\alpha)A\end{aligned} \tag{4.60}$$

which shows that

$$\hat{S}(s) \sim \frac{\lambda L^2}{2D}\Gamma(1-\alpha)As^{\alpha-1}. \tag{4.61}$$

Finally, by Karamata's Theorem as $t\to\infty$

$$\mathbb{P}\{T_{SD}>t\} \sim \frac{\lambda L^2}{2D}At^{-\alpha}. \tag{4.62}$$

With the above we can calculate the mean first passage time for the subdiffusive Switching Diffusion. Notice that that for $t\to\infty$

$$\int_0^t \mathbb{P}\{T_{SD}>u\}du \sim \frac{\lambda L^2}{2D(1-\alpha)}At^{-\alpha+1}. \tag{4.63}$$

Since

$$\mathbb{E}[T_{FTD}] = \lim_{t\to\infty}\int_0^t \mathbb{P}\{T_{SD}>u\}du, \tag{4.64}$$

we thus conclude that for any $\alpha\in(0,1)$ the mean first passage time for subdiffusive Switching Diffusion is infinite.

4.3.2 FPT for SD using Regular Variation

Our calculations in the previous section cover a scenario where σ has distribution that is a mixture of exponentials. We decide to approach the FPT problem again but using different techniques and prove same result for a broader class of distributions. The main observation here is that due to the structure of immobilization events we can represent an exit time for Switching Diffusion as a time that Brownian motion takes to exit the interval plus the amount of time which the particle spends immobilized, i.e.,

$$T_{SD} = T_{BM} + \sum_{i=1}^{K} \sigma_i, \tag{4.65}$$

where K, conditionally on T_{BM}, is a Poisson random variable with rate λT_{BM}.

Theorem 4.3.1. *Let $\{\sigma_i\}_{i \in \mathbb{N}}$ be i.i.d. sequence of random variables with cdf F such that $\bar{F} \in RV_{-\alpha}$, $\alpha \in (0, 1)$. Then survival function of FPT of Switching Diffusion has the following form as $t \to \infty$*

$$\mathbb{P}\{T_{SD} > t\} \sim \frac{\lambda L^2}{2D} t^{-\alpha} L(t), \tag{4.66}$$

where $L(t)$ is some slowly varying function.

Proof. Notice that we can write

$$T_{SD} = T_{BM} + \sum_{i=1}^{K} \sigma_i, \tag{4.67}$$

where K, conditionally on T_{BM}, is a Poisson random variable with mean λT_{BM}. First, observe that

$$\mathbb{E}[K] = \mathbb{E}[\mathbb{E}[K|T_{BM}]] = \lambda E[T_{BM}] = \frac{\lambda L^2}{2D} < \infty. \tag{4.68}$$

By Proposition 2.1 in [35] we can write

$$\mathbb{P}\{K > k\} = \lambda \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} S(t) dt, \tag{4.69}$$

where $S(t)$ is a survival function for Brownian motion (4.2). This shows that $\mathbb{P}\{K >$

$k\} = o(\mathbb{P}\{\sigma_i > k\})$. Now Proposition 4.1 from [36] asserts that

$$\begin{aligned} \mathbb{P}\Big\{\sum_{i=1}^{K}\sigma_i > t\Big\} &\sim \mathbb{E}[K]\mathbb{P}\{\sigma_i > t\} \\ &\sim \frac{\lambda L^2}{2D}t^{-\alpha}L(t). \end{aligned} \tag{4.70}$$

Now, we show

$$\mathbb{P}\Big\{T_{BM} + \sum_{i=1}^{K}\sigma_i > t\Big\} \sim \mathbb{P}\Big\{\sum_{i=1}^{K}\sigma_i > t\Big\}. \tag{4.71}$$

This statement would be trivial if T_{BM} and $\sum_{i=1}^{K}\sigma_i$ were independent. In our case K depends on T_{BM}. Clearly,

$$\mathbb{P}\Big\{T_{BM} + \sum_{i=1}^{K}\sigma_i > t\Big\} \geq \mathbb{P}\Big\{\sum_{i=1}^{K}\sigma_i > t\Big\}. \tag{4.72}$$

If $0 < \delta < 1/2$ then from

$$\Big\{T_{BM}+\sum_{i=1}^{K}\sigma_i > t\Big\} \subset \Big\{\sum_{i=1}^{K}\sigma_i > (1-\delta)t\Big\}\cup\Big\{T_{BM} > (1-\delta)t\Big\}\cup\Big\{\sum_{i=1}^{K}\sigma_i > \delta t, T_{BM} > \delta t\Big\},$$

it follows that

$$\begin{aligned} \mathbb{P}\Big\{T_{BM} + \sum_{i=1}^{K}\sigma_i > t\Big\} \leq \mathbb{P}\Big\{\sum_{i=1}^{K}\sigma_i > (1-\delta)t\Big\} + \mathbb{P}\Big\{T_{BM} > (1-\delta)t\Big\} \\ + \mathbb{P}\Big\{\sum_{i=1}^{K}\sigma_i > \delta t, T_{BM} > \delta t\Big\}. \end{aligned} \tag{4.73}$$

From (4.2) we obtain

$$\mathbb{P}\Big\{T_{BM} > t\Big\} = o\Big(\mathbb{P}\Big\{\sum_{i=1}^{K}\sigma_i > (1-\delta)t\Big\}\Big). \tag{4.74}$$

Let $f_{T_{BM}}$ denote the density of T_{BM} and write

$$\begin{aligned}\mathbb{P}\bigg\{\sum_{i=1}^{K}\sigma_i > \delta t, T_{BM} > \delta t\bigg\} &= \mathbb{P}\{T_{BM} > \delta t\}\mathbb{P}\bigg\{\sum_{i=1}^{K}\sigma_i > \delta t\bigg| T_{BM} > \delta t\bigg\} \\ &= \mathbb{P}\{T_{BM} > \delta t\}\int_{\delta t}^{\infty}\mathbb{P}\bigg\{\sum_{i=1}^{K}\sigma_i > \delta t\bigg| T_{BM} = s\bigg\} f_{T_{BM}}(s)ds \\ &\sim \mathbb{P}\{T_{BM} > \delta t\}\mathbb{P}\{\sigma_i > \delta t\}\int_{\delta t}^{\infty}\mathbb{E}[K|T_{BM} = s] f_{T_{BM}}(s)ds \\ &= \mathbb{P}\{T_{BM} > \delta t\}\mathbb{P}\{\sigma_i > \delta t\}\int_{\delta t}^{\infty}\lambda s f_{T_{BM}}(s)ds.\end{aligned}$$

This shows that

$$\mathbb{P}\bigg\{\sum_{i=1}^{K}\sigma_i > \delta t, T_{BM} > \delta t\bigg\} = o\Bigg(\mathbb{P}\bigg\{\sum_{i=1}^{K}\sigma_i > (1-\delta)t\bigg\}\Bigg). \tag{4.75}$$

Therefore, we can write

$$\mathbb{P}\bigg\{T_{BM} + \sum_{i=1}^{K}\sigma_i > t\bigg\} \le \mathbb{P}\bigg\{\sum_{i=1}^{K}\sigma_i > (1-\delta)t\bigg\}(1+o(1)). \tag{4.76}$$

Finally,

$$1 \le \lim_{t\to\infty}\frac{\mathbb{P}\bigg\{T_{BM} + \sum_{i=1}^{K}\sigma_i > t\bigg\}}{\mathbb{P}\bigg\{\sum_{i=1}^{K}\sigma_i > t\bigg\}} \le (1-\delta)^{-\alpha}, \tag{4.77}$$

which proves (4.71) upon letting $\delta \downarrow 0$. □

4.4 Quantiles of the FPT distribution

The time it takes for a certain fraction of the population to escape a region in a given time is

$$T_\theta := \inf\{t : \mathbb{P}\{T > t\} \le 1-\theta\}, \tag{4.78}$$

where T is a FPT for some stochastic process. This statistic has been rarely used in physics and mathematics but is somewhat common in the viral infectivity literature (see for example [4]).

If we denote the distribution function of T as F then we can write

$$\begin{aligned} T_\theta &= \inf\{t : \mathbb{P}\{T > t\} \leq 1 - \theta\} \\ &= \inf\{t : 1 - F(t) \leq 1 - \theta\} \\ &= \inf\{t : F(t) \geq \theta\}. \end{aligned} \tag{4.79}$$

Now, let us introduce the right-continuous inverse of a monotone function.

Definition 4.4.1. Suppose $H : \mathbb{R} \to (a, b)$ is a nondecreasing function on $\mathbb{R}$ with range (a, b), where $-\infty \leq a < b \leq \infty$. With the convention that the infimum of an empty set is ∞, we define the (right-continuous) inverse $H^{\leftarrow} : (a, b) \to \mathbb{R}$ of H as

$$H^{\leftarrow}(\theta) = \inf\{t : H(t) \geq t\}. \tag{4.80}$$

Therefore, we observe

$$T_\theta = F^{\leftarrow}(\theta), \tag{4.81}$$

which is the quantile function of the random variable T. The quantile function is another way of describing a probability distribution and is an alternative to the cumulative distribution function.

In our work we deal with random variables that have regularly varying tails. As an example we calculate the quantile function of a Pareto distribution. Recall that a random variable has the Pareto distribution with shape parameter $\alpha \in (0, \infty)$ if it has a continuous distribution on $[1, \infty)$ with distribution function F given by

$$F(t) = 1 - t^{-\alpha} \quad t \in [1, \infty). \tag{4.82}$$

Then its quantile function $F^{\leftarrow}(\theta)$ is

$$F^{\leftarrow}(\theta) = (1-\theta)^{-1/\alpha} \quad \theta \in [0,1), \tag{4.83}$$

which is easily computed from solving $F(t) = \theta$ for t in terms of θ.

Now, we want to calculate the quantile function for the first passage time of Switching Diffusion- T_{SD}. As shown in Theorem 4.3.1 the distribution of T_{SD} has a regularly varying tail therefore we expect that the quantile function should be somewhat similar to one obtained in 4.83. Unfortunately we only know the asymptotic distribution of T_{SD} therefore we are going to investigate only the asymptotic behavior of its quantile function.

Theorem 4.4.2. *Let T_{SD} has a survival function as in Theorem 4.3.1. Then as $\theta \uparrow 1$*

$$T_\theta \sim \left(\frac{\lambda L^2}{2D}\right)^{1/\alpha} (1-\theta)^{-1/\alpha} L_*((1-\theta)^{-1}), \tag{4.84}$$

where L^* is a slowly varying function.

Proof. Let F be the distribution function of T_{SD}. Therefore, by Theorem 4.3.1 we have

$$1 - F(t) \sim \frac{\lambda L^2}{2D} t^{-\alpha} L(t) \in RV_{-\alpha}, \tag{4.85}$$

and this implies the following

$$\frac{1}{1-F}(t) \sim \frac{2D}{\lambda L^2} t^{\alpha} \frac{1}{L(t)} := \frac{2D}{\lambda L^2} H(t). \tag{4.86}$$

Now, since $H \in RV_\alpha$ and increasing to ∞, we can use Proposition A.1.1 (v) which gives us

$$H^{\leftarrow} \in RV_{1/\alpha}, \tag{4.87}$$

i.e. $H^{\leftarrow}(y) \sim y^{1/\alpha} L_*(y)$ as $y \to \infty$, for some slowly varying function L_*. Finally, from

Proposition A.1.1 (vi) we obtain

$$\left(\frac{1}{1-F}\right)^{\leftarrow} \sim \left(\frac{2D}{\lambda L^2}\right)^{-1/\alpha} H^{\leftarrow}. \tag{4.88}$$

Now, notice that for $y > 1$

$$\begin{aligned}\left(\frac{1}{1-F}\right)^{\leftarrow}(y) &= \inf\left\{t : \frac{1}{1-F}(t) \geq y\right\} \\ &= \inf\left\{t : F(t) \geq 1 - \frac{1}{y}\right\} \\ &= F^{\leftarrow}\left(1 - \frac{1}{y}\right).\end{aligned} \tag{4.89}$$

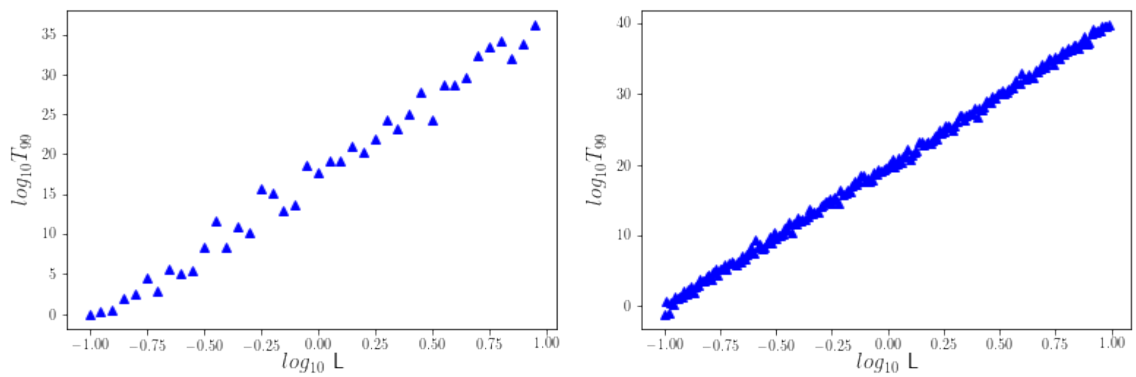
If we make substitution $y = (1-\theta)^{-1}$ in Equation 4.88 we arrive at

$$T_\theta = F^{\leftarrow}(\theta) \sim \left(\frac{\lambda L^2}{2D}\right)^{1/\alpha} (1-\theta)^{-1/\alpha} L_*((1-\theta)^{-1}) \quad \text{as } \theta \uparrow 1, \tag{4.90}$$

where L_* is a slowly varying function associated with $H^{\leftarrow}$ above. □

4.5 T_θ - Simulations

In this section we would like to present simulation results for T_θ for the Switching Diffusion. In the light of Proposition 4.4.2 T_θ scales with $L^{2/\alpha}$ for values of θ close to one. That was confirmed also through our simulations as seen in Figure 4.1. However our further numerical investigation suggests that this scaling is true even for small values of θ if the size of the interval is sufficiently large. Moreover we noticed very interesting behavior of the Switching Diffusion. It exhibits "switchover" between diffusive and subdiffusive regimes. We will see in this section that the switchover point depends on the values of θ and L. We will present our results for two simulation procedures: a "Full" and a "Toy" model. The Full model simulates the exact behavior of the Switching Diffusion whereas in the Toy model we make some simplifications



(a) Full model, $\alpha = 0.1$

(b) Toy model, $\alpha = 0.1$

Figure 4.1: The $L^{2/\alpha}$ scaling with the T_{99} for Full and Toy Model.

that significantly reduce running times.

4.5.1 Full Model

In this section we present the simulation procedure for the Full model. The T_θ is computed in the following way. First, we simulate 1000 paths of Switching diffusion. Each path starts from zero and we record the time that it takes to exit an interval of size $2L$, where $L \in [0.1, 10]$. In order to generate these paths we first simulate the FPT of Brownian motion. We use a standard approach where we simulate brownian paths through a random walk. We fix a step size $dt = 0.001$, $\lambda = 1$, and $D = 1$. We generate consecutive steps of the random walk as normal random variables with mean equals to 0 and standard deviation $\sqrt{dt}$. The cumulative sum of the random variables represents the spatial position of the random walk. This procedure is done until the absolute value of the position is greater than L or equivalently until random walk leaves the interval. The recorded time is the FPT for Brownian motion. Now, K, which is the number of immobilizations conditional on the length of the brownian path is Poisson distributed. Therefore for each path we generate a Poisson random variable with mean $\lambda * T_{BM}$. The length of each immobilization event, σ_i , is drawn from a hyperexponential distribution as explained in Section 2.2. The FPT for Switching diffusion is obtained by adding the FPT for Brownian motion and K hyperexponential immobilization times, i.e.,

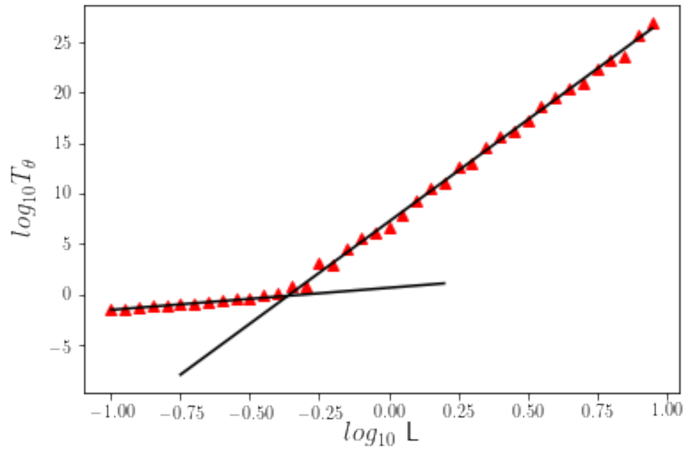
$$T_{SD} = T_{BM} + \sum_{i=1}^{K} \sigma_i. \tag{4.91}$$

The final step of the simulation procedure is to calculate T_θ which is a θ-percentile of the collection of 1000 simulated T_{SD}s.

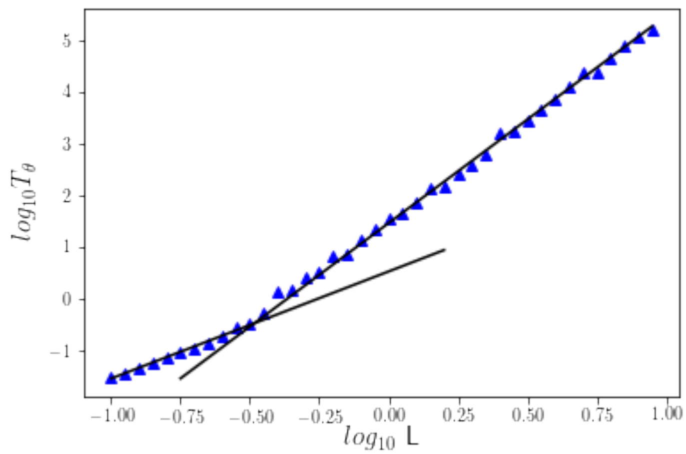
First, we test the Full model simulation procedure for $\theta = 0.99$ (Figure 4.1(a)). As we predicted, the simulation confirms that for values of θ close to 1 the T_θ scales with $L^{2/\alpha}$ for all values of L. In the next simulation we choose intermediate value $\theta = 0.9$ which is more likely to be used in real-world applications. Also, we would like to point out that for the purpose of these simulations the choice of diffusivity parameter D and rate λ is arbitrary and does not have a significant impact on the phenomenons (by Proposition 4.4.2).

Figure 4.2 represents the comparison of the change in scalings for T_{90} with L for three different values of $\alpha \in \{0.1, 0.5, 0.9\}$. It is noticeable that in this case scaling with L is no longer the same on the entire range. We can see that for small values of L, T_{90} scales with L^2; the slope of the fitted bottom line is 2.1927 in Figure 4.2(a), 2.0727 in Figure 4.2(b), and 2.1356 in Figure 4.2(c). Therefore in this range of L the particles exhibit diffusive-like behavior. Now, if we let L to be larger, T_{90} scales again with $L^{2/\alpha}$; the slope of the fitted upper line is 20.2492 for $\alpha = 0.1$, 4.01 for $\alpha = 0.5$ and 2.2101 for $\alpha = 0.9$ which is consistent with Proposition 4.4.2.

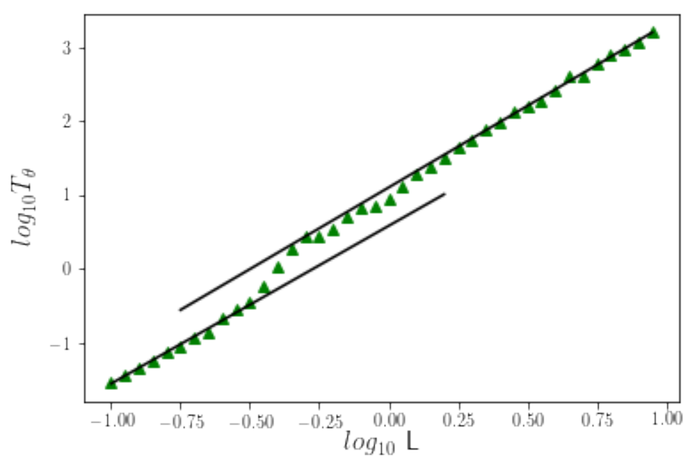
An interesting object to study is a "switchover" point L_*- the point where T_θ changes its scaling with L. To further investigate its behavior we generate the same plots but for different values of θ (Figure 4.3). We observe that for $\theta \to 1$ the L_* approaches 0. On the other hand even for very small θ's there is a sufficiently large L such that there is a switch to the subdiffusive regime. Even though we have not shown this in our calculations but it seems pretty reasonable that increasing the interval width extends the time that a particle stays within the boundaries and



(a) $\alpha = 0.1, D = 1, \lambda = 1$



(b) $\alpha = 0.5, D = 1, \lambda = 1$



(c) $\alpha = 0.9, D = 1, \lambda = 1$

Figure 4.2: The comparison of the change in scalings for T_{90} with L for different α's using Full model.

consequently has more time to exhibit subdiffusive behavior.

The transition in scaling happens faster for smaller α's (Figure 4.3(a)) and notice that if $\alpha \to 1$ then $2/\alpha \to 2$, therefore it is much harder to detect the switchover point for larger α's (Figure 4.3(c)).

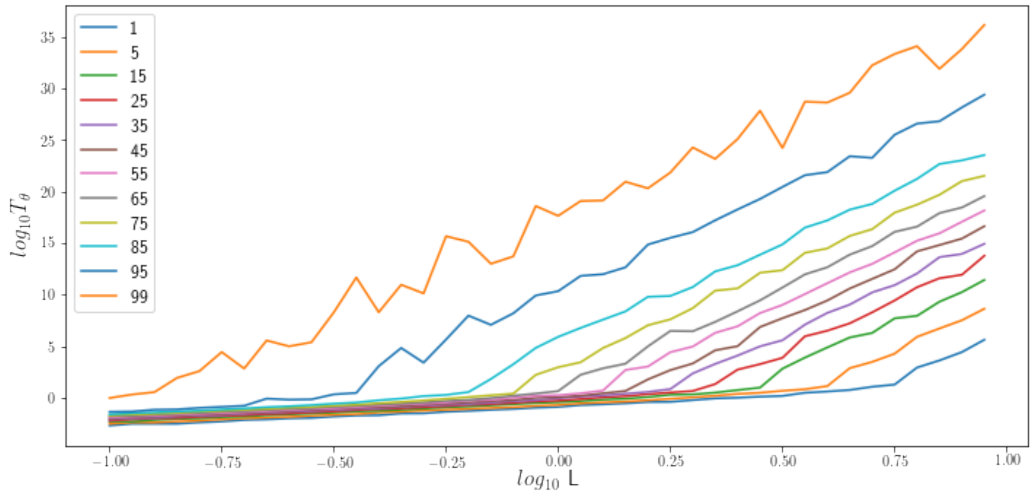
4.5.2 Toy Model

As mentioned before, the Full model simulations have very long running times and therefore the number of simulations and accuracy are limited. Therefore we created simplified procedure - Toy model which takes advantage of the structure of T_{SD}. Let's recall that we can write T_{SD} as

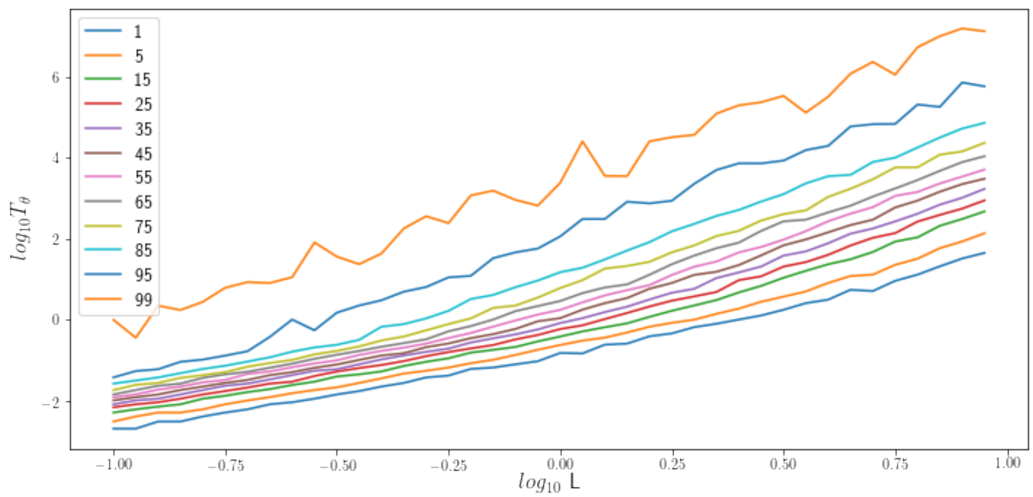
$$T_{SD} = T_{BM} + \sum_{i=1}^{K} \sigma_i, \tag{4.92}$$

where K, conditional on T_{BM} is Poisson distributed with rate λT_{BM}. The main simplification here is that we assume T_{BM}, FPT for Brownian Motion, is exponential with rate $L^2/2D$. This can be made by Equation 4.2 and the assumption that the event of leaving the boundaries is a tail event. Furthermore the immobilization times, σ_i, are generated from Pareto instead of hyperexponential distribution. Due to this simplifications, the running times are much shorter and therefore we were able to run more simulations. For each point in Figures 4.4 and 4.5 we generate 10000 T_{SD}'s.

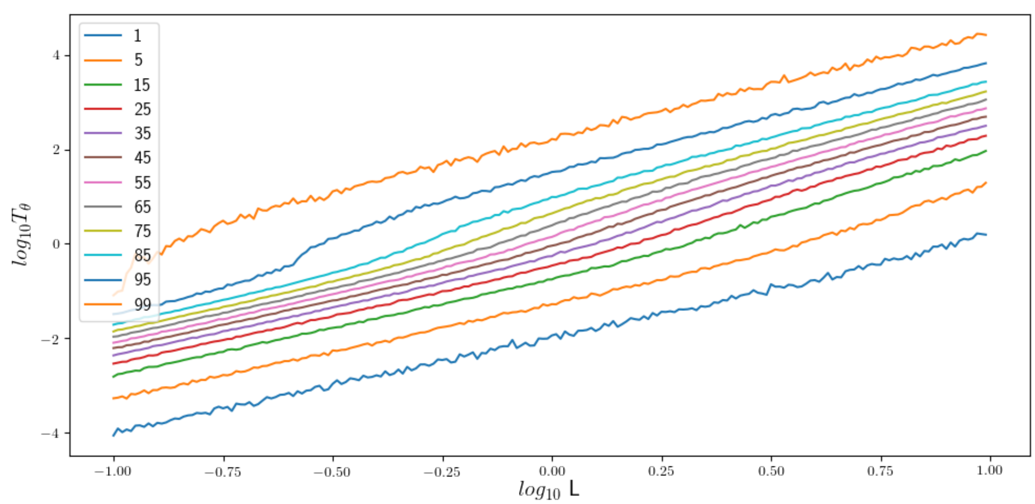
We start with Toy model simulation procedure for $\theta = 0.99$ (Figure 4.1(b)). The simulation shows that for θ close to 1 the T_θ scales with $L^{2/\alpha}$ for all values of L, similarly to Full model. Figure 4.4 represents the comparison of the change in scalings for T_{90} with L for three different values of $\alpha \in \{0.1, 0.5, 0.9\}$. Agin, we can notice that scaling with L is no longer the same on the entire range. For small values of L, T_{90} scales with L^2, i.e. displays diffusive behavior; the slope of the fitted blue line is 2.1120 in Figure 4.4(a), 2.1266 in Figure 4.4(b), and 2.0213 in Figure 4.4(c).



(a) $\alpha = 0.1, D = 1, \lambda = 1$

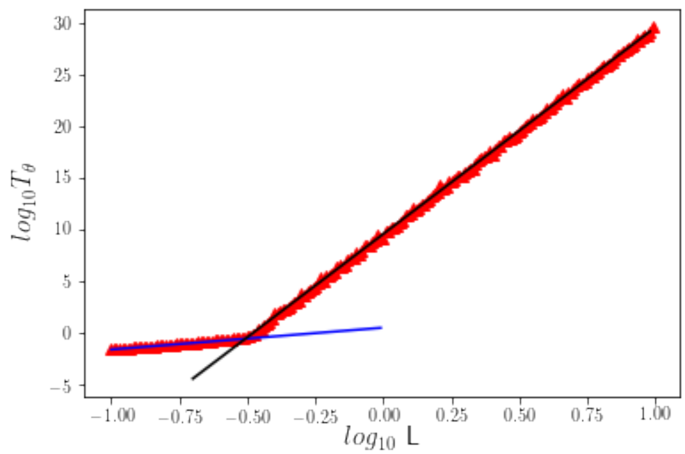


(b) $\alpha = 0.5, D = 1, \lambda = 1$

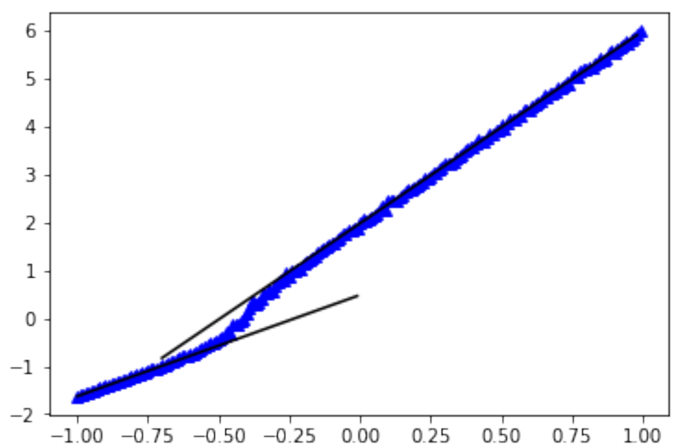


(c) $\alpha = 0.9, D = 1, \lambda = 1$

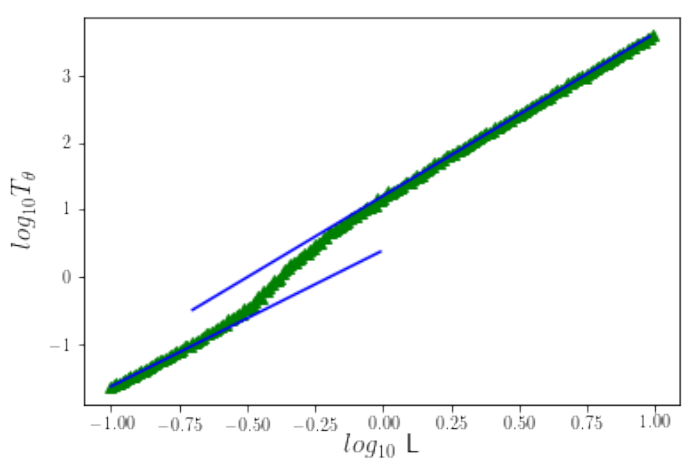
Figure 4.3: The comparison of the change in scalings for T_θ with L for different percentiles and α's using Full model.



(a) $\alpha = 0.1, D = 1, \lambda = 1$



(b) $\alpha = 0.5, D = 1, \lambda = 1$

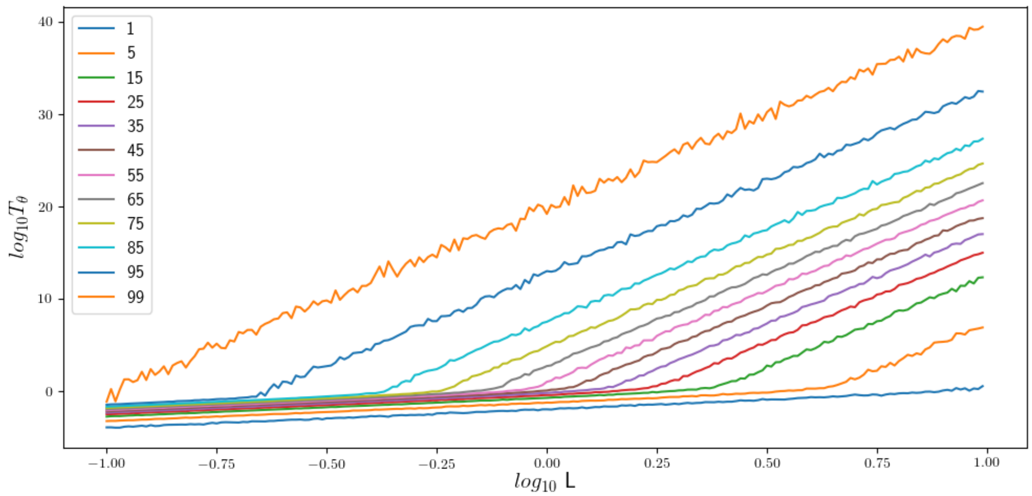


(c) $\alpha = 0.9, D = 1, \lambda = 1$

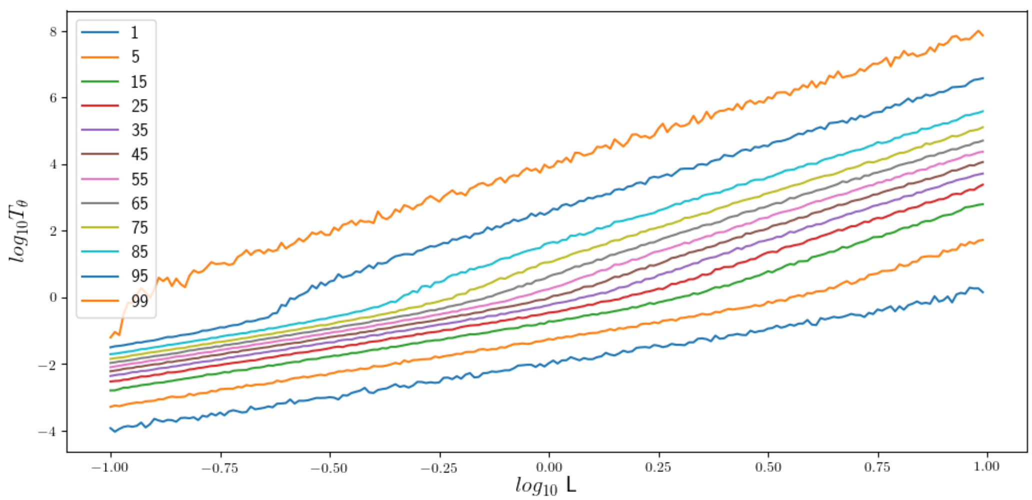
Figure 4.4: The comparison of the change in scalings for T_{90} with L for different α's using Toy model.

For larger L's, T_{90} scales again with $L^{2/\alpha}$; the slope of the fitted line is 19.9930 for $\alpha = 0.1$, 4.0073 for $\alpha = 0.5$ and 2.4163 for $\alpha = 0.9$. A study of the switchover point leads to the similar result as seen in Figure 4.5.

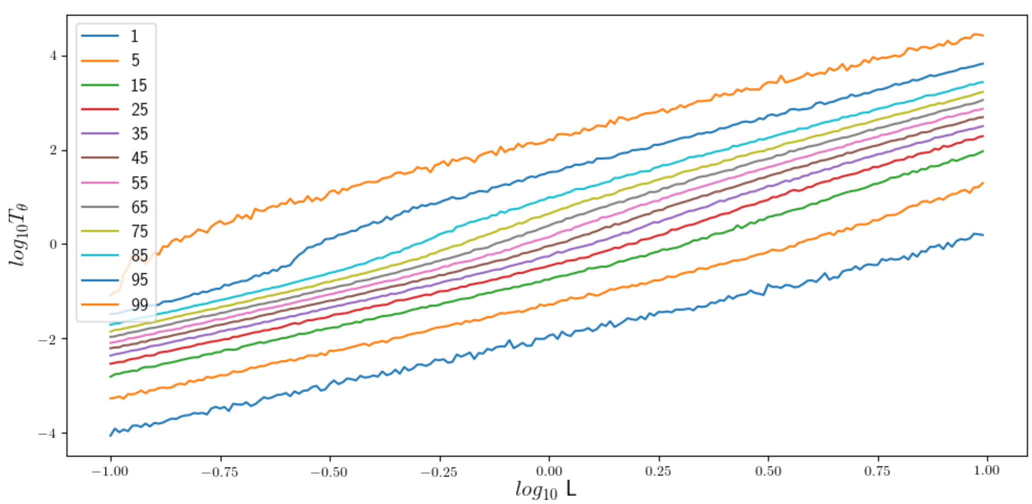
Comparing above with the Full model results it seems that Toy model is reasonable procedure in order to study switchover behavior of Switching diffusion. The simplification did not impact the phenomenon in a detrimental way and allowed us to achieve higher accuracy due to much lower running times.



(a) $\alpha = 0.1, D = 1, \lambda = 1$



(b) $\alpha = 0.5, D = 1, \lambda = 1$



(c) $\alpha = 0.9, D = 1, \lambda = 1$

Figure 4.5: The comparison of the change in scalings for T_θ with L for different percentiles and α's using Toy model.

Appendix A

Additional topics

A.1 Regular Variation

Asymptotic estimates are widely encountered in applications (mathematical biology, insurance mathematics and mathematical finance). Very often it is easier to establish asymptotic of transforms (Laplace, Laplace-Stiltjes). This allows the use of classical Abel-Tauber theory. Started by Karamata [37] and imported to probability through Feller [11] the theory of regularly varying functions is a very elegant setting for discussion of heavy-tailed phenomena. We will summarize some of definitions and results, relevant for our applications. Full treatment and more detail can be found in [7].

Definition A.1.1. A positive. measurable function L on $(0, \infty)$ is called slowly varying at infinity ($L \in RV_0$) if

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1, \text{ for all } t > 0. \tag{A.1}$$

Definition A.1.2. A positive. measurable function U is called regularly varying at infinity with index $\alpha \in \mathbb{R}$ ($U \in RV_\alpha$) if

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^\alpha, \text{ for all } t > 0. \tag{A.2}$$

Note that if $U \in RV_\alpha$ then $U(x)/x^\alpha \in RV_0$, and setting $L(x) = U(x)/x^\alpha$, we can see that it is always possible to represent a regularly varying function as $x^\alpha L(x)$. An important result is the fact that the convergence in (A.1) is uniform on each compact subset of $(0, \infty)$.

Theorem A.1.3 (Uniform convergence theorem for regularly varying functions)**.** *If $U \in RV_\alpha$ for $\alpha \in \mathbb{R}$, then*

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^\alpha \tag{A.3}$$

locally uniformly in t on $(0, \infty)$. If $\alpha < 0$, then uniform convergence holds on intervals of form $(b, \infty), b > 0$. If $\alpha > 0$, uniform convergence holds on intervals $(0, b]$ provided f is bounded on $(0, b]$ for all $b > 0$.

The following results essentially says that integrals of regularly varying functions are again regularly varying.

Theorem A.1.4 (Karamata's Theorem)**.** *Let L be slowly varying and locally bounded in $[0, \infty)$. Then*

- *for $\alpha > -1$*

$$\int_0^x t^\alpha L(t)dt \sim \frac{x^{\alpha+1}L(x)}{(\alpha+1)},\ x \to \infty,$$

- *for $\alpha < -1$*

$$\int_x^\infty t^\alpha L(t)dt \sim -\frac{x^{\alpha+1}L(x)}{(\alpha+1)},\ x \to \infty.$$

Next result is essential for the differentiation of regularly varying functions.

Theorem A.1.5 (Monotone density theorem)**.** *Let $U(x) = \int_0^x u(y)\,dy$ (or $\int_x^\infty u(y)\,dy$) where u is ultimately monotone. If*

$$U(x) \sim x^\alpha L(x),\ x \to \infty. \tag{A.4}$$

with $\alpha \in \mathbb{R}$ *and* $L \in RV_0$*, then*

$$u(x) \sim \alpha x^{\alpha-1} L(x),\ x \to \infty. \tag{A.5}$$

Next result due to Karamata is extremely useful in determining asymptotic properties of a function.

Theorem A.1.6 (Karamata's Tauberian theorem)**.** *Let* U *be a non-decreasing right-continuous function defined on* $[0, \infty)$*. If* $L \in RV_0$*,* $\alpha \geq 0$*, then the following are equivalent:*

- $U(x) \sim (1/\Gamma(1+\alpha)) x^{\alpha-1} L(x),\ x \to \infty.$
- $\hat{u}(s) = \int_0^\infty e^{-sx} dU(x) \sim s^{-\alpha} L(1/s),\ s \downarrow 0.$

We will present a list of useful properties of regularly varying functions. For the following list, it is convenient to define rapid variation of regular variation with index ∞.

Definition A.1.7. We say $U : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying with index ∞ ($U \in RV_\infty$)if for every $x > 0$

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\infty := \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{if } x = 1, \\ \infty & \text{if } x > 1. \end{cases} \tag{A.6}$$

Similarly, $U \in RV_{-\infty}$ if

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^{-\infty} := \begin{cases} \infty & \text{if } x < 1, \\ 1 & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases} \tag{A.7}$$

Proposition A.1.1. *(i) If $U \in RV_\alpha$, $-\infty \le \alpha \le \infty$, then*

$$\lim_{x\to\infty} \frac{\log U(x)}{\log x} = \alpha$$

so that

$$\lim_{x\to\infty} U(x) = \begin{cases} 0 & \text{if } \alpha < 0, \\ \infty & \text{if } \alpha > 0. \end{cases}$$

(ii) (Potter bounds) Suppose $U \in RV_\alpha$, $\alpha \in \mathbb{R}$. Take $\epsilon > 0$. Then there exists t_0 such that for $x \ge 1$ and $t \ge t_0$,

$$(1-\epsilon)x^{\alpha-\epsilon} < \frac{U(tx)}{U(t)} < (1+\epsilon)x^{\alpha+\epsilon}. \tag{A.8}$$

(iii) IF $U \in RV_\alpha$, $\alpha \in \mathbb{R}$, and $\{a_n\}, \{b_n\}$ satisfy $0 < b_n \to \infty$, $0 < a_n \to \infty$, and $b_n \sim ca_n$ as $n \to \infty$ for $0 < c < \infty$, then

$$U(b_n) \sim c^\alpha U(a_n). \tag{A.9}$$

If $\alpha \neq 0$, the result also holds for $c = 0$ or ∞. Analogous results hold with sequences replaced by functions.

(iv) If $U_1 \in RV_{\alpha_1}$ and $U_2 \in RV_{\alpha_2}$, $\alpha_2 < \infty$, and $\lim_{x\to\infty} U_2(x) = \infty$, then

$$U_1 \circ U_2 \in RV_{\alpha_1\alpha_2}. \tag{A.10}$$

(v) Suppose U is nondecreasing, $U(\infty) = \infty$, and $U \in RV_\alpha$, $0 \le \alpha \le \infty$. Then

$$U^{\leftarrow} \in RV_{1/\alpha}. \tag{A.11}$$

(vi) Suppose U_1, U_2 are nondecreasing and α-varying, $0 < \alpha < \infty$. Then for $0 \le$

$c \leq \infty$,

$$U_1(x) \sim cU_2(x), \quad x \to \infty, \tag{A.12}$$

iff

$$U_1^{\leftarrow}(x) \sim c^{-1/\alpha} U_2^{\leftarrow}(x), \quad x \to \infty, \tag{A.13}$$

(vii) If $U \in RV_\alpha$, $\alpha \neq 0$, then there exists a function U^* that is absolutely continuous, strictly monotone, and

$$U(x) \sim U^*(x), \quad x \to \infty. \tag{A.14}$$

In our work mostly the role of U will be playing the tail of a cumulative distribution- $\bar{F}(x) := 1 - F(x)$. Below we rewrite the Karamata's Tauberian theorem in the language of probability distribution functions.

Theorem A.1.8. *Suppose F is a cumulative distribution function with Laplace-Sieltjes transform $\hat{f}$. For $\alpha \in (0,1)$ and $L \in RV_0$, the following are equivalent.*

- $\bar{F}(x) \sim (1/\Gamma(1+\alpha))x^{\alpha-1}L(x)$, $x \to \infty$.
- $1 - \hat{f}(s) \sim s^\alpha L(1/s)$, $s \downarrow 0$.

A.2 Mittag-Leffler Function and Fractional Operators

MittagLeffler functions are connected with analytical solutions to linear fractional differential equations just like exponential functions are related to linear differential equations. In fact one parameter Mittag-Leffler function is a generalization of exponential function, defined by the following power series,

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \tag{A.15}$$

to which it reduces for $\alpha = 1$.

In particular in our work we are interested in the function

$$e_\alpha(t) := E_\alpha(-t^\alpha) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)}, \quad t > 0, \quad 0 < \alpha \leq 1, \tag{A.16}$$

that appears in the solution of the fractional diffusion equation. Let us very briefly introduce the fractional operators commonly used in the literature. Recall the initial value problem

$$\frac{du}{dt} = -u(t), \quad t \geq 0, \quad \text{with} \quad u(0^+) = 1 \tag{A.17}$$

whose solution is

$$u(t) = \exp(-t) = e_1(t). \tag{A.18}$$

In the literature one can find two alternatives. For $\alpha \in (0, 1)$

$$\frac{du}{dt} = -D_t^{1-\alpha} u(t), \quad t \geq 0, \quad \text{with} \quad u(0^+) = 1. \tag{A.19}$$

Here $D_t^{1-\alpha}$ denotes the fractional derivative of order $1 - \alpha$ in the Riemann-Liouville sense. For generic order $\mu \in (0, 1)$ and for 'nice' function $f(t)$ with $t > 0$ the above derivative is defined as

$$D_t^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \left[\int_0^t \frac{f(\tau)}{(t-\tau)^\mu} d\tau \right]. \tag{A.20}$$

Also for $\alpha \in (0, 1)$

$$_*D_t^\alpha u(t) = -u(t), \quad t \geq 0, \quad \text{with} \quad u(0^+) = 1, \tag{A.21}$$

where ${}_*D_t^\alpha$ denotes the fractional derivative of order α in the Caputo sense. For generic order $\mu \in (0,1)$ and for 'nice' function $f(t)$ with $t > 0$ the above derivative is defined as

$$ {}_*D_t^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\mu} d\tau. \tag{A.22}$$

We have the following relationship between the two derivatives

$$ {}_*D_t^\mu f(t) = D_t^\mu f(t) - f(0^+) \frac{t^{-\mu}}{\Gamma(1-\mu)}. \tag{A.23}$$

We can solve Equation (A.19) and (A.21) using the Laplace transform techniques. In both cases the transform of the solution comes out as

$$ \hat{u}(s) = \frac{s^{\alpha-1}}{s^\alpha + 1}, \tag{A.24}$$

which is the Laplace transform of

$$ u(t) = e_\alpha(t) := E_\alpha(-t^\alpha). \tag{A.25}$$

This can be shown by transforming the power series representation of $e_\alpha(t)$ in the Equation (A.16).

One is often interested in the asymptotic behavior of the function $e_\alpha(t)$. This function is known to have an exponential decay as $t \to 0^+$. The short time approximation is

$$ e_\alpha(t) = 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \cdots \sim \exp\left[- \frac{t^\alpha}{\Gamma(1+\alpha)} \right], \quad t \to 0^+. \tag{A.26}$$

The long time approximation was derived by Erdélyi [38] from the asymptotic power series representation of $e_\alpha(t)$. We get

$$ e_\alpha(t) \sim \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{-\alpha} n}{\Gamma(1-\alpha n)}, \quad t \to \infty, \tag{A.27}$$

so that, at the first order it matches negative power law

$$e_\alpha(t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \to \infty. \tag{A.28}$$

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Biography

The author was born in Gdynia in 1988 year and graduated from University of Gdansk with a masters degree in 2012. The author started the PhD program at the Tulane University mathematics department in August 2012, eventually completing the program in May 2019.