Graded and Dynamic Categories

An abstract

Submitted on the third day of May 2019

to the department of Mathematics

in partial fulfillment of the requirements

of the School of Science and Engineering

of Tulane University

for the degree

of

Masters of Mathematics

by

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Abstract

In this thesis, I define and study the foundations of the new framework of *graded category theory*, which I propose as just one structure that fits under the general banner of what Andree Eheresman has called “dynamic category theory” [1]. Two approaches to defining graded categories are developed and shown to be equivalent formulations by a novel variation on the Grothendieck construction.

Various notions of graded categorical constructions are studied within this framework. In particular, the structure of graded categories in general is then further elucidated by studying so-called “variable-object” models, and a version of the Yoneda lemma for graded categories.

As graded category theory was originally developed in order to better understand the intuitive notions of *absolute* and *relative* cardinality – these notions are applied to the problem of vindicating the Skolemite thesis that “all sets, from an absolute perspective, are countable”. Finally, I discuss some open problems in this framework, discuss some potential applications, and discuss some of the relationships of my approach to existing approaches in the literature.

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To my friends, family, and loved ones – all that have stood by me and believed that I could make it this far.

Acknowledgments

There are many people here that I should thank for their role, both directly and indirectly, in helping me shape this thesis. Necessarily, just due to the sheer number of those who have had an impact on my personal and intellectual development throughout the years, there will have to be some omissions. Moreover, there are those in my life whose impact has been so profound that it is hard to put into words, at least in so public of a declaration. And so, to quote one of my favorite philosophers, Ludwig Wittgenstein (who, incidentally, has also provided me with some ample inspiration for this thesis):

"Whereof one cannot speak, thereof one must be silent."

However, of those that I *can* speak, there are still many to name. I'd like first off to thank my parents for always encouraging me in my intellectual pursuits when I was younger, no matter how unorthodox they were. I'd like to also thank all of my personal mentors throughout the years for much the same reason: Jeremy Coulson, Bryan Contreras, Mr. Lump, Dr. Cook – thank you for always being there, and supporting me in all of my passions.

In more recent years, I'd like to thank my adviser, Mike Mislove, who first encouraged me to turn my fledgling paper on graded categories into the topic for a master's thesis project, as well as the other members of my thesis committee – Tai Ha, and Rafał Komendarczyk. I'd also like to thank Mahir Can, since although we was not in the end an official member of my thesis committee, he provided me with many useful discussions about my work. Amongst those I have met in the category theory community, I'd also like to thank Bert Lindenhovious, Valdimir Zamdzhiev, Emily Riehl, Steve Awodey, Paul Levy, Noson Yanofksy, Alex Kavvos, and Harley Eades III for their helpful comments, discussions, and interest in my work.

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Contents

Chapter 1

Graded Categorical Foundations

1.1 An Introduction to Elementary and Partial Graded Categories

There are many mathematical structures in the literature that have been endowed with some sort of a graded structure. Since this is a thesis on graded categories, we will begin by first considering some of the more familiar kinds of gradings that appear in the literature. Perhaps the first such notion to come to the reader's mind, might be that of a graded ring:

Definition 1.1.1. A graded ring consists of a ring R which can be decomposed into a disjoint union of abelian groups R_n for each $n \in \mathbb{N}$ such that for all n, m we have $R_n R_m \subseteq R_{n+m}$.

with the intuition being that elements r of "grade n" in R and elements s of "grade s" multiply to give elements rs of "grade $n + m$" in the ring.

In theoretical computer science, various notions of graded monads and comonads have also been studied. Perhaps the simplest such notion to describe is that of an $\mathcal{M}$-graded (co)monad, where $\mathcal{M}$ is a monoidal category. This is simply (in the case of a $\mathcal{M}$-graded monad) a lax monoidal functor $T : \mathcal{M} \to [\mathcal{C}, \mathcal{C}]$, where $[\mathcal{C}, \mathcal{C}]$ denotes the monoidal category of endofunctors on $\mathcal{C}$ (with the monoidal product given by functor composition) [2]. If we use the convenient notation T_n for $T(n)$, where $n \in \mathcal{M}$, it is easy to see this definition amounts to the following diagrams commuting:

$$
\begin{tikzcd}
T_n T_m T_k \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T_n T_{m+k} \arrow[d, "\mu"] \\
T_{n+m} T_k \arrow[r, "\mu"'] & T_{n+m+k}
\end{tikzcd}
\qquad
\begin{tikzcd}
T_n \arrow[r, "\eta T"] \arrow[dr, "1_{T_n}"'] & T_1 T_n \arrow[d, "\mu"] \\
& T_n
\end{tikzcd}
\qquad
\begin{tikzcd}
T_n \arrow[r, "T\eta"] \arrow[dr, "1_{T_n}"'] & T_n T_1 \arrow[d, "\mu"] \\
& T_n
\end{tikzcd}
$$

And hence, it is now easier to see why this notion deserves to be called a graded monad. Moreover, by duality we obtain similar results for $\mathcal{M}$-graded comonads when we replace lax monoidal functors with oplax monoidal functors (in other words, we simply reverse all the arrows in the diagrams above). There are also some useful generalizations of these which involve a richer grading structure (i.e. over semirings, or more generally weakly distributive categories instead of merely monoids and monoidal categories) which can be used to fruitfully model various schemes of *bounded linear logic* (for more on all of these notions, as well as additional references, the reader may consult [3]).

However, when dealing with algebras over a graded monad, things become more complicated, and the definitions are more involved. For instance, Fujii et. al. give the following definition for a graded T-algebra:

Definition 1.1.2. [4] Given a graded monad $T : \mathcal{M} \to [\mathcal{C}, \circ]$, a graded T-algebra is a pair (A, h) consisting of a functor $A : \mathcal{M} \to \mathcal{C}$ and a family of morphisms

$$h_{m,n} : T_m(A_n) \to A_{m \otimes n}$$

natural in m, n and making the following diagrams commute for all objects m, n of

$\mathcal{M}$:

$$
\begin{tikzcd}
 & T_n A_m \arrow[dr, "h_{I,m}"] & \\
A_m \arrow[ur, "\eta"] \arrow[rr, "\mathrm{id}"'] & & A_m
\end{tikzcd}
\qquad
\begin{tikzcd}
T_m T_n A_p \arrow[r, "T_m(h_{n,p})"] \arrow[d, "\mu_{m,n}"'] & T_m A_{n\otimes p} \arrow[d, "h_{m,n\otimes p}"] \\
T_{m\otimes n} A_p \arrow[r, "h_{m\otimes n,p}"'] & A_{m\otimes n\otimes p}
\end{tikzcd}
$$

which, although different from the naïve way of taking the usual categorical definition of T-algebras and "annotating it" with grades, as one can essentially do in the case of graded (co)monads, at least can be seen as a sort of generalization of definitions such as 1.1.1.

However, the definition of the kleisli category that Fuji et. al. give is even more difficult to compare (at least intuitively) with the "un-graded" definition, involving a coend formula which we will not reproduce here. This is, for instance, different from the structure that Abramsky considers in [5].

Though it is outside the scope of the present thesis, we hope that with future development, the notion of a graded category might help to clarify and further elucidate these examples from the literate.

1.1.1 Variable sets: The "dynamics" of graded categories

Since at least Gottlob Frege, philosophers and mathematicians have been trying to formalize the intuitive notion of a *set*. The basic idea of Frege's was that sets are *predicates in extension*. In other words, sets can be defined defined by a *membership predicate* (for an example of this, think of the usual set-builder notation $\{x \mid P(x)\}$), and moreover we treat these membership predicates *extensionally* – in other words, if $P(x) \iff Q(x)$ for all x, then $\{x \mid P(x)\} = \{x \mid Q(x)\}$. Sets are not determined up to some finer (syntactic) equivalence relation on the predicates defining them.

Another way of thinking about this "extensionality" principle is that *sets are defined by their elements*, and nothing else (such as the particular ways the predicates $P(x)$ and $Q(x)$ are defined).

This idea however, formulated in terms of these two supposedly self-evident axioms of *extensionality* and *comprehension*, infamously led to *Russell's paradox*, and hence, needs to be modified in some way in order to be a useful theory of sets.

One approach, which has been favored by intuitionists over the years (for instance, both by Bishop and Martin Löf) is to consider a set as instead being determined by:

1. A description for *how to build* members of a set.

2. The criteria for two members of a set to be considered equal.

The above objects are sometimes referred to as *setoids* or *Bishop sets*, and are written as a pair $(A, \sim)$. Such a definition avoids the perils of Russell's paradox by defining sets in terms of *how to construct elements of a set* directly, rather than (in the "naïve" Fregean approach) defining sets in terms of arbitrary predicates, which may or not make constructive (or even logical!) sense.

In this thesis, we will consider a slightly different view of sets – one that takes into account *dynamics*, and *change*. Perhaps the easiest (or at least the most intuitive) example of this is to imagine sets varying in (discrete) time – which we can view abstractly as the poset $(\mathbb{N}, \leq)$:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$$

with the "dynamics" of how the sets evolve in time being given by the functions in the diagram above. Such objects are well-studied in the literature, especially in relation to topos theory, as in fact, these "variable sets" are nothing more than objects in the pre-sheaf topos $\mathbf{Set}^{\mathbb{N}}$ (where $\mathbb{N}$ here is viewed in the standard way as a posetal category) [6, 7]. However, in this thesis, we would like to study a more general class of objects. We imagine that now, instead of our sets themselves varying over time, our *identity criterion for elements of the sets* itself is what varies over time. This gives rise to what we will call a *graded setoid* $(A, \sim_n)_{n \in \mathbb{N}}$.

Example 1. Suppose we wish to model the observations of an agent A about his world as a graded setoid. In particular, to keep this example simple, we will restrict our attention to the set X of (descriptions of) entities in agent $A's$ world. So, for instance, I might look something like:

$$X = \{A, \texttt{the\_morning\_star}, \texttt{the\_evening\_star}, \texttt{Venus}, \ldots$$

We might further assume that initially ($n = 0$), A assumes that the descriptions "morning star", "evening star", and "Venus" all refer to distinct entities. However, upon developing further astronomical capabilities (say at $n = 1$), A in fact discovers that $\texttt{the\_evening\_star} \sim_1 \texttt{the\_morning\_star}$ – even though $\texttt{the\_evening\_star} \not\sim_0 \texttt{the\_morning\_star}$. If we suppose that $A's$ knowledge remains static after this point, then this gives an example of a graded setoid over $\mathbb{N}$.

Sets by themselves do not have much interesting structure. So, we will now make a further methodological assumption. One of the main "philosophical" ideas behind category theory is that to study a collection of *objects* (of what have you – sets, groups, rings, fields, modules, etc...), instead of looking at the "internal structure" of the objects directly, we instead study the *external relationships* between different objects of the same type by studying properties of the *morphisms* between them. Thus, if we wish to synthesize this with the idea behind the "variable sets" (more properly, *variable setoids*) considered above, a natural definition is the following:

Definition 1.1.3. (tentative): A *graded category* consists of a category $\mathcal{C}$, a join semi-lattice $\mathcal{M}$ and a grading functor $G : \mathcal{C} \to \mathcal{M}$.

Remark. Note that we make the standard identification in this thesis between a monoid $\mathcal{M}$, and the category with a single object $*$ with $\mathrm{Hom}(*, *) \cong \mathcal{M}$.

The idea behind the above being that a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ with $G(f) = n$ gives us some construction, or some relationship between the objects A and B at stage n – where the particular interpretation of what a "stage" means will, of course, depend on the specific application that one has in mind. We choose $\mathcal{M}$ to be a join-semilattice, because this gives us the nice interpretation that the composition of a morphism at "stage n" with a morphism at "stage m" gives us one at "stage $n \vee m$". And so, one can think of composing two morphisms of a graded category by first "lifting both morphisms" to the least stage that is greater than both n and m, and then composing the two morphisms there. This gives us an interesting "confidentiality" property, in the sense that "lower" morphisms in a graded category are not affected by (i.e. do not "leak information to") morphisms of a strictly higher grade.

Remark. Note that throughout this thesis we will write $f : A \to_n B$ to denote a morphism of grade n with domain A and codomain B, whenever the graded category that f belongs to is clear.

There are several problems with this first tentative definition, however. The first (and most important) issue being that we would like to have explicit maps in our graded categories witnessing the "lifting both morphisms to the same grade" interpretation we gave above. We call these maps, fittingly enough, *lifts*.

Definition 1.1.4. A graded category with lifts is a graded category (as defined above), such that for every object A, and for every grade $n \in \mathcal{M}$, there exists a morphism $\mathrm{lift}_A^n : A \to_n A$ such that for all morphisms $f : A \to_m B$:

1. If $G(f) = m \geq n$, then $\mathrm{lift}_B^n \circ f = f \circ \mathrm{lift}_A^n = f$.

2. $\mathrm{lift}_B^n \circ f = f \circ \mathrm{lift}_A^n$

In other words, lifting morphisms can also be thought of as "identity morphisms up to grading", since they act as identity morphisms for all morphisms above a fixed grade. This is particularly important to the framework of graded category theory, as it turns out that for many notions that we want to consider, the most that we can expect is that the usual notions hold "up to lifts". For example, in a graded category, requiring morphisms to be *strictly invertible* is often too stringent. To see this, consider any morphism $f : A \to_n B$ of grade $n \neq 0$ (the identity element of $\mathcal{M}$) – then by the "confidentiality" property of the grading, it is impossible for f to be an isomorphism in the standard sense. It is for this reason that we will later define the weaker notion of a "pseudoisomorphism".

The next issue with our definitions above is less of a problem than our first issue. Rather than being a fundamental deficiency in our definition (since the lack of a suitable replacement for an identity morphism for non-trivial grades seriously limits our ability to build a non-trivial theory of graded categories analogously to "vanilla" category theory), the following example instead shows that in order to consider more general types of situations, we need to generalize our notion of grading over a join-semilattice to more general posets.

Example 2. Suppose you are a biologist studying a newly reported species of poisonous dart frog, and you have two competing hypotheses:

1. $(\mathcal{H}_1)$: The species belongs to genus Colostethus.

2. $(\mathcal{H}_2)$: The species belongs to genus Silverstoneia.

Being an astute researcher (and follower of David Spivak's perhaps), you want to keep an OLOG (i.e. a small category $\mathcal{C}$) with objects which represent classes of various entities that you care about (e.g. Species, Genus, etc...), and whose morphisms represent various relationships between these classes [8]. (For instance, $\texttt{is\_a\_member\_of} :$ Species $\to$ Genus).

We can generalize this notion of an OLOG to take into account different "possible worlds" (representing different possible states of knowledge) by viewing it as a graded category. In this context, we can use morphisms of bottom grade ($\perp$) to represent a priori relationships between the entities in our OLOG. For instance, although we may not know *what* genus the species S belongs to, we can reason a priori that it must belong to *some* genus, and hence, the composition $\texttt{is\_a\_member\_of} \circ S$ makes sense as a morphism of grade $\perp$.

However, if we view $\mathcal{H}_1$ and $\mathcal{H}_2$ as two other grades (where morphisms at these grades represent relationships we can infer given the hypothesis $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively), then we should have

$$\mathrm{lift}^{\mathcal{H}_1} \circ \texttt{is\_a\_member\_of} \circ S = \texttt{Colostethus}$$

and

$$\mathrm{lift}^{\mathcal{H}_2} \circ \texttt{is\_a\_member\_of} \circ S = \texttt{Silverstoneia}$$

Now, if our set of all grades is to form a join-semilattice, then the composition $\mathrm{lift}^{\mathcal{H}_1 \vee \mathcal{H}_2} \circ \texttt{is\_a\_member\_of} \circ S$ should also be well-defined – but this is incoherent. $\mathcal{H}_1 \vee \mathcal{H}_2$ does not make sense as a valid "stage", because $\mathcal{H}_1$ and $\mathcal{H}_2$ are inconsistent – they are *incompatible.*

To treat examples like the above, we need to generalize to a *partial* version of join-semilattices, where joins only sometimes exist between elements of the poset. The notion we will use is referred to in the order-theory literature as a *bounded-complete partial order*:

Definition 1.1.5. A poset $\mathcal{P}$ is said to be bounded complete if every bounded subset $B \subset \mathcal{P}$ (i.e. there exists $x \in \mathcal{P}$ such that $b \leq x$ for all $b \in B$) has a join $\bigvee B$. In this case we say that $\mathcal{P}$ is a bounded complete partial order (BCPO).

However, as we have seen from the example above, if we want to grade over bounded complete partial orders $\mathcal{P}$ instead of join-semilattices, and we have morphisms $f : A \to_a B$ and $g : B \to_b C$ with incompatible grades a, b (i.e. grades where the join $a \vee b$ doesn't exist in $\mathcal{P}$), then the composition $g \circ f$ will not exist. Thus, for general BCPOs $\mathcal{P}$ we technically need to consider *partial graded categories* $\mathcal{C}$ (i.e. categories where the composition operation is only partially defined), where the well-definedness of $g \circ f$ is determined by the grading (i.e. $g \circ f$ exists if and only if the morphisms have compatible grades). We define this as follows:

Definition 1.1.6. A partial category $\mathcal{C}$ consists of a collection of objects, denoted $\mathrm{Ob}(\mathcal{C})$, for all $A, B \in \mathrm{Ob}(\mathcal{C})$ a collection $\mathrm{Hom}(A, B)$ of morphisms, together with for all objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ a partial binary operation

$$\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \nrightarrow \mathrm{Hom}(A, C)$$

such that for all f, g, h for which $(f \circ g) \circ h$ is well-defined, $f \circ (g \circ h)$ is also well-defined, and moreover $f \circ (g \circ h) = (f \circ g) \circ h$. (We will use the notation $g \circ f \downarrow$ to mean that the composition $g \circ f$ is well-defined.)

A (total) functor F between partial categories $\mathcal{C}, \mathcal{D}$ consists of a (total) map on both objects and morphisms (i.e. $\mathrm{Hom}(A, B) \mapsto \mathrm{Hom}(FA, FB)$) which preserves well-defined compositions of morphisms and identities (if they exist).

From which we can then define:

Definition 1.1.7. Given a partial category $\mathcal{C}$, and a partial semigroup $\mathcal{M}$, we call $\mathcal{C}$ a *graded category* if it comes equipped with a functor $G : \mathcal{C} \to \mathcal{M}$ of partial categories, where $\mathcal{M}$ is a bounded complete partial order, and G reflects definedness of composition, in the sense that:

$$g \circ f \downarrow \iff G(g) \vee G(f) \downarrow$$

Aside: Spectra and generalizations of graded categories

Before we move on to the rest of the content of this thesis, there is one final issue we should mention. Since this issue was not discovered until recently, in the rest of the thesis, we will *not* in fact treat the theory of graded categories at this level of generality. We simply bring this issue to attention here, and a full exposition will have to be relegated to a later date, as it turns out that this new context significantly complicates some of the proofs that were discovered earlier in the development in this thesis.

To see how this one last technical issue can arise, we only need consider the structure we aim to generalize with our graded categories – the graded setoids that we introduced above, since it turns out that the *hom sets* of a graded category with lifts naturally have the structure of a graded setoid. In particular, the issue that arises is that graded setoids allow sets to be *empty* up to a particular stage, and then later non-empty, and this can lead to some issues with our proposed definition of grading above. In other words, in the terminology we introduce below, objects in a graded category can potentially have non-trivial (i.e. non-full) *point-spectrum*:

Definition 1.1.8. Given a graded category $(\mathcal{C}, G : \mathcal{C} \to \mathcal{M})$ and an object $A \in \mathcal{C}$, we define the point-*spectrum* (or simply the *spectrum*, if otherwise understood) to be the set

$$\mathrm{Spec}^+(A) = \mathrm{Spec}(A) = \{m \in \mathcal{M} \mid \exists f : X \to_m A\}$$

We moreover define the *copoint spectrum* to be the set

$$\mathrm{Spec}^-(A) = \{m \in \mathcal{M} \mid \exists f : A \to_m X\}$$

Finally, the *total spectrum* of an object A is the union of these two sets – $\mathrm{Spec}^t(A) = \mathrm{Spec}^+(A) \cup \mathrm{Spec}^-(A)$.

Now, for a specific example illustrating how objects in a graded category with a non-full spectrum can be problematic, suppose we have a category $\mathcal{C}$ graded over $(\mathbb{N}, \vee)$, and let A be an object with $\mathrm{Spec}^t(A) = \{1, 2, 3 \dots\}$. Since $\mathrm{Spec}(A)$ does not contain 0, A does not have an identity morphism. However, it is still natural to require that all objects $A \in \mathcal{C}$ have lifting morphisms for every grade in their spectrum. So, we can let $\mathrm{lift}^1_A : A \to_1 A$. However, since the spectrum of A only contains elements of grade ≥ 1, by definition of a lifting map, lift^1_A acts as an identity morphism for *all* morphisms with domain or codomain A. Hence, by the uniqueness of identities in a category, we have $\mathrm{lift}^1_A = 1_A$. Hence $G(\mathrm{lift}^1_A) = G(1_A) = 0$. But also $G(\mathrm{lift}^1_A) = 1$, a contradiction!

As the example above shows, if we want to have a graded category containing objects with non-full spectra (which again, we will not consider here, for the sake of simplicity), in general we must require G to be a *semi-functor* (i.e. a mapping between categories like a functor, which only preserves compositions (where defined), not necessarily identities), not a *functor*.

The reader should note here that at this point, so long as we do not wish to consider examples of graded categories whose object have non-full point spectra, we do not need to make any modifications to our previous examples, as the existence of lifts for all grades implies that all object will have full spectrums.

As we will see later, this is a less-than-ideal set of circumstances, as it happens that categories whose objects have non-full spectra in fact arise quite naturally in our theory. However, due to time constraints, we will have to leave the development of this more complete theory as future work.

1.1.2 Graded and Indexed Categories

From this point on, we will call structures of the form $(\mathcal{C}, \mathcal{M}, G : \mathcal{C} \to \mathcal{M})$ *elementary graded categories*, to contrast them with the *indexed structures* we will introduce later in this section.

The following example (which essentially takes a category $\mathcal{C}$ and "trivially" makes it into a graded category with lifts) is useful for constructing examples/counterexamples to explore various properties of graded categories, so we introduce it here:

Example 3. Given a category $\mathcal{C}$ and a monoid $\mathcal{M}$, we define a category $\mathcal{C}_{\mathcal{M}}$ whose objects are the same as the objects of $\mathcal{C}$, and whose morphisms are pairs $(n, f) : A \to B$, with $n \in \mathcal{M}$, and $f : A \to B$ a morphism of $\mathcal{C}$, and composition defined by $(m, g) \circ (n, f) = (m \vee n, g \circ f)$. This gives us a category graded over $(\mathcal{M}, \cdot)$ by defining $G(n, f) = n$. We call this the category $\mathcal{C}$ *formally graded* over $\mathcal{M}$.

Generally speaking, the natural notion of a morphism between graded categories is that of a (strictly) grade-preserving functor. That is – given two graded categories with grading functors $G : \mathcal{C} \to \mathcal{M}$ and $G' : \mathcal{D} \to \mathcal{M}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is (strictly) grade-preserving if it makes the following diagram commute

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{D} \arrow[dl, "G'"] \\
& \mathcal{M} &
\end{tikzcd}$$

Although, as implied by the name, some times this kind of morphism will be too strict for our purposes. Thus, in addition to this we will define another natural notion of grade-preserving functor later.

Remark. Elementary graded structures on their own do not have much intrinsically interesting categorical structure, as they are essentially a way of annotating the grades of a category with additional information in a compositional way. However, such structures have still found some important uses in the literature, for instance in Ehresmann's [1], where she defines such a structure in order to model the firing delay of individual neurons (which are modeled collectively as a category).

Remark. Notice that a graded category with lifts in our original sense (i.e. a *total* category graded over a monoid) is necessarily one graded over a join-semilattice (or, in the partially graded case, a *partial join-semilattice*) – since if we take $f = \mathrm{lift}_A^m$ in either one of the first conditions, we obtain $\mathrm{lift}_A^m \circ \mathrm{lift}_A^m = \mathrm{lift}_A^m$. And with the naturality condition we can take $f = \mathrm{lift}_A^n$ we obtain $\mathrm{lift}_A^n \circ \mathrm{lift}_A^m = \mathrm{lift}_A^m \circ \mathrm{lift}_A^n$. Furthermore, using both of these properties, we obtain

$$\begin{aligned} \mathrm{lift}_A^n \circ \mathrm{lift}_A^m &= (\mathrm{lift}_A^n \circ \mathrm{lift}_A^m) \circ \mathrm{lift}_A^{n \vee m} \\ &= \mathrm{lift}_A^n \circ \mathrm{lift}_A^{n \vee m} \\ &= \mathrm{lift}_A^{n \vee m} \end{aligned}$$

for any $n, m \in \mathcal{M}$. Hence, given the necessity of lifting morphisms in our theory, we are very well justified in restricting our attention (in the total case) to join-semilattices, beyond just our intuitive justification of this choice in the previous section. In other words, if we wish to study gradings over more general monoids, this will not be possible in our current framework.

Notice also that, if a lifting map of grade n exists, it is unique. Suppose ℓ_n and ℓ_n' are two candidates for lift_A^n. Then, by our properties we have $\ell_n' \circ \ell_n = \ell_n$ (ℓ_n' acts as a left identity for all morphisms of grade $\geq n$) and similarly we have $\ell_n' \circ \ell_n = \ell_n'$ (ℓ_n acts as an identity on the right for all morphisms of grade $\geq n$). Thus, we conclude $\ell_n = \ell_n'$.

It turns out that, as alluded to in the previous section with our discussion of lifting morphisms, many of the usual concepts that one works with in "vanilla" category theory need to be tweaked slightly to work the way one would expect them to in the context of graded categories. For instance, instead of monomorphisms and epimorphisms, we have:

Definition 1.1.9. Given a graded category $\mathcal{C}$, we say that $f : A \to B$ is a (graded) epimorphism if for all grades n and for all morphisms $g, h : B \to_n C$, $g \circ f = h \circ f \implies g = h$.

The notion of a (graded) monomorphism is defined dually.

Remark. It turns out that there are several more-or-less weak versions of graded epi/monomorphisms one can define in the context of graded categories, not all of which turn out to be equivalent, but for our purposes in this thesis, we will stick with this definition. More properly, these should probably be called *fiber-wise* graded mono/epi morphisms.

In addition to this, we also have the important notion of a *pseudoisomorphism*, which we discussed earlier, and will make ample use of in the next chapter:

Definition 1.1.10. Let $\mathcal{C}$ be a graded category over $\mathcal{M}$ with lifts. We say that $f : A \to_n B$ is a *pseudoisomorphism* if there exists a morphism $g : B \to_m A$ such that $g \circ f = \mathrm{lift}_A^{n \vee m}$ and $f \circ g = \mathrm{lift}_B^{n \vee m}$.

Example 4. In $\mathcal{C}_{\mathcal{M}}$, a graded epimorphism is precisely a morphism of the form (n, f), where f is an epimorphism in $\mathcal{C}$, and similarly for monomorphisms.

The notion of a graded monomorphism allows us to define the so-called *strong lifting property*:

Definition 1.1.11. Let $\mathcal{C}$ be a graded category with lifts. We say that $\mathcal{C}$ has the *strong lifting property* if every lifting morphism lift_A^n is a graded monomorphism.

Intuitively this says that strong lifting morphisms actually act as "embeddings" of the set of "lower morphisms" into the set of "higher morphisms", and thus, it is easy to see that this will not hold in general.

Example 5. $\mathcal{C}_{\mathcal{M}}$ has the strong lifting property.

Remark. There are some intuitive properties we would like the lifting morphisms to satisfy which require the strong lifting property in order to prove. For instance, suppose we have a commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "g"] & & A \\ & A \arrow[ul, "\mathrm{lift}_A^n"] \arrow[ur, "\mathrm{lift}_A^m"'] & \end{tikzcd}$$

where g is a morphism of grade k. Then:

$$\begin{aligned} & g \circ \mathrm{lift}_A^n = \mathrm{lift}_A^m \\ \implies & g \circ \mathrm{lift}_A^n \circ \mathrm{lift}_A^m = \mathrm{lift}_A^m \circ \mathrm{lift}_A^m \\ \implies & g \circ \mathrm{lift}_A^n = \mathrm{lift}_A^m \\ \implies & g \circ \mathrm{lift}_A^n = \mathrm{lift}_A^k \circ \mathrm{lift}_A^m \\ \implies & g = \mathrm{lift}_A^k \ \ \text{(By the strong lifting property)} \end{aligned}$$

Much like we have considered weakened versions of the standard notions of mono, epi, and isomorphism above, in the context of graded category theory, it is also natural to weaken our notion of a grade preserving functor. In fact, the more natural morphism between graded categories with lifts actually turns out to be a particular kind of *semifunctor* (though, as the reader should keep in mind, this does not change the fact that in the context we are working in for this thesis, we still require the grading functor itself to be an honest-to-goodness *functor*):

Definition 1.1.12. Let $G : \mathcal{C} \to \mathcal{M}$, $G' : \mathcal{D} \to \mathcal{M}$ be two elementary graded categories with lifts. We say that a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is a *weakly grade-preserving* if the following diagram commutes for some grade n – which we call the *grade* of the functor:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "G'"] \\
\mathcal{M} \arrow[r, "n"'] & \mathcal{M}
\end{tikzcd}$$

Note that in general, for this definition to be non-trivial, F in fact needs to be a semifunctor, as for a weakly grade-preserving semifunctor of grade n, id_A will have to map to a morphism of grade n, and hence cannot possibly get mapped to id_{FA}, which is another morphism of grade zero. Thus, we say that F is additionally a weakly grade-preserving *functor* if it satisfies the next best condition: namely, that $F(\mathrm{id}_A) = \mathrm{lift}^n_{FA}$ for all A in the category.

Remark. The reader should take a moment here to ensure they understand the terminology that we are using here. The important notion for us is that of a *weakly grade-preserving functor* as defined above. While this is not a functor in the strict sense, we still use this terminology since a weakly grade-preserving functor can essentially be thought of as a functor *up to lifts*.

Part of the justification for the above definition is that, in general, since we cannot expect identity morphisms to show up in equations involving morphisms of non-trivial (i.e. non-zero) grade, all of the relevant categorical constructions we consider in this thesis should generally be *up to lifting maps*.

And thus, in order to be able to define categorical notions (for instance, such as the colimit of a diagram) *up to grading* in terms of functors between graded categories (analogously to how this is done in standard category theory), our functors should also, generally speaking, be *up to lifts*. In other words, we wish to consider different sorts of *weakened* universal properties which only hold after our morphisms are lifted to some particular grade, and thus our morphisms between graded categories should reflect this by not being *strictly* grade-preserving.

While in addition to this we could in principle consider even weaker notions of grade preservation (i.e. grade-monotone maps), and the categories of graded categories with such weak morphisms likely deserves further study, we will not consider this here.

Our restriction to this less general setting of weakly-grade preserving functors is helpful because of the following nice result:

Proposition 1.1.13. The category $\mathbf{GCat}^{\ell}_{\mathcal{M}}$ of graded categories with lifts, and (weakly) grade-preserving functors is itself a graded category with lifts over $\mathcal{M}$.

Proof. Define the grade of a functor as defined in definition 1.1.12 to be the n such that $F(\mathrm{id}_A) = \mathrm{lift}^n_A$, it is then straightforward to verify that this defines a graded category over $\mathcal{M}$. The lifting functors are defined by $\mathrm{Lift}_{\mathcal{C}}(f) = f \circ \mathrm{lift}^n_A$. □

Remark. The category $\mathbf{GCat}^{\ell}_{\mathcal{M}}$ can be defined just as well in the partial case, provided we replace all of the notions with their partial analogs. Notice, however, that in this case, our weakly grade-preserving functors of grade n actually need to be *partial* functors (defined only for grades compatible with n), rather than total functors between partial categories.

We say that this is a "nice result", because it mirrors the result in standard category theory that the category **Cat** of categories is again a category (in fact a 2-category). In fact, this analogy goes even deeper, because it turns out that $\mathbf{GCat}^{\ell}_{\mathcal{M}}$ is actually a "graded *2-category* with lifts" – a notion which we expect will be useful in further developments of the theory of graded categories, but will not formally define here.

Indexed Structures

One particularly useful way of looking at the more general BCPO-graded categories with lifts we considered earlier is as a certain sort of (strict) indexed category. In fact, the main result of this section is to show the equivalence between the 2-categories $\mathbf{GCat}^{\ell}_{\mathcal{M}}$ and $\mathbf{IndCat}_{\mathcal{M}}$. Notice that, in particular, the presence of lifts is important here – the construction we will consider in this section does not work for graded categories without lifts. Hence, we define the following:

Definition 1.1.14. An indexed structure for a graded category is a strict indexed category (i.e. a functor $\mathbb{C} : \mathcal{P} \to \mathrm{Cat}$) where $\mathcal{P}$ is a (connected) bounded complete poset, and the functors (which we call transition functors) $\mathbb{C}_i^j : \mathbb{C}_i \to \mathbb{C}_j$ are identities on objects – where here, we say that a poset $\mathcal{P}$ is *connected* if the Hasse diagram of $\mathcal{P}$ is connected as an undirected graph. In other words, more formally, we say that $\mathcal{P}$ is connected if there is a "path of zig-zags" $p_1 \leq p_2 \geq p_3 \leq \cdots \geq p_n$ between any two elements of $\mathcal{P}$.

Remark. Here we will sometimes denote $\mathbb{C}(\leq_i^j)$ as $\mathbb{C}_i^j$, where $\leq_i^j$, if it exists, is the unique morphism in $\mathcal{P}$ from i to j.

The intuition with this definition being that $\mathbb{C}_n$ is the collection of *objects and morphisms of grade* n of the graded category. We can then define the composition of two morphisms $f \in \mathbb{C}_n(A, B)$ and $g \in \mathbb{C}_m(B, C)$ (say, if $n \leq m$) by applying the transition functor from n to m to f, and then composing with g – i.e. $g \circ \mathbb{C}_n^m(f)$ in order to derive an elementary graded category from this structure.

Remark. While generally speaking, our preferred reserved notation is to use $\mathcal{P}$ for posets (viewed as a posetal category) used as the domain of the functor defining an indexed category, and $\mathcal{M}$ for join-semilattices (more generally, BCPOs) interpreted as single object (partial) categories in the standard way, we will often abuse notation and use (for instance) $\mathcal{M}$ to refer to both it's representation as a single object category, and it's representation as a posetal category. This should cause no confusion, as the former use will always arise when $\mathcal{M}$ appears as the codomain of a grading functor $G : \mathcal{C} \to \mathcal{M}$, and the latter when $\mathcal{M}$ is the domain of a functor defining an indexed category – e.x. $\mathbb{C} : \mathcal{M} \to \mathbf{Cat}$.

The condition in the above definition that the functor defining an indexed structure be *identity on objects* may seem a little arbitrary – if we interpret $\mathbb{C}_i$ as the "category of objects and morphisms at stage i", then this restriction essentially means that we *never introduce any new objects to our graded category.* In other words, *every object that exists, already exists at stage* 0.

Arguably, we could generalize this in different ways, and in future work, it may be fruitful to study such generalizations. However, this definition is in fact quite natural in light of the structure we are trying to generalize here – the notion of a graded setoid. Recall that for graded setoids, rather than the *objects* of the set themselves varying over some poset, it is instead the identity criterion itself that varies. Similarly, for a graded category, the set of objects itself is constant across all grades – what changes is merely the set of morphisms.

The most natural definition of a morphism between indexed structures (i.e. an *indexed functor* is simply a natural transformation between the two indexed categories – in other words, a collection of maps from the objects and morphisms of grade n in $\mathbb{C}$, to the objects and morphisms of grade n in $\mathbb{D}$ that commute with the transition functors. This is the "strict" notion of functor between indexed structures.

In the case of $\mathcal{M} = \mathbb{N}$ (which we will often use for simple visualizations of indexed structures, even when in full generality, the theory we develop here all holds over general BCPOs), such a functor between graded categories $\mathbb{C}$ and $\mathbb{D}$ may be pictured as a commutative diagram as follows:

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r] & \mathbb{D}_1 \arrow[r] & \mathbb{D}_2 \arrow[r] & \mathbb{D}_3 \arrow[r] & \dots \\
\mathbb{C}_0 \arrow[r] \arrow[u, "F_0"] & \mathbb{C}_1 \arrow[r] \arrow[u, "F_1"] & \mathbb{C}_2 \arrow[r] \arrow[u, "F_2"] & \mathbb{C}_3 \arrow[r] \arrow[u, "F_3"] & \dots
\end{tikzcd}$$

As was the case with elementary graded categories, however, this notion is not always the most natural one. In the context of indexed categories, the more general thing to do is to define a *partial functor* between indexed structures – that is, given some subset $U \subseteq \mathcal{P}$ of the poset $\mathcal{P}$, a natural transformation:

$$
\begin{tikzcd}
\mathcal{P} \arrow[r, "\mathbb{C}"] & \mathbf{Cat} \\
\mathcal{U} \arrow[u, hook] \arrow[r, hook] & \mathcal{P} \arrow[u, "\mathbb{D}"'] \arrow[ul, Rightarrow, "F"']
\end{tikzcd}
$$

This defines a category of indexed structures (which we denote $\mathbf{IndCat}_{\mathcal{P}}$) with the composition of two partial functors between indexed structures defined by the following pasting diagram:

$$
\begin{tikzcd}
 & & U \cap V \arrow[dl, hook] \arrow[dr, hook] & & \\
 & U \arrow[dl, hook] \arrow[dr, hook] \arrow[rr, phantom, "="] & & V \arrow[dl, hook] \arrow[dr, hook] & \\
\mathcal{P} \arrow[drr, "\mathbb{C}"'] \arrow[rr, phantom, "\overset{F}{\Rightarrow}"] & & \mathcal{P} \arrow[d, "\mathbb{D}"] \arrow[rr, phantom, "\overset{G}{\Rightarrow}"] & & \mathcal{P} \arrow[dll, "\mathbb{E}"] \\
 & & \mathcal{M} & &
\end{tikzcd}
$$

Such a functor can be pictured (as the name suggests) as the usual commutative diagram used to visualize functors between indexed structures, except where some of the required natural transformations are missing, for instance:

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r] & \mathbb{D}_1 \arrow[r] & \mathbb{D}_2 \arrow[r] & \mathbb{D}_3 \arrow[r] & \ldots \\
\mathbb{C}_0 \arrow[r] \arrow[u, dash, dashed] & \mathbb{C}_1 \arrow[r] \arrow[u, dash, dashed] & \mathbb{C}_2 \arrow[r] \arrow[u, "F_2"] & \mathbb{C}_3 \arrow[r] \arrow[u, "F_3"] & \ldots
\end{tikzcd}
$$

However, in order to find an equivalence here between elementary graded categories and indexed structures, we do not want to consider arbitrary subsets U as possible domains of definition for our partial functors. Given a poset $\mathcal{P}$, we say that a subset

$U \subset \mathcal{P}$ is an *upperset* if $x \in U$ and $y \in \mathcal{P}$ with $y \geq x$ implies $y \in U$. The collection of uppersets forms the set of opens for a topology on $\mathcal{P}$, called the *Alexandrov topology* , which we will denote by $\mathcal{A}(\mathcal{P})$. The so-called *principal upper set* of an element $a \in \mathcal{P}$ is the set $\uparrow a = \{x \mid x \geq a\}$, which is a basic example of an upperset. Moreover, the collection of all principal upper sets in fact forms a basis for the Alexandrov topology.

This topology is important in the study of graded categories with lifts, as essentially, the existence of lifting morphisms guarantees us (in general), that

> "If something is defined at a lower grade, it might as well be defined at all higher grades as well"

In other words, our partial functors "might as well" be defined on *upper sets*. Moreover, we will also consider partial morphisms between indexed structures to be defined only on *principal up-sets*. Since these are in one-to-one correspondence with the elements of $\mathcal{P}$, this gives us the analogous situation to the one we had when defining a notion of "weak" morphisms between (elementary) graded categories – where the category of graded categories over $\mathcal{M}$ (with lifts) is again a graded category over $\mathcal{M}$ (with lifts).

We state and prove this result for more than merely **Cat**-valued functors, as we will need this result in more generality later:

Proposition 1.1.15. Let $\mathcal{P}$ be a (connected) BCPO viewed as a posetal category, and $\mathcal{P} /\!\!/ \mathcal{C}$ denote the category with functors $\mathcal{P} \to \mathcal{C}$ as objects, and morphisms as partially defined natural transformations as discussed above (i.e. defined on subsets of the form $\mathcal{U} = \uparrow a$ of $\mathcal{P}$) as morphisms. Then $\mathcal{P} /\!\!/ \mathcal{C}$ is a graded category over $\mathcal{P}$ with lifts.

Proof. It will be easiest to show that this is an *elementary* graded category – and so given the equivalence between elementary graded categories and indexed structures we will prove below, we show this here. Define the grade of a partial morphism between indexed structures to be the $a \in \mathcal{P}$ such that the partial morphism is defined on the set $\uparrow a$. It is then easy to show that this defines the structure of a graded category over $\mathcal{P}$ (noting that $\uparrow a \cap \uparrow b = \uparrow(a \vee b)$ if $a \vee b$ exists, and that since we are in a BCPO, the intersection is the empty set otherwise). To define lifting morphisms, we can simply take restrictions of the identity natural transformation between two indexed structures to the relevant set $\uparrow a$. $\square$

Corollary 1.1.16. $\mathbf{IndCat}_{\mathcal{P}}$ (which is equivalently the full subcategory of $\mathcal{P} /\!\!/ \mathbf{Cat}$ consisting of functors $\mathcal{P} \to \mathbf{Cat}$ whose transition functors are all identity-on-objects, where $\mathcal{P}$ is here being viewed as a posetal category) is a graded category over $\mathcal{P}$ with lifts.

1.1.3 2-categories of graded categories and indexed structures

Now, let us consider some of the possible definitions one might have for a natural transformation between two functors between elementary graded categories. If F and G are grade-preserving functors, for instance, and $\eta_A : FA \to GA$ is a family of morphisms such that the usual naturality square commutes, then under some fairly mild conditions (i.e. there exists a morphism either from A to B, or from B to A of grade 0), η_A and η_B will necessarily have the same grade. Thus, it seems like a reasonable enough condition to assume that our natural transformations have components with a uniform grade n.

The situation when we consider natural transformations with weakly grade-preserving functors is slightly different, but notice here that if the grade of F is n and the grade of G is n, then since we can always factor $F(f)$ as $\mathrm{lift}^n_{FA} \circ F(f) = F(f) \circ \mathrm{lift}^n_{FB}$ and $G(f)$ as $\mathrm{lift}^m_{GA} \circ G(f) = G(f) \circ \mathrm{lift}^m_{GB}$, we might as well assume the components of such a natural transformation have grade at least $n \vee m$ – and furthermore, for the reasons already mentioned, we might as well assume also that all component morphisms must be of equal grade.

For indexed categories in general, a natural transformation between two functors $F, G : \mathbb{C} \to \mathbb{D}$, where $\mathbb{C}, \mathbb{D} : \mathcal{P} \to \mathbf{Cat}$ is defined as a family of natural transformations such that, for all $n \leq m$, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{D}_n \arrow[r] & \mathbb{D}_m \\
\mathbb{C}_n \arrow[u, bend left, "G_n"] \arrow[u, bend right, "F_n"'] \arrow[u, phantom, "\Longrightarrow"] \arrow[r] & \mathbb{C}_m \arrow[u, bend left, "G_m"] \arrow[u, bend right, "F_m"'] \arrow[u, phantom, "\Longrightarrow"]
\end{tikzcd}$$

This definition, for instance, is used in Crole [9]. We will denote the set of natural transformations between two indexed functors $F, G : \mathbb{C} \to \mathbb{D}$ by $\mathrm{Nat}(F, G)$. To extend this definition to partial functors $F : \mathbb{C} \to_n \mathbb{D}, G : \mathbb{C} \to_m \mathbb{D}$ between indexed structures, we can simply define a natural transformation between indexed categories to be a family of natural transformations η^i for $i \in {\uparrow}n \cap {\uparrow}m$, satisfying the condition above for all $i \leq j$ with $i, j \in {\uparrow}n \cap {\uparrow}m$.

For natural transformations between elementary graded categories, we have a simpler definition, based on our discussion above:

Definition 1.1.17. Let $\mathcal{C}$ and $\mathcal{D}$ be elementary graded categories and $F : \mathcal{C} \to_a \mathcal{D}, G : \mathcal{C} \to_b \mathcal{D}$ be (weakly) grade preserving functors. A natural transformation $\alpha : F \to_n G$ of grade $n \geq a \vee b$ consists of, for every object $A \in \mathcal{C}$ a family of morphisms $\alpha_A : FA \to_n GA$ such that for all $f : A \to B$ in $\mathcal{C}$, the usual naturality condition holds – i.e. the following diagram commutes:

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\eta_A"'] & FB \arrow[d, "\eta_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

1.1.4 The Equivalence of Graded Categories and Indexed Structures

It is a well-known fact of topos theory that there is an equivalence between *fibrations*, and *indexed categories*, given by the so-called *Grothendieck construction*. In our context, we have a similar result, but given the fact that we only consider a vary particular class of indexed categories in this section, the correspondence is not exactly the same as the one from the theory of fibrations. Thus, we will now outline a concrete construction showing the equivalence of graded categories and indexed structures.

Definition 1.1.18. Given an indexed structure $\mathbb{C} : \mathcal{P} \to \mathrm{Cat}$, where $\mathcal{P}$ is a (connected) BCPO, we can define a graded category with lifts $\mathcal{E}(\mathbb{C})$ (graded over $\mathcal{P}$, now viewed as a single object partial category) as follows:

- The objects are given by:

$$\mathrm{Ob}(\mathcal{E}(\mathbb{C})) := \mathrm{Ob}(\mathbb{C}_i) \text{ for any } i \in \mathcal{P}$$

- For any $A, B \in \mathrm{Ob}(\mathcal{E}(\mathbb{C}))$, we define:

$$\mathrm{Hom}_{\mathcal{E}(\mathbb{C})}(A, B) := \coprod_{j \in \mathcal{P} \ A,B \in \mathcal{E}(\mathbb{C})} \mathrm{Hom}_{\mathbb{C}_j}(A, B)$$

- We define $G(f) := j$, where $f \in \mathrm{Hom}_{\mathbb{C}_j}(A, B)$.

- Two morphisms f, g with $G(f) = i, G(g) = j$ are composable if and only if i and j are compatible elements of $\mathcal{P}$. In this case, let $k = i \vee j \in \mathcal{P}$ and define:

$$g \circ f := \mathbb{C}_j^k(g) \circ \mathbb{C}_i^k(f)$$

From the last two conditions is is easy to see that this in fact defines a partially graded category over $\mathcal{P}$ in the sense of definition 1.1.7, with lifting maps given by $\mathrm{lift}_A^n := \mathrm{id}_A \in \mathrm{Hom}_n(A, A)$.

Definition 1.1.19. Given a partial graded category $(\mathcal{C}, G : \mathcal{C} \to \mathcal{P})$ with lifts, we define an indexed category $\mathbb{I}(\mathcal{C})$ by:

- $\mathbb{I}(\mathcal{C})_i$ is the sub-category of $\mathcal{C}$ consisting of the objects of $\mathcal{C}$, together with all of morphisms of grade i. We will also call this category the ith fiber over the element $i \in \mathcal{P}$, and will sometimes also denote this by $G^{-1}(i)$.

- $\mathbb{I}(\mathcal{C})_i^j$ is given by the identity map on objects, and by mapping $f \mapsto f \circ \mathrm{lift}_A^j$ on morphisms.

It is then straightforward to see that this defines an indexed structure over $\mathcal{P}$.

Proposition 1.1.20. The constructions outlined in definitions 1.1.18 and 1.1.19 are mutually inverse, i.e:

$$\mathbb{I}(\mathcal{E}(\mathbb{C})) \cong \mathbb{C} \text{ for any indexed category } \mathbb{C} : \mathcal{M} \to \mathbf{Cat}$$

and

$$\mathcal{E}(\mathbb{I}(\mathcal{C})) \cong \mathcal{C} \text{ for any partial graded category } (\mathcal{C}, G : \mathcal{C} \to \mathcal{M})$$

Proof. Let $(\mathcal{C}, G : \mathcal{C} \to \mathcal{M})$ be a graded category with lifts over the connected BCPO $\mathcal{M}$, and note that $\mathcal{E}\mathbb{I}\mathcal{C}$ has the same set of objects as $\mathcal{C}$ by construction. Moreover, for its set of morphisms we have

$$\mathrm{Hom}_{\mathcal{E}\mathbb{I}\mathcal{C}}(A,B) := \coprod_{i\in\mathcal{M}} \mathrm{Hom}_{\mathcal{C}_i}(A,B) \cong \mathrm{Hom}_{\mathcal{C}}(A,B)$$

where $\mathcal{C}_i$ denotes $\mathbb{I}\mathcal{C}_i$. Thus, to show that these are isomorphic as categories, it remains to see that the composition operations of both of these categories are defined in the same way. Given $f : A \to_n B$ and $g : B \to_m C$ in $\mathcal{C}$, the composition is defined as simply $g \circ f$. In $\mathcal{E}\mathbb{I}\mathcal{C}$ the composition is given (under our identification of the homsets of these categories above) by $g \circ_{\mathcal{E}\mathbb{I}\mathcal{C}} f = (\mathrm{lift}_C^{n\vee m} \circ g) \circ (\mathrm{lift}_B^{n\vee m} \circ f)$, which by the properties of lifting morphisms we know is equal to $g \circ f$.

Now, let $\mathbb{C} : \mathcal{P} \to \mathbf{Cat}$, and consider the indexed structure $\mathbb{I}\mathcal{E}\mathbb{C}$. Again, it is easy to see that $\mathrm{Ob}(\mathbb{C}_i) = \mathrm{Ob}(\mathbb{I}\mathcal{E}\mathbb{C}_i)$ for all $i \in \mathcal{P}$, so it remains to show that:

1. The set of morphisms for each of the categories $\mathbb{C}_i$ is the same as the set of morphisms for $\mathbb{I}\mathcal{E}\mathbb{C}_i$ (and the composition operations in these categories are defined in the same way).

2. The transition functors for $\mathbb{C}$ and $\mathbb{I}\mathcal{E}\mathbb{C}$ are defined in the same way.

1. is easy to see given the decomposition

$$\mathrm{Hom}_{\mathcal{E}\mathbb{C}}(A,B) = \coprod_{i\in\mathcal{P}} \mathrm{Hom}_{\mathcal{E}\mathbb{C}_i}(A,B) \cong \mathrm{Hom}_{\mathbb{C}_i}(A,B)$$

For 2, let $\mathbb{C}_i^j$ be an arbitrary transition functor and consider how these transition functors act on morphisms $f \in \mathbb{C}_i(A,B)$ in it's image $\mathbb{I}\mathcal{E}(\mathbb{C}_i^j)$. Now, notice that in viewing $\mathbb{C}$ as an elementary graded category (which we denote by $\mathcal{E}\mathbb{C} = \mathcal{C}$), we have by our construction $\mathrm{lift}_B^j \circ f := \mathbb{C}_i^j(f)$. Thus, we see that the left-hand side (which is how we define the transition functors in $\mathbb{I}\mathcal{E}\mathbb{C}$) is defined in the same way as the transition functors $\mathbb{C}_i^j$ are defined. $\square$

Proposition 1.1.21. The maps $\mathbb{I}$ and $\mathcal{E}$ extend to functors between $\mathbf{IndCat}_{\mathcal{P}}$ and $\mathbf{GCat}^{\ell}_{\mathcal{P}}$ by defining $\mathbb{I}F$ component-wise by

$$(\mathbb{I}F)_i = F|_{\mathbb{I}\mathcal{C}_i}$$

where $F : \mathcal{C} \to_n \mathcal{D}$ is a (weakly) grade-preserving functor – and by defining $\mathcal{E}(F)$ (where $F : \mathbb{C} \to \mathbb{D}$ is now a functor between indexed structures) by mapping $f \in \mathrm{Hom}_i(A, B)$ to $F_{i\vee n}(\mathbb{I}\mathcal{C})_i^{i\vee n}(f) \in \mathrm{Hom}_{i\vee n}(FA, FB)$ for all grades $i \in \mathcal{P}$ such that $i \vee n$ exists (and leaving the functor undefined otherwise).

Proof. The only non-trivial fact to check here is that both of these operations above preserve composition and identities of morphisms in $\mathbf{IndCat}_{\mathcal{P}}$ and $\mathbf{GCat}^{\ell}_{\mathcal{P}}$ respectively, but both of these facts are easy enough to check directly by our definitions. $\square$

Putting this all together, we can now show:

Theorem 1.1.22. There is an equivalence of categories between $\mathbf{IndCat}_{\mathcal{P}}$ and $\mathbf{GCat}^{\ell}_{\mathcal{P}}$ induced by the functors defined above.

Proof. By the isomorphisms we showed in proposition 1.1.20 we are almost all the way to a categorical equivalence. What it remains to show is that both of these are *natural isomorphisms*. In other words, for any weakly grade preserving functor $F : \mathcal{C} \to_n \mathcal{D}$ we need to show that the following diagram commutes

$$\begin{tikzcd}
\mathcal{E}\mathbb{I}\mathcal{C} \arrow[r, "\mathcal{E}\mathbb{I}F"] & \mathcal{E}\mathbb{I}\mathcal{D} \\
\mathcal{C} \arrow[u] \arrow[r, "F"'] & \mathcal{D} \arrow[u]
\end{tikzcd}$$

where the vertical morphisms in the above diagram denote the identifications that were implicit in our proof of proposition 1.1.20. To see why this indeed commutes, note that this essentially amounts to checking that under our identification, $\mathcal{E}\mathbb{I}F$ is defined the same way as F, but this is easy to see from our construction in proposition 1.1.21. Similarly, verifying our other naturality condition

$$\begin{tikzcd} \mathbb{I}\mathcal{E}\mathbb{C} \arrow[r, "\mathbb{I}\mathcal{E}F"] & \mathbb{I}\mathcal{E}\mathbb{D} \\ \mathbb{C} \arrow[u] \arrow[r, "F"'] & \mathbb{D} \arrow[u] \end{tikzcd}$$

is just as straightforward. □

Remark. Oftentimes we can characterize graded categorical properties in a simpler or more convenient way by passing through this equivalence. For instance, it is easy to see that an elementary graded category $(\mathcal{C}, G : \mathcal{C} \to \mathcal{M})$ satisfies the strong lifting property if and only if the transition functors $\mathbb{C}^m_n$ in its equivalent indexed structure are all faithful, since if f and g are two morphisms of grade m, then $\mathrm{lift}^n_B \circ f = \mathrm{lift}^n_B \circ g$ if and only if $\mathbb{C}^n_m(f) = \mathbb{C}^n_m(g)$.

Additional Properties of Graded Categories and Natural Transformations

In this section we will explore some of the basic constructions that one can preform on the different 2-categories of graded categories that one might want to consider. We first recall some of the basic properties of functors from plain category theory:

Definition 1.1.23. Given a functor between categories $F : \mathcal{C} \to \mathcal{D}$, we say:

1. F is full if the function induced by F mapping $\mathcal{C}(A, B)$ to $\mathcal{D}(FA, FB)$ is surjective.

2. F is faithful if the function induced by F mapping $\mathcal{C}(A, B)$ to $\mathcal{D}(FA, FB)$ is injective.

3. F is essentially surjective if for all objects $X \in \mathcal{D}$, there exists an object $A \in \mathcal{C}$ such that X is isomorphic to FA.

The first two notions can easily be defined just as well for a strictly grade-preserving functor if we consider $\mathcal{C}(A, B)$ to be the disjoint union of the graded hom sets $\mathcal{C}_i(A, B)$ for all i. However, for *weakly grade-preserving functors* these conditions are too strong. Also, since in a graded category, *pseudoisomorphism* is the more important notion than *isomorphism*, we will want to modify the last definition slightly as well.

Definition 1.1.24. Given two graded categories $\mathcal{C}, \mathcal{D}$ and a graded semifunctor $F : \mathcal{C} \to \mathcal{D}$, we say:

1. F is pseudo-full if for for all objects A, B and for all grades n the function induced by F mapping $\mathcal{C}_n(A, B)$ to $\mathcal{D}_{Fn}(FA, FB)$ is surjective.

2. F is pseudo-faithful if for for all objects A, B and for all grades n the function induced by F mapping $\mathcal{C}_n(A, B)$ to $\mathcal{D}_{Fn}(FA, FB)$ is injective.

3. F is (pseudo)-essentially surjective if for all objects $X \in \mathcal{D}$ there exists an object $A \in \mathcal{C}$ such that X is pseudoisomorphic to FA.

Example 6. Given a graded category $\mathcal{C}$ with lifts, and $n \in \mathcal{M}$, consider the weakly grade-preserving functor $F : \mathcal{C} \to \mathcal{C}$ defined by mapping $F(f) = f \circ \mathrm{lift}^n_A$. If $\mathcal{C}$ has the strong lifting property, then this functor is pseudo-faithful (and also pseudo-full) but it is neither full, nor faithful in the traditional sense.

1.2 Variable Object Models and the Graded Yoneda Lemma

Given a category $\mathcal{C}$ with a terminal object, it is often convenient to interpret morphisms $1 \to X$ as "points" or "elements" of X. More generally, even in categories without terminal objects, we can view morphisms $A \to X$ as so-called "generalized elements" of X. This observation is essentially what allows us to define a "Cayley representation" of every (small) category as a category of sets and functions (theorem 1.6 of Awodey [10]). However, if we look at graded categories and want to prove a similar representation theorem, we need to take a slightly different approach. The basic observation is, given a morphism $f : A \to_n B$ of grade n in a graded category $\mathcal{C}$ and a "point" $x : X \to_m A$ of grade m, the composition $f \circ x$ gives us a "point" $X \to_{n \vee m} A$ of grade $n \vee m$. Since generalized points in a graded category come with a grade, a natural approach to the representation of the objects of a graded category is to interpret the objects A as an indexed family $\{A_n\}_{n \in \mathcal{M}}$ of the set of "points" of grade n of A. Then, given our observation of how composition works in a graded category, a morphism in this representation should be a family of morphisms $f_m : A_m \to B_{m \vee n}$. This is the basic idea of our model.

The following construction works for general elementary graded categories (i.e. not nescesarialy with lifts, or over a join-semilattice):

Definition 1.2.1. Given a category $\mathcal{C}$ and an ordered monoid $\mathcal{M}$, we define the category of $\mathcal{M}$-variable objects in $\mathcal{C}$ and graded functions, denoted $\mathbf{Var}_{\mathcal{M}}(\mathcal{C})$ whose objects consist of functors $A : \mathcal{M} \to \mathcal{C}$, and whose morphisms (of grade n) consist of indexed families f of morphisms $f_m : A_m \to B_{n \cdot m}$ in $\mathcal{C}$, where given such a morphism f of grade n composition is defined

$$(g \circ f)_m = g_{n \cdot m} \circ f_m$$

The key theorem in this section is that we can define a "Cayley" representation functor into this category of variable objects.

Definition 1.2.2. For any (small) graded category $\mathcal{C}$ (again, not necessarily with lifts, and graded over some monoid $\mathcal{M}$), we define its Cayley representation $\mathscr{R}(\mathcal{C})$ by mapping objects $A \in \mathcal{C}$ to variable sets

$$\mathscr{R}(A)_n = \{x : X \to_n A \mid X \in \mathcal{C}\}$$

and by mapping morphisms $f : A \to_n B$ to the family of functions from $\mathscr{R}(A)_m \to \mathscr{R}(B)_{n \cdot m}$ given by precomposition of points with f.

Theorem 1.2.3. Every elementary graded category $\mathcal{C}$ is isomorphic to its representation $\mathscr{R}(\mathcal{C})$.

Proof. Clearly $\mathcal{R}$ is a grade-preserving functor from $\mathcal{C}$ onto $\mathcal{R}(\mathcal{C})$ (i.e. it is full and surjective on objects by construction). For $\mathcal{R}$ to be an isomorphism of graded categories, we need it to be a bijection on objects and morphisms, and hence, we need to show that it is injective on objects, as well as faithful.

1. Suppose A and B are distinct objects of $\mathcal{C}$, then the identity morphisms id_A and id_B are also distinct. Hence, since $\mathrm{id}_A \in \mathcal{R}(A)$ but $\mathrm{id}_A \notin \mathcal{R}(B)$ we conclude that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are distinct sets. Hence, $\mathcal{R}$ is faithful on objects.

2. Let $f, g : A \to_n B$. Suppose that $\mathcal{R}(f) = \mathcal{R}(g)$, then for all morphisms x we have $f \circ x = g \circ x$, and hence, $f = g$ since in particular we can take $x = \mathrm{id}_A$.

$\square$

Representations via the graded Yoneda embedding

In the case that we are grading over a join semilattice and our graded category has lifts, our representation can be simplified somewhat. This is because given a morphism $f : A \to_n B$ and a point $x : X \to_m A$ we have

$$f \circ x = (f \circ \mathrm{lift}_A^{n \vee m}) \circ (\mathrm{lift}_A^{n \vee m} \circ x)$$

– in other words, in this case, we only ever need to consider families of morphisms of the form $f_i : A_i \to B_i$. Once we generalize this to BCPOs, this leads us to our second representation theorem, which is more categorical in flavor, and based on a graded version of the Yoneda lemma.

In plain category theory, the analogous representation of small categories as "concrete categories" – i.e. subcategories of the category of sets and functions can be understood more abstractly as a representation of the objects of the category as *presheaves* – i.e. functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. We can do the same in the context of graded categories, but we must first make some initial observations. First, we have another application of proposition 1.1.15:

Corollary 1.2.4. Let $\mathcal{M}$ be a BCPO, viewed as a posteal category, then the category $\mathcal{M} \mathbin{/\!\!/} \mathbf{Set}$ of $\mathcal{M}$-variable sets and partial maps between them forms a graded category over $\mathcal{M}$.

Now, notice that in ordinary categories $\mathrm{Hom}(A, B)$ is always a set, so in the Yoneda lemma we consider set-valued pre-sheaves $\mathrm{Hom}(\_, C) : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$. In the case of graded categories, we have more structure. Specifically, if $\mathcal{C}$ is a graded category over $\mathcal{M}$ with lifts, then $\mathrm{Hom}(A, B)$ is an object of the category $\mathcal{M} \mathbin{/\!\!/} \mathbf{Set}$ (with objects given by $\mathrm{Hom}_m(A, B)$, and transition functions given by pre/post compositon by lifting morphisms), which by corollary 1.2.4 is again a graded category over $\mathcal{M}$ (with lifts). In the remainder of this section we will show that this category plays an analogous role to $\mathbf{GCat}^{\ell}_{\mathcal{M}}$ as $\mathbf{Set}$ plays to $\mathbf{Cat}$. In other words, by making use of this fact, we can prove a "graded Yoneda lemma".

Definition 1.2.5. We define the *graded Yoneda embedding* as a functor between indexed structures $よ : \mathbb{C}^{\mathrm{op}} \to \mathcal{M} \mathbin{/\!\!/} \mathbf{Set}$ by setting $(よB)(A)_i = \mathrm{Hom}_i(A, B)$ – i.e. the set of morphisms of grade i between A and B in $\mathbb{C}$ (with transition functions given by pre/post composition by lifting morphisms), and on morphisms by defining

$(\text{よ}C)(f : A \to_n B)$ to be the partial map induced by pre-composition by f – i.e. for all $i \in {\uparrow}n$, we define $(\text{よ}C)(f)_i$ by mapping $x \in \mathrm{Hom}_i(B, C)$ to $x \circ f \in \mathrm{Hom}_i(A, C)$.

Theorem 1.2.6 (Graded Yoneda Lemma)**.** Let $F : \mathcal{C}^{\mathrm{op}} \to_n \mathcal{M} \,/\!\!/\, \mathbf{Set}$ be an indexed pre-sheaf, were $\mathcal{M}$ is a connected BCPO, then for all $C \in \mathcal{C}$ there is a pseudo-natural isomorphism of variable sets (in other words, a natural transformation for which all of the component morphisms are pseudo-isomorphisms):

$$\mathrm{Nat}(\text{よ}C, F) \cong FC$$

Proof. First, a note about the interpretation of this result. FC is a variable set, so we must first explain in what way we are viewing $\mathrm{Nat}(\text{よ}C, F)$ as a variable set. This is done by first passing through the equivalence between (elementary) graded categories and indexed structures. In the context of indexed structures, natural transformations between functors of indexed structures are a family of natural transformations between (ordinary) functors – one natural transformation for each grade $i \geq n$ (where n here is the grade of the natural transformation in question). Thus, we can view $\mathrm{Nat}(\text{よ}C, F)$ as a variable set by setting $\mathrm{Nat}(\text{よ}C, F)_i$ to be empty if $i \notin {\uparrow}n$, and otherwise set $\mathrm{Nat}(\text{よ}C, F)_i$ to be the set of all natural transformations between $\text{よ}C_i$ and F_i arising from the definition of a natural transformation between the indexed functors corresponding to $\text{よ}C$ and F. Since a natural transformation between two indexed functors consists of a family of regular natural transformations, the transition functions here are defined by mapping a natural transformation $\eta^i \in \mathrm{Nat}(\text{よ}C_i, F_i)$ to the "corresponding one" $\eta^j \in \mathrm{Nat}(\text{よ}C_j, F_j)$.

Now, note that it is easy to check that in the category $\mathcal{M} \,/\!\!/\, \mathbf{Set}$, a morphism is a pseudoisomorphism if and only if each of the component functions is an isomorphism. Thus, we can make use of the fact that by the usual Yoneda lemma, there are isomorphisms of sets $\mathrm{Nat}(\text{よ}C_i, F_i) \cong \text{よ}C_i$, natural in C for all $i \in {\uparrow}n$ (since F is a morphism of grade n, F_i is not defined otherwise). Thus, it remains to show that the morphisms we get of variable sets $\phi : \mathrm{Nat}(よC, F) \to FC$ and $\phi^{-1} : FC \to \mathrm{Nat}(よC, F)$ by setting their component functions to the natural isomorphisms we obtain via Yoneda are indeed (partial) morphisms of variable sets. In other words, we need to verify that the following diagram

$$\begin{tikzcd}
\mathrm{Nat}(よC_i, F_i) \arrow[r] \arrow[d, "\phi^i_C"'] & \mathrm{Nat}(よC_j, F_j) \arrow[d, "\phi^j_C"] \\
FC_i \arrow[r] & FC_j
\end{tikzcd}$$

commutes for all $i \leq j$ (as well as the corresponding diagram for ϕ^{-1}). To see this in both cases is an easy verification based on the properties of lifting morphisms once one looks at the concrete definitions of ϕ^i and $(\phi^i)^{-1}$ in the standard proof of the Yoneda lemma (see [10] for details). □

Corollary 1.2.7. The indexed Yoneda embedding is (pseudo) full and faithful.

Proof. For all objects A and B, and all grades $n \in \mathcal{M}$, by the general graded Yoneda lemma we have a pseudo-natural isomorphism of sets:

$$\mathrm{Hom}_n(A, B) = (よ_n B)(A) \cong \mathrm{Nat}_n(よA, よB)$$

□

1.3 Graded Categorical Structures

Now that we have elaborated on some of the basic definitions of graded categories, we will briefly consider some of the theory of graded categorical constructions. Intuitively speaking, these should be analogs of the usual notions of standard category theory that are "up to grading", rather than "strict". Moreover, as hinted at earlier, many of the usual categorical notions do not make much sense in the context of graded category theory without some alterations. For instance, the following example gives credence to the fact that that the standard definition of a categorical product is too strict for our purposes:

Example 7. Given a category $\mathcal{C}$ with products, consider the category $\mathcal{C}_{\mathbb{N}}$ we defined in example 3. Suppose that $A, B \in \mathrm{Ob}(\mathbb{C})$, and that:

$$A \xleftarrow[p]{} A \times B \xrightarrow[q]{} B$$

is a product in $\mathcal{C}$. Then $(A \times B, (0, p), (0, q))$ is not a product in $\mathcal{C}_{\mathbb{N}}$. To see this, take any pair of morphisms $f : X \to A$, $g : X \to B$ in $\mathcal{C}$, and consider the universal property for the pair of morphisms $(f, 0)$ and $(g, 1)$ in $\mathcal{C}_{\mathbb{N}}$ – then by grade considerations alone, there is no morphism $\langle (f, 0), (g, 1) \rangle : X \to A \times B$ making the diagram for the product commute.

Unfortounately, although there are some simple fixes to this problem that address examples like the one above, it turns out that finding the "correct" notion of limit/colimit in general graded categories with lifts turns out to be a very subtle issue. In cases where there is not some condition involving the composition of morphisms in the category (i.e. in the case of initial and terminal objects), there is a obvious solution to this problem: Namely, we say that $1 \in \mathcal{C}$ is a graded terminal object if *for every grade* n there is a unique morphism $!^n_X : 1 \to_n X$. We call such a universal construction "universal" or "fiberwise" if it is defined in this way (i.e. uniqueness holds when restricted to morphisms of a particular grade). However, things become more subtle when we look at more involved definitions, which actually involve the properties of composition in a graded category. Just to illustrate some of the proliferation of the different notions of limits/colimits one might consider in a graded category, some other natural conditions one might require of graded universal constructions are:

1. **Compatibility**: All morphisms satisfying some universal property lift to some unique morphism of higher grade satisfying the same universal property. (Note that this property may or may not require fiber-wise uniqueness).

2. **Lax Uniqueness**: There exists a unique morphism which *lifts* to all the morphisms satisfying some universal property.

In addition to the above properties, we have the issue of which set of morphisms our universal construction applies to, and whether for that class of morphisms we require commutativity either *on the nose* or merely *up to lifts*. Ignoring the conditions of compatibility and lax uniqueness (as well as any other potential types of "up to grading" uniqueness that one might think of), one natural approach to studying graded limits/colimits is to treat a diagram as a functor $F : \mathcal{D} \to_n \mathcal{C}$, and to treat "cones over a diagram with apex $X \in \mathcal{C}$" to be natural transformations from one of the "constant" X-valued diagrams $\Delta^a_X : \mathcal{D} \to_a \mathcal{C}$ of grade a to F.

In this "graded comma category", the limiting cone $\eta : \Delta_X \to_m F$ satisfies an appropriate universal property. For instance, if we want to consider "on the nose" commutativity, then we could require for all other cones $\zeta : \Delta_Y \to_k F$, there exists a unique natural transformation ("unique" here being unique up to grading in whatever sense we like) such that the following diagram commutes:

$$
\begin{tikzcd}
\Delta_X \arrow[r, "\eta"] & F \\
\Delta_Y \arrow[u, dotted] \arrow[ur, "\zeta"'] &
\end{tikzcd}
$$

In the case of "vanilla" categories, this is straightforward, and the limit and colimit of a diagram can be defined as the terminal object in the comma category (or, for colimits, the initial object in the cocomma category) above. This is difficult for us in our current framework, as if $\mathcal{C}$ is a (full-spectrum) graded category with lifts, then the "graded comma categories" we can consider are not generally full-spectrum categories, and thus, since at the moment we do not we do not have the usual tools (for instance, the graded Yoneda lemma) needed to be able to work with and prove theorems in the context of graded categories with objects of a non-full spectrum, we will not say any more about this approach here – except to note that this all likely merits further study in future work.

Instead, to avoid these complications, we will discuss some more concrete definitions of graded limits/colimits. But first, we should mention a few more issues that can arise when thinking about limits/colimits of diagrams in a graded category in this more concrete way – and how these issues are avoided by the definitions we considered discussed above. In particular, what we need to be wary of is that our diagrams are *uniform* – i.e. that all morphisms in the diagram are of the same grade. If this is not the case, then non-trivial things can happen with the spectra of the objects involved. For instance, if we take the limit over a diagram consisting soley of the morphisms 1_A and lift^n_A, then this should correspond to the subset of A consisting of elements of at least grade n – an object with non-full grade.

For the remainder of this thesis, we will primarialy be considering the fiber-wise notions of categorical constructions in a graded category – since even though they are not quite as general as some of the other notions that one might consider, they will still suffice for our purposes. However, while one can of course also consider the notion of a simple fiber-wise exponential object in the natural way in a graded category, we will also need to consider the following *restricted* notion in the next chapter, so we define it here:

Definition 1.3.1. Given a graded category $\mathcal{C}$ with lifts over $\mathcal{M}$ with fiber-wise products, we say that $(C^B)_n$ is a *restricted exponential object* of grade n if there exists a morphism $\epsilon : (C^B)_n \times B \to_n C$ such that for all morphisms $f : A \times B \to_m C$ of grade $m \leq n$, there exists a unique morphism $\hat{f} : A \to_m (C^B)_n$ such that the following diagram commutes:

$$\begin{tikzcd}
(C^B)_n \times B \arrow[r, "\epsilon"] & C \\
A \times B \arrow[u, "\hat{f} \times \mathrm{id}_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

and such that all morphisms $g : X \to_k (C^B)_n$ of grade $k \geq n$ factor as $g = f \circ \mathrm{lift}_X^k$ for some $f : X \to_m (C^B)_n$ of grade $m \leq n$.

This allows us to define a weaker notion of Cartesian closed category in the context of graded categories:

Definition 1.3.2. Given a graded category $\mathcal{C}$ with lifts over a BCPO $\mathcal{M}$, we say that $\mathcal{C}$ is a weakly Cartesian closed graded category if it has:

- A fiber-wise terminal object.
- Fiber-wise cartesian products.
- Restricted exponential objects for all grades.

Chapter 2

Applications of Graded Categories

2.1 Absolute and Relative Cardinality

> "Perhaps we could be pushed in the end to say that all sets are countable...but really pleasant axioms have not been produced by me or anyone else, and the suggestion remains speculation."
>
> – Dana Scott [11]

One of my original motivations for the theory of graded categories was to come up with a general framework which would allow one to explore the notion of *absolute* cardinality from a categorical perspective. In particular, the end goal being the construction of a model with some nice categorical properties such that all objects are "absolutely countable".

This language of "absolute" and "relative" cardinality has its roots in the so called *Skolem's paradox*, which is essentially the observation that there are *countable models of set theory* (i.e. models in which all of the sets have countably many elements), which, since they are still models of ZFC, and Cantor's theorem is a theorem of ZFC, must have sets which "internally" are uncountable, even though "externally" we see that each of the sets in our model in fact only has at most countably many elements.

This becomes less mystifying once one realizes that, in some formal system (such as ZFC) what "the infinite set A is uncountable" really means is "there is no bijection from A onto the natural numbers". To be even more explicit we could say "there is no bijection *in the model under consideration* from A onto the natural numbers." This is why it is perfectly valid for us to see that "external to the model", all sets are in fact countable. In our meta-theory we can evidently construct a bijection between the elements of A and the elements of the natural numbers, since both sets are countable – but this does not contradict Cantor's theorem (applied to this model of set theory), because this is a statement about what bijections we can construct *inside the model.*

Thus, since graded categories naturally encode the "level" of different morphisms, they give us a natural framework for understanding Skolem's paradox – namely, we can construct a category with:

- **Objects**: Sets in our model of ZFC.
- **Morphisms of grade 0**: Functions that we can witness "internal" to the model.
- **Morphisms of grade 1**: Functions that we can witness "external" to the model.

Now, in this setup, although a set such as $\mathbb{N}^{\mathbb{N}}$ (the set of all morphisms within the model from $\mathbb{N} \to \mathbb{N}$) is still not *isomorphic* to $\mathbb{N}$ – it *can* be *pseudoisomorphic* to $\mathbb{N}$, as nothing in Cantor's theorem prohibits such a bijection from existing "outside" the model (i.e. in our case, at grade 1).

So certainly we can express the notion of "absolute countability" in our framework of graded categories, as well as even the notion of two objects in a graded category having the same "relative cardinality" (i.e. up to some particular grade) – but notice that in order for this to work, we actually need to distinguish between the sets $(\mathbb{N}^{\mathbb{N}})_0$ (which consists of functions of grade 0 from $\mathbb{N}$ to $\mathbb{N}$) and $(\mathbb{N}^{\mathbb{N}})_1$ (which consists of functions of grade 1 from $\mathbb{N}$ to $\mathbb{N}$) – which then motivates our use of the notion of a *restricted exponential object* that we defined in the previous chapter. Thus, it turns out that formalizing the details of such a model in a satisfactory way brings with it a number of technical challenges which we will discuss further in the conclusion of this thesis.

2.1.1 A word on Lawvere's fixed point theorem and inconsistencies

Since this is a thesis in category theory, and not in set theory – and since we are here proposing a generalizaton of the notion of Cartesian closed categories to the case of graded categories in order to allow for a context where we can meaningfully and non-trivially model a theory where "all sets are countable", it is worth pointing out:

1. How Cantor's theorem (a result about the Cartesian closed category **Set**) relates to the more general framework of graded categories.

2. Why we in fact need to go beyond the theory of Cartesian closed categories in order to non-trivially model this notion of absolute countability – i.e. why can't we simply have a Cartesian closed category in which the isomorphisms themselves express the notion of *absolute countability*? Why do we need pseudoisomorphisms? (beyond the nice interpretation in terms of "what level" morphisms sit at in some heierarchy of metatheories)

The first question can be answered by a (relatively) little-known result by Lawvere [12, 13], which generalizes a number of known *diagonalization results* by lifting them to the more general context of Cartesian closed categories:

Theorem 2.1.1 (Lawvere's Fixed Point Theorem)**.** If $\mathcal{C}$ is a Cartesian closed category, $A, B \in \mathrm{Ob}(\mathcal{C})$, and there exists a point surjective morphism $\phi : B \to A^B$ (i.e. the induced map $\mathrm{Hom}(1, B) \to \mathrm{Hom}(1, A^B)$ mapping x to $\phi \circ x$ is surjective), then A has the fixed point property – that is, every morphism $f : A \to A$ has a fixed point.

which might seem more familiar to the reader in its contrapositive form:

Corollary 2.1.2. If $\mathcal{C}$ is a Cartesian closed category, $A, B \in \mathrm{Ob}(\mathcal{C})$, and there exists a morphism $f : A \to A$ without a fixed point, then there does not exist a weakly point surjective morphism $\phi : B \to A^B$.

The traditional interpretation of this theorem with application to self-referential paradoxes can be understood once one realizes that the property that every morphism $f : A \to A$ has a fixed point is a *trivializing* condition on A, and that conversely the existence of a morphism $f : A \to A$ without a fixed point is a kind of non-degeneracy condition on A. For instance, in **Set** any singleton set $\{*\}$ satisfies the fixed point property, whereas for any set containing at least two distinct elements, we can clearly construct a map without fixedpoints.

Even so, a natural question that presents itself given Lawvere's theorem is: Is there a way of modeling the sort of countable universe that we're after simply by *allowing* such fixed points to occur – even if they are not natural from the perspective of our set theoretic intuitions? For instance, if we take $A = B = \mathbb{N}$, is it really harmful to have every morphism $f : \mathbb{N} \to \mathbb{N}$ have a fixed point? Wouldn't it be OK if our model had some *non-standard* natural numbers which are fixed by every morphism $\mathbb{N} \to \mathbb{N}$?

The short answer is: Yes, it is possible (in categories other than **Set**) for an object to satisfy the fixed point property, and yet still have some non-trivial structure. For instance, this can be seen in categorical models of the *untyped lambda calculus* as a *reflexive object* in a Cartesian closed category. Such a model consists of an object U of a Cartesian closed category which is interpreted as an (infinite) collection of untyped lambda terms – and since we can apply any lambda term to any other lambda term, we need to have an isomorphism (or at least a retraction) $U \cong U^U$. In this sort of model, fixedpoints correspond to *non-terminating* lambda terms, which is fine, since we're trying to model the untyped lambda calculus, in which not all computations terminate.

However, the moment you try to incorporate additional structure into your category, such fixedpoints pose a number of problems. The following results are from [14], where a category is said to be *inconsistent* if it is equivalent to the trivial category, and is said to have *fixedpoints* if every object A has the fixedpoint property from Lawvere's theorem:

Proposition 2.1.3. Let $\mathcal{C}$ be a Cartesian closed category with fixedpoints, then:

1. If $\mathcal{C}$ has an initial object then $\mathcal{C}$ is inconsistent.

2. If the coproduct $1 + 1$ exists, then $\mathcal{C}$ is inconsistent.

3. If $\mathcal{C}$ has equalizers, then $\mathcal{C}$ is inconsistent.

4. If $\mathcal{C}$ has a natural numbers object N, then $N \cong 1$.

In combination with Lawvere's theorem, these sorts of results prove to be detrimental towards our goal of modeling a category in which "everything is countable".

Beyond inconsistency, another more subtle type of degeneracy that can arise because of fixedpoints (as well as for other reasons) is the condition of $\mathcal{C}$ actually being a preorder category (i.e. each hom set $\mathcal{C}(A, B)$ contains at most one element). The fixedpoints arising from Lawvere's theorem can also lead to this kind of degeneracy if we're not careful:

Proposition 2.1.4. Let $\mathcal{C}$ be a bicartesian closed category with an object D such that 2^D is a retract of D, then $\mathcal{C}$ is a preorder category with $D \cong 1$.

Thus, to summarize, if we wish to model a **Set**-like category (and thus, to have a Cartesian closed structure of some sort, a natural number object, as well as cocartesian structure, at the very least) where "all sets are countable" – in particular, that there is a point-surjective morphism $\phi : \mathbb{N} \to 2^{\mathbb{N}}$, we cannot use the standard framework of category theory.

2.2 Conclusions and Further Research

When I started work on this thesis, one of my central results was:

Proposition 2.2.1. There exists a non-trivial (weakly) Cartesian-closed graded category $\mathcal{C}$ such that every infinite object in $\mathcal{C}$ has the same absolute cardinality as $\mathbb{N}$.

The construction witnessing this result was simple:

1. Start with a Cartesian closed "base category" $\mathcal{C}_0$ with natural numbers object whose morphisms correspond to (possibly higher-order) primitive recursive functions.

2. At each stage $\mathcal{C}_n$, freely adjoin to the set of morphisms of $\mathcal{C}_n$ an enumeration map $\mathrm{enum}_n : \mathbb{N} \to_{n+1} (\mathbb{N}^{\mathbb{N}})_n$

3. Close this under products and (weak) exponential objects to form a new category $\mathcal{C}_{n+1}$, into which $\mathcal{C}_n$ naturally embeds.

By construction then (ignoring the fact that "closing under products and weak exponential objects" is already a somewhat involved construction if one wishes to flesh out the details), this graded category certainly satisfies the desired "all infinite objects are absolutely countable" property. However, one issue with this construction is that it is actually a construction of an indexed category – and the transition functors of this indexed category are merely *injective on objects*, not *identity on objects*, so this is not, in the sense of this thesis, an *indexed strucure* for a graded category.

While this is perhaps salvagable, at least in the sense that if we allow graded categories that have objects with non-full spectra, the correspondence outlined in section 1.1.4 likely generalizes to the case of injective on object indexed categories, I am much less confident that the graded Yoneda lemma carries over in this case – and hence, this still does not show that we have a relatively "nice categorical approach" for talking about absolute cardinality, which was one of the hopes for this thesis.

In addition, the notion of graded exponential object I used in the above construction was slightly different (and not as well-thought out) as the notion I ended up settling for in this thesis. However, both the notion of fiber-wise product, and "restricted" exponential object I ended up using here are still not entirely satisfying from a theoretical perspective, and much work remains to be done in order to better understand the nature of the more general kinds of "graded universal constructions" discussed in section 1.3, and how they relate to notions definable in the 2-categories $\mathbf{GCat}^{\ell}_{\mathcal{M}}$ and $\mathbf{IndCat}_{\mathcal{M}}$ (either via standard conical, or by weighted limits). Thus, rather than trying to cobble together some hot fix of my original construction that would fit with the new definitions that I decided to use for my thesis, I'll instead merely be honest, and say: "There is much work that remains to be done."

And so, my original aspirations for this thesis with respect to some of my original motivations for "graded categories" has, I think to some extent fallen flat. I took Dana Scott's speculation that "perhaps we could be pushed in the end to say that all sets are countable," and I ran with it.

But as it turns out, it's quite hard to get around general incompleteness phenomena (which, arguably Cantor's theorem is – being akin to both Gödel's incompleteness theorem and the undecidability of the halting problem via their generalization in Lawvere's fixed point theorem) in a satisfactory way. Eventually, I came to the realization that the use of restricted exponential objects to come up with a model where "everything is absolutely countable" is interesting, although perhaps "cheating" in a way.

What can be said about the case when more honest-to-goodness “full” and “up to grading” notions of exponential object are used rather than the restricted exponential objects? Is it still possible for everything to be “absolutely countable” in this context?

With regards to this question, what I’d conjecture is this: *Even in the prescense of more “full” graded exponential objects, we can consistently say that “all sets are absolutely countable” – but this should not be formalizable within the graded category itself!* So in other words, what I’d expect is that what we *can* formulate is that with regards to *full exponential objects in a graded category*, we get a sort of *undefinability theorem* with regard to the notion of “absoluteness”. Then, the reason why it is possible to detect all infinite objects having the same absolute cardinality in the case of only allowing restricted exponential objects is simply that (over nice enough posets) we can always move (within the system) to a higher grade where we can witness the absolute countability at that stage.

Zach Weber ends his paper on paraconsistent naïve set theory [15] (in which he, interestingly enough, also cites the remark by Dana Scott which I opened this chapter with) by saying:

> The few pages of this paper have not destroyed all cardinals. Rather, the proofs are robust evidence that Cantor’s ideas about the transfinite are true, and that this is so independently of classical logic. His ideas are true in a different light than previously seen

which, I think essentially sums up my approach to tackling this problem of coming up with a coherent framework in which one can do mathematics where “everything is absolutely countable.”

I am not ultimately trying to run from the incompleteness that Cantor derived in his classic theorem – I am merely attempting to understand the same fundamental phenomenon in a different way – in a way that takes seriously the fact that mathematics is constructed in stages, and the fact that results (such as countability!) *change* according to the various perspectives one might take. This is what I hoped (and still hope!) to model with the framework of graded categories.

However, on the subject of my aspirations for the theory of graded categories (with regards to their application in understanding this notion of *absolute contability*, I'd like to bring up one more quote from the paper by Weber, and then to leave with some intuition that I think best sums up my current aspirations for graded categories in this area – and shows how perhaps what I am doing might perhaps even be related to Weber's approach:

> A hierarchy, be it of iterative sets, truth predicates, or transfinite cardinals, can be understood as a way to control otherwise inconsistent information, by separating it over distinct levels... Once we forgo consistency, the hierarchy of ascending cardinalities is merely tracking a contradiction.

Consider: If we want to model some "uncountable" collection like $\mathbb{N}^{\mathbb{N}}$ in a graded category – then perhaps a better intuition is not that "everything is countable", but rather that "everything is *fiberwise countable*". With this in mind, and viewing $\mathbb{N}^{\mathbb{N}}$ in terms of our representation of the objects of graded categories as variable sets – $\mathbb{N}^{\mathbb{N}}$ should look (without loss of generality) something like

$$\mathbb{N}_0 \xrightarrow{\quad f_1 \quad} \mathbb{N}_1 \xrightarrow{\quad f_2 \quad} \mathbb{N}_2 \xrightarrow{\quad f_3 \quad} \dots$$

where each of the transition functions $f_1, f_2, f_3 \dots$ can be interpreted as "keeping track of a contradiction". (Here $\mathbb{N}_i$ are all countable sets) In other words, if at stage 0 we assume (wrongly – so, a contradiction) that $\mathbb{N}^{\mathbb{N}}$ can be exhausted at this level by a countable set, we have to embed $\mathbb{N}_0$ into the "larger" set $\mathbb{N}_1$ (for instance, by the map $n \mapsto 2n$ – so $\mathbb{N}_0$ gets mapped into a proper part of $\mathbb{N}_1$) to make room for all the "new elements" that now are required to inhabit this set by the diagonal construction.

However, with all that said, I still think it is interesting to consider the question of what "doing mathematics" using a framework where only restricted exponentiation in a graded category is allowed "looks like". I.e. what do results in fields like topology, group theory, analysis, etc... which depend on the fact that some infinite collections are larger than others look like when we replace exponentiation and powersets with their restricted analogues, thus forcing everything to be countable?

This is not quite as outlandish as it may at first seems, as some foundations of mathematics (such as predicativism) do not allow *unrestricted powersets* (for instance, see [16]). And in particular, some recent work by Palmgren [17] towards developing a theory of "predacative toposes" seems very akin to my approach of only allowing restricted exponentiation.

There are also many interesting ideas and examples that, due to time limitations, I was not able to include in the final draft of my thesis. In particular, it is rather unfortunate that the "dynamic categories" aspect of my thesis had to be mainly relegated to some motavational remarks towards the beginning of the thesis. This is a shame, because although this material is less developed than some of my other ideas – I do believe that they could have a lot of potential, for instance: In artificial intelegence, or linguistics, where one has to deal with an evolving system of ontologies, if one could find a way to conviniently represent such an evolving system as a graded category in a nice way, then perhaps some of the weaker notions of universal constructions "up to grading" that one can formulate in graded categories might have some explanatory power that would not otherwise be possible in "static" category theory.

It is interesting to note that Andree Eheresmann (a remark by whom inspired at least the terminology of the "dynamic categories" aspect of this thesis!) uses some graded categorical structures in her work on applying category theory to neuroscience – although these are $(\mathbb{N}, +)$-gradings, not the "idempotent-type" gradings that I considered in this thesis.

The relationship between these different types of gradings, however, is just one more potential avenue to be explored. I discovered some preliminary results in this direction while I was working with Harley Eades III on the semantics of graded modal type theories (which uses a grading over semirings – the interesting result being that one of the type theories I was considering for the internal language of my graded categories turned out to essentially be a special case of the "resource-semiring graded" type theories we considered, when join-semilattices are viewed as degenerate semirings). However, it would be interesting to explore more in the future exactly what the relationships are between these types of gradings. In general, it seems to me that these more general (non-idempotent) gradings should be more difficult to deal with, because they do not allow one to have lifting morphisms in my sense, as is the case for the idempotent gradings considered in this thesis. How to deal with this, and whether or not any interesting graded categorical constructions can be formulated in this more general setting (and whether or not they would be useful) is, I think, an interesting open question to consider. However, it should be noted that Paul Blain Levy has shared with me in private communication some of his unpublished work in what he calls "locally graded categories" – which is a generalization of my work in the sense that he works with gradings over general *monoidal categories*.

The original draft of this thesis (before I started cutting things down in order to get something that was actually feasiable to edit with my time constraints) was 266 pages at it's max. Thus, even including this summary, there are still many more topics I have left out. However, I hope that both with this, and with the rest of the thesis, I have convinced the reader that graded categories are an interesting new framework with a number of interesting applications. Or, at the very least, I hope that I have shown that this is a framework that merits further, and more systematic study.

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Biography

Nathan Bedell was born and raised in Winchester, Virginia. After a long childhood of exploring a variety of different academic interests such as music theory, linguistics, and computer science – just to name a few – he eventually fell in love with, and decided to major in Mathematics. Nathan graduated with his B.S. from Liberty University in December of 2016, and is graduating this May with his M.S. in Mathematics.