Counting Borel Orbits In Classical Symmetric Varieties

AN ABSTRACT
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FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
BY

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Abstract

Let G be a reductive group, B be a Borel subgroup, and let K be a symmetric subgroup of G. The study of B orbits in a symmetric variety G/K or, equivalently, the study of K orbits in a flag variety G/B has importance in the study of Harish-Chandra modules; it comes with many interesting Schubert calculus problems. Although this subject is very well studied, it still has many open problems from combinatorial point of view. The most basic question that we want to be able to answer is that how many B orbits there are in G/K. In this thesis, we study the enumeration problem of Borel orbits in the case of classical symmetric varieties. We give explicit formulas for the numbers of Borel orbits on symmetric varieties for each case and determine the generating functions of these numbers. We also explore relations to lattice path enumeration for some cases. In type A, we realize that Borel orbits are parameterized by the lattice paths in a $p \times q$ grid moving by only horizontal, vertical and diagonal steps weighted by an appropriate statistic. We provide extended results for type C as well. We also present various t-analogues of the rank generating function for the inclusion poset of Borel orbit closures in type A.

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Chapter 1

Introduction

Let G be a complex reductive algebraic group and let K be a closed subgroup of G. By a reductive group we mean a linear algebraic group with the trivial unipotent radical. Let B be a Borel subgroup, that is a maximal closed connected solvable subgroup of G. The subgroup consisting of lower triangular matrices in the general linear group $GL(n, \mathbb{C})$ is an example of a Borel subgroup.

A normal G-variety X is called spherical, if B has a dense and open orbit in X. The subgroup K is called a spherical subgroup if G/K is a spherical G-variety. In [1], Brion showed that a normal G-variety X is spherical if and only if there are only finitely many B orbits in X. Spherical varieties appear in several areas of mathematics including algebraic geometry, representation theory, and invariant theory. Torus embeddings are among the best known and studied examples.

A subgroup K is called a symmetric subgroup if there exists an involutory automorphism on G such that $K = G^{\theta}$, the fixed subgroup. Springer [2] and Matsuki [3] showed that symmetric subgroups are spherical subgroups. In particular, G/K has only finitely many B orbits. The combinatorics and geometry of symmetric varieties are well studied for example see [4].

The classification problem of the real forms of simple complex Lie algebras over an algebraically closed fields is solved by E. Cartan. Subsequently, by the works of Araki [5] and Helminck [6], Cartan's theory of involutions is extended over arbitrary fields (see [2], also). In this thesis, by a classical symmetric variety we mean one of the following homogeneous varieties;

Type	Symmetric variety
AI	$SL(2n,\mathbb{C})/SO(n,\mathbb{C})$
AII	$SL(n,\mathbb{C})/Sp(2n,\mathbb{C})$
AIII	$SL(n,\mathbb{C})/S(GL(p,\mathbb{C})\times GL(q,\mathbb{C}))$
BI	$SO(2n+1,\mathbb{C})/S(O(2p,\mathbb{C})\times O(2q+1,\mathbb{C}))$
CI	$Sp(2n,\mathbb{C})/GL(n,\mathbb{C})$
CII	$Sp(2n,\mathbb{C})/(Sp(2p,\mathbb{C})\times Sp(2q,\mathbb{C}))$
DI(i)	$SO(2n,\mathbb{C})/S(O(2p,\mathbb{C})\times O(2q,\mathbb{C}))$
DI(ii)	$SO(2n,\mathbb{C})/S(O(2p+1,\mathbb{C})\times O(2q-1,\mathbb{C}))$
DIII	$SO(2n,\mathbb{C})/GL(n,\mathbb{C})$

Table 1.1: Classical Symmetric Varieties

(Note that in Table 1.1 whenever n, p, q appear in the same row we assume that $n = p + q$. We follow the convention of Chapter 11 in [7] and [8].)

In this thesis, we are concerned with the combinatorics of the number of Borel orbits in the case of classical symmetric varieties given in Table 1.1. We note here that our work is based on the fact that B orbits in G/K are in one-to-one correspondence with the K orbits in G/B. Due to the importance of K orbits of a flag variety in representation theory, it is natural to analyze the parametrization of double cosets $K\backslash G/B$. This can be done by Bruhat decomposition for the case where $K = B$. Although there is no explicit parametrization of the Borel orbits for the general spherical varieties, for G and K from Table 1.1 it is fascinating that K orbits in a flag variety G/B can be parametrized naturally by combinatorial objects, called clans. This combinatorial notion first appeared in [3] and then used in [9] for special cases.

The first three varieties given in Table 1.1 are of type A. Let S_n denote the symmetric group of permutations on $1, 2, \ldots, n$. The K orbits in type AI are parametrized by the set of all involutions in S_n. The K orbits in type AII are parametrized the set of all fixed point free involutions in S_n. The K orbits in type AIII are parametrized the set of all signed (p, q) involutions for S_n. The numbers of involutions and fixed point free involutions are well-investigated and many combinatorial properties of them found their ways into textbooks. See, for example, Bona's [10]. Although the set of all signed (p, q) involutions is well-studied as well, see [3] and [11], there is no explicit formula for them. In particular, a complete characterization of the maximal chains of an arbitrary lower order ideal in any of these sets are given in [12].

In this thesis, we present various formulas for the numbers of Borel orbits in all the classical symmetric varieties in Table 1.1, except for type DIII. This case is still in progress and we will leave this for a future project.

This thesis is organized as follows: After introducing the notation and background material that we use in the sequel in Chapter 1, we analyze the number of Borel orbits in type AIII by counting the signed (p, q) involutions in Chapter 2. In Section 2.4, we prove that Borel orbits in type AIII are parametrized by weighted Delannoy lattice paths in $p \times q$ grid. Note that as a consequence of our construction there is a way to reinterpret and study the action of the "Richardson-Springer monoid of the symmetric group" in terms of weighted lattice paths. This connection is particularly exciting, because intersection theory on Borel orbit closures in type A is determined by the action of Richardson-Springer monoid, see [13] and [14].

Each of these remaining chapters are constructed in a similar manner and their organization is as follows: For each symmetric variety G/K, we analyze the number of Borel orbits. We then determine the corresponding generating series and provide detailed examples in each case. In addition, for some cases we explore relations to lattice path enumeration by applying our combinatorial program which is introduced in Chapter 2.

In order to not break the flow of our present exposition we postpone the messy calculations for some generating functions to the appendix.

1.1 Notation

In this section, we introduce our basic notation and background for involutions.

Let us begin by introducing some matrices which will be used throughout the paper. We denote the $n \times n$ identity matrix by id_n and $H_{n,m}$ denotes

$$H_{n,m} = \begin{pmatrix} 0 & id_n \\ -id_m & 0 \end{pmatrix}.$$

Moreover,

$$S_{n,m} = \begin{pmatrix} 0 & s_n \\ -s_m & 0 \end{pmatrix}, \qquad J_{p,q} := \begin{pmatrix} 0 & 0 & s_q \\ 0 & id_{p-q} & 0 \\ s_q & 0 & 0 \end{pmatrix},$$

where s_n is the $n \times n$ matrix with 1's on the anti-diagonal and 0's elsewhere and $0 < q < p$ are two positive integers such that $n = p + q$ for the latter. Also,

$$I_{p,q,r} = \begin{pmatrix} id_p & 0 & 0 \\ 0 & -id_q & 0 \\ 0 & 0 & id_r \end{pmatrix}, \qquad K_{p,q} = \begin{pmatrix} J_{p,q} & 0 \\ 0 & J_{p,q} \end{pmatrix}.$$

Our main reference for the classical symmetric groups will be [7] and for K orbits in flag varieties we refer to [13] and [8]. By "classical group" we mean here one of the following:

$$SL(n,\mathbb{C}) = \{g \in GL(n,\mathbb{C}) \mid \det(g) = 1\},$$
$$SO(n,\mathbb{C}) = \{g \in SL(n,\mathbb{C}) \mid gg^\top = id_n\},$$
$$Sp(2n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C}) \mid H_{n,n}(g^\top)^{-1}H_{n,n} = g\},$$

where $\top$ stands for the transpose of a matrix.

Now, let σ denote an involutory automorphism on G. Define $T \subset B$ to be a σ-stable maximal torus of a Borel subgroup B of G and S to be the maximal torus of K. Here, σ-stable means that $\sigma \cdot T = T$. The Weyl group W of G is the finite group $N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. In this setting, we have the following notation:

$$\Phi := \text{the set of the roots of } T,$$
$$\Phi^+ := \text{the set of positive roots associated with } \Delta,$$
$$\Phi^- = -\Phi^+ := \text{the set of negative roots associated with } \Delta,$$
$$\Delta(G,T) := \text{the set of simple roots determined by } (G,T).$$

Here, recall that a subset $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ is a set of simple roots if every $\gamma \in \Phi$ can be written uniquely as

$$\gamma = n_1\alpha_1 + \cdots + n_l\alpha_l, \text{ with } n_1, \ldots, n_l, \text{ integers all of the same sign.}$$

Also for $\alpha \in \Phi$, $s_\alpha \in W$ denotes the corresponding simple reflection. Throughout this paper, P_α denotes the minimal parabolic subgroup associated to α.

The notation $\mathbb{N}$ stands for the set of natural numbers, which includes 0. Let us treat $+$ and $-$ as two symbols rather than arithmetic operations. Throughout this paper, the notation $\mathbb{P}$ stands for the set $\{+,-\} \cup \mathbb{N}$ and its elements are called symbols.

In combinatorics, it is customary to denote by $[n]$ the set $\{1, \ldots, n\}$. We write the elements of the symmetric group S_n in cycle notation using parentheses, as well as in one-line notation using brackets. We omit brackets in one-line notation if there is no danger of confusion. For example, $w = 4213 = [4, 2, 1, 3] = (1, 4, 3)$ is the permutation that maps 1 to 4, 2 to 2, 3 to 1, and 4 to 3.

Recall that a permutation π is an involution, that is to say $\pi^2 = \mathrm{id}$, if and only if every cycle of π is of length at most 2. The set of involutions in S_n is denoted by I_n.

Let $\pi \in I_n$ be an involution. The standard way of writing π is as a product of transpositions (2-cycles). Since we often need the data of fixed points (1-cycles) of π, we are always going to include them in our notation. Thus, our standard form for π is

$$\pi = (a_1, b_1)(a_2, b_2) \cdots (a_k, b_k) d_1 \ldots d_{n-2k},$$

where $a_i < b_i$ for all $1 \leq i \leq k$, $a_1 < a_2 < \cdots < a_k$, and $d_1 < \cdots < d_{n-2k}$.

It is known that the Bruhat-Chevalley order on S_n is a ranked poset and its grading is given by $\ell : S_n \to \mathbb{Z}$, $\ell(\pi) =$ the number of inversions in π.

The Bruhat order on I_n is also a ranked poset; for $\pi = (a_1, b_1) \cdots (a_k, b_k) d_1 \ldots d_{n-2k}$ in I_n, the length $L(\pi)$ is defined by

$$L(\pi) := \frac{\ell(\pi) + k}{2},$$

where $\ell(\pi)$ is the length of π in S_n, and k is the number of transpositions that appear in the standard form of π. The Bruhat order on I_n has a minimal element $\alpha_n := \mathrm{id}$, and a maximal element $\beta_n := w_0$, where $w_0 = [n, n-1, \ldots, 2, 1]$, the longest permutation. We drop the subscript n when it is clear from context.

We will also be dealing with a flag - that is chains of subspaces of a given vector space V. $F_\bullet$ will denote a flag

$$\{0\} \subset F_1 \subset \cdots \subset F_n = V.$$

We also need the following notation;

$$F_\bullet = \langle v_1, v_2, \ldots, v_n \rangle$$

which means that F_i is the linear span $\mathbb{C} \cdot \langle v_1, \ldots, v_i \rangle$, when we need to be specific about the components of F_i for each $i = 1, \ldots, n$. Moreover, let $E_p = \mathbb{C} \cdot \langle e_1, \ldots, e_p \rangle$ be the span of the first p standard basis vectors and similarly let $E_q = \mathbb{C} \cdot \langle e_{p+1}, \ldots, e_n \rangle$ be the span of the last q standard basis vectors. Let $pr : \mathbb{C}^n \to E_p$ be the projection onto E_p.

Although all the definitions will be given in Section 1.3, here we introduce the notation for clan collections and the corresponding cardinalities as follows:

$$\begin{array}{cc}
A(p,q) := \{(p,q) \text{ clans}\} & A_{p,q} := |A(p,q)| \\
B(p,q) := \{\text{symmetric } (2p, 2q+1) \text{ clans}\} & B_{p,q} := |B(p,q)| \\
\Gamma(p,q) := \{\text{symp-symmetric } (2p, 2q) \text{ clans}\} & \Gamma_{p,q} := |\Gamma(p,q)| \\
\Delta(n) := \{\text{skew symmetric } (n,n) \text{ clans}\} & \Delta_n := |\Delta(n)| \\
\Theta(p,q) := \{\text{symmetric } (2p, 2q) \text{ clans}\} & \Theta_{p,q} := |\Theta(p,q)| \\
M(p,q) := \{\text{symmetric } (2p+1, 2q-1) \text{ clans}\} & M_{p,q} := |M(p,q)|.
\end{array}$$

Table 1.2: Sets of Clans and their Cardinalities

By using the bijection between clans and signed involutions, which we give its detailed proof in Chapter 2, we proceed to denote by $\alpha_{k,p,q}$, $\beta_{k,p,q}$, $\gamma_{k,p,q}$, $\delta_{k,n}$, $\theta_{k,p,q}$ and by $\mu_{k,p,q}$ the number of clans for each type whose corresponding involution has exactly k transpositions as a permutation, respectively. Thus, it is clear that we have

$$A_{p,q} = \sum_k \alpha_{k,p,q} \quad \text{and} \quad B_{p,q} = \sum_k \beta_{k,p,q},$$
$$\Gamma_{p,q} = \sum_k \gamma_{k,p,q} \quad \text{and} \quad \Delta_n = \sum_k \delta_{k,n},$$
$$\Theta_{p,q} = \sum_k \theta_{k,p,q} \quad \text{and} \quad M_{p,q} = \sum_k \mu_{k,p,q}.$$

We aim to present various formulas and determine generating functions (or series) and lattice path interpretations for $\alpha_{k,p,q}$, $\beta_{k,p,q}$, $\gamma_{k,p,q}$ $\delta_{k,n}$, $\theta_{k,p,q}$ and $\mu_{k,p,q}$ and more importantly, for $A_{p,q}$, $B_{p,q}$, $\Gamma_{p,q}$, Δ_n, $\Theta_{p,q}$ and $M_{p,q}$, by using the standard combinatorial techniques.

1.2 Orbits

In this section, we give some facts about K orbits in G/B. Let us start with the closed orbits. One of the many characterizations of the closed K orbits can be given by twisted involutions and the Richardson-Springer map, see [13], for more.

Recall that an element w of the Weyl group of G is called an twisted involution if $w = \sigma(w)^{-1}$ and we will denote the set of all twisted involutions by $\mathcal{I}$. Now, consider a morphism $\tau : G \to G$ which maps g to $g\sigma(g)^{-1}$ and define $\mathcal{V}$ to be the set of all $g \in G$ such that $\tau(g) \in N_G(T)$.

Define $\rho : G \to N_G(T)$ to be the canonical projection. One can verify that $\mathcal{V} \subset G$ is closed under the following action of $T \times K$ on G;

$$(t,k) \cdot g = tgk^{-1},$$

for t in T, k in K, and g in G. The orbit set $T\backslash\mathcal{V}/K$ is in bijective correspondence with $K\backslash G/B$.

Let us denote the set of $T \times K$ orbits in $\mathcal{V}$ by V and for each element $v = TgK$ of V denote the corresponding K orbit $K \cdot g^{-1}B$ by $\mathcal{O}_v$. Now, consider the map $\phi : T\backslash\mathcal{V}/K \to W$ defined by $\phi(g) = \rho(\tau(g))$. Observe that this map is constant on $T \times K$, so we have a map from V to the Weyl subgroup W that we still denote by ϕ. It can be verified that ϕ is a map actually from V to the set of involutions $\mathcal{I}$. Moreover, one can show that ϕ can also be considered as a map $K\backslash G/B \to \mathcal{I}$ defined by $\phi(\mathcal{O}_v) = \phi(v)$. This map is called Richardson-Springer map and it plays an important role in the characterization of K orbits in a flag variety. For further details, see [4] and [13]. Next, we can state one of those characterizations as follows:

Proposition 1.2.1 ([13], Proposition 1.4.3)**.** For $w \in W$, the K orbit $\mathcal{O} = K \cdot wB$ is closed if and only if $\phi(\mathcal{O}) = 1$.

There is a partial ordering on $K\backslash G/B$ given by inclusion of the orbit closures, details can be found in [13] and in the exposition [15]. In order to define the other K orbits in G/B, let α be a simple root from $\Delta(G, T)$ and $\mathcal{O}$ be from $K\backslash G/B$. Now, consider the natural projection

$$\Pi_\alpha : G/B \to G/P_\alpha.$$

In this case, one can show that $\Pi_\alpha^{-1}(\Pi_\alpha(\mathcal{O}))$ has a dense K orbit. Then,

- if $\dim(\Pi_\alpha(\mathcal{O})) < \dim(\mathcal{O})$, then the dense orbit on $\Pi_\alpha^{-1}(\Pi_\alpha(\mathcal{O}))$ is nothing but $\mathcal{O}$ itself and

- if $\dim(\Pi_\alpha(\mathcal{O})) < \dim(\mathcal{O})$, the dense K orbit will be another orbit which is one dimension higher than $\mathcal{O}$ and will be denoted by $\mathcal{O}' = s_\alpha \cdot \mathcal{O}$.

Lastly, we are ready to give the definition of the weak (closure) order as follows: The weak order on $K\backslash G/B$ is generated by relations of the form $\mathcal{O} < \mathcal{O}'$ if and only if $\mathcal{O}' = s_\alpha \cdot \mathcal{O}$, with $\dim(\mathcal{O}') = \dim(\mathcal{O}) + 1$. Its explicit combinatorial description will be given in the next section after we introduce our combinatorial tool, called clan. See [8], for more details.

1.3 Symbolic Parametrizations: Clans

In this section, we give a brief introduction to our main combinatorial object, called clan, which plays a crucial role in the parametrization of B orbits in classical symmetric varieties. Our main references for this combinatorial notion are [3], [9] and [16].

As we said before, $K = S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$ orbits in the flag variety $G = SL(n,\mathbb{C})/B$ are parametrized by the set of all signed (p,q) involutions, where B is a Borel subgroup consisting of all lower triangular matrices of G. In addition, we have the following;

Fact 1.1. $K = S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$ orbits in the flag variety $G = SL(n,\mathbb{C})/B$ are parametrized by (p,q) clans.

In this section, our aim is to analyze clans enough to be able to understand the one-to-one correspondence between them and signed involutions. Let us begin by giving the definition of clans.

Definition 1.3.1. A (p,q) clan γ is a string of $n = p+q$ symbols from $\mathbb{P}$ such that

1. there are $p-q$ more $+$'s than $-$'s (if $p \geq q$, if not there are $q-p$ more $-$ signs than $+$'s)

2. if a natural number appears once it must appear twice.

The concept of a clan is quite vague. Before we go further, in order to have a better understanding let us go through some examples.

Example 1.2. Consider the string $\gamma = 2\ 8\ 2\ 8$ with length 4. Observe that each natural numbers appear in γ appears exactly twice and also there is no $\pm$ symbols.

It follows that the number of $+$ signs equals to $-$'s, and thus γ is an example of a $(2,2)$ clan. We can say that $\gamma \in A(2,2)$, see Table 1.2 for the notation.

Now, consider $\gamma = 1\ 7\ +\ +\ -\ +\ 7\ 1$. It can be easily seen that this is an example of a $(5,3)$ clan. In fact, there are exactly 2 more $+$'s than $-$'s and each (distinct) natural number exactly appears twice in the string.

A string $\gamma = 1\ -\ -\ 2\ 1\ 2\ +$ can be considered as an another example of a clan. γ is a $(3,4)$ clan, and so it is an element of the set $A(3,4)$. Here, note that the number of $-$ signs is exactly one more that the number of $+$'s.

On the flip side, it is not hard to see that the string $\gamma = 1\ +\ +\ 2\ 2$ is not a clan because the natural number 1 appears only once.

To avoid confusion and to be consistent among complicated calculations, we make the following convention.

Convention 1.3. Throughout this paper, it is assumed that the number of $+$ symbols is greater than or equal to the number of $-$ symbols.

Example 1.4. Consider the following two clans; $\tilde{\gamma} = 1\ 2\ 3\ 1\ 2\ 3$ and $\gamma = 7\ 8\ 6\ 7\ 8\ 6$. It is crucial to observe here that these are actually the same clans. Indeed, each natural number appearing in the clan appears twice at corresponding spots. We consider such strings only up to equivalence. To be precise, we say two clans are equivalent if and only if their pairs of matching numbers occur at the same position.

Thus, it is not the natural numbers themselves which determine the clans; it is the positions of matching natural numbers.

Example 1.5. In the case where $p = 2, q = 1$, all the possible different clans are

$$+ + -,\ + - +,\ - + +,\ 1\ 1\ +,\ 1 + 1,\ + 1\ 1,$$

up to equivalence. Thus, the cardinality of $A(2,1)$ is $A_{2,1} = 6$.

Now, we are ready to state the following theorem which has an important play in our computations.

Theorem 1.3.2 ([9], Theorem 2.2.8)**.** *For a given clan $\gamma = c_1, \ldots, c_n$, define $\mathcal{O}_\gamma$ to be the set of all flags $F_\bullet$ having the following three properties for all i, j $(i < j)$:*

1. *dim$(F_i \cup E_p)$ = the total number of $+$ signs and pairs of equal natural numbers occurring among $c_1, \ldots, c_i$;*

2. *dim$(F_i \cup E_q)$ = the total number of $-$ signs and pairs of equal natural numbers occurring among $c_1, \ldots, c_i$;*

3. *dim$(pr(F_i) \cup F_j) = j + r$, where r is the number of pairs of equal natural numbers $c_s = c_t \in \mathbb{N}$ with $s \leq i < j < t$.*

For each clan, $\mathcal{O}_\gamma$ is nonempty and stable under K. In fact, $\mathcal{O}_\gamma$ is a single K orbit on G/B. Conversely, every K orbit on G/B is of the form $\mathcal{O}_\gamma$ for some (p, q) clan γ.

Now, let us outline the algorithm to understand the relation between clans and orbits. This algorithm was proposed by Yamamoto, see [9] for further details and also [8]. Let $\gamma = c_1 \ldots c_n$ be a clan.

Algorithm 1.3.3.

1. First, assign opposite signs to each matching pair. The sign is called the *signature* of the symbols $c_i \in \mathbb{P}$. For instance, for a given clan $\gamma = 1\ 1\ +\ -$, the corresponding signed clan is either $1_+\ 1_-\ +\ -$ or $1_-\ 1_+\ +\ -$. To be consistent, we always assign $+$ sign to the first of the matching numbers.

2. For such a signed clan, choose a permutation π as follows:

   - $1 \leq \pi(i) \leq p$ if the signature of $c_i = +$,

   - $p + 1 \leq \pi(i) \leq n$ if the signature of $c_i = -$.

3. Next, construct the corresponding flag for each signed clan.

$$v_i = \begin{cases} e_{\pi(i)} & \text{if } c_i = \pm, \\ e_{\pi(i)} + e_{\pi(j)} & \text{if } c_i \in \mathbb{N} \text{ has signature } + \text{ and } c_i = c_j, \\ -e_{\pi(i)} - e_{\pi(j)} & \text{if } c_i \in \mathbb{N} \text{ has signature } - \text{ and } c_i = c_j. \end{cases}$$

4. Finally, define $\mathcal{O}_\gamma$ to be the set of all flags $F_\bullet =< v_1, \ldots, v_n >$ where the v_i's are defined as above.

Let us illustrate this algorithm by an example.

Example 1.6. Consider the case where $\gamma = 1\,2\,1\,2$. For the associated orbit, we could choose $\pi = 1243$. It would give us the flag $F_\bullet =< e_1 + e_3, e_2 + e_4, -e_4 - e_1, -e_3 - e_2 >$.

Although our goal in this paper is not studying the poset structure, only for this case we give the description of the weak order on the set of orbits in a combinatorial way as follows: Let γ be a clan of the form $\gamma = c_1 \cdots c_n$ and let α_i be a simple root with an associated orbit $\mathcal{O}_\gamma$. Then $s_{\alpha_i} \cdot \mathcal{O}_\gamma \neq \mathcal{O}_\gamma$ if and only if one of the following holds:

1) c_i and c_{i+1} are distinct natural numbers and the mate of c_i is to the left of the mate of c_{i+1},

2) c_i is a sign, c_{i+1} is a natural number and the mate of c_{i+1} is to the right of c_{i+1},

3) c_i is a natural number, c_{i+1} is a sign and the mate of c_i is to the left of c_i,

4) c_i and c_{i+1} are opposite signs.

Moreover, if γ' is a clan corresponding to the orbit $\mathcal{O}_{\gamma'} = s_{\alpha_i} \cdot \mathcal{O}_\gamma \neq \mathcal{O}_\gamma$, then γ' can be obtained from γ by interchanging c_i and c_{i+1} in the first three cases. For the latter case, γ' can be obtained from γ by replacing the opposite signs c_i and c_{i+1} by a pair of same natural numbers, see [8]. Here is an example to visualize this combinatorial construction by applying it to the set of all $(2, 1)$ clans.

Example 1.7. Consider the case when $p = 2$, $q = 1$. Then,

a) First, taking $i = 1$ gives us;

$$\begin{aligned} + - + &\mapsto 1\,1\,+ \\ - + + &\mapsto 1\,1\,+ \\ +\,1\,1 &\mapsto 1 + 1 \end{aligned}$$

b) Next, take $i = 2$ to get;

$$\begin{aligned} + - + &\mapsto +\,1\,1 \\ + + - &\mapsto +\,1\,1 \\ 1\,1\,+ &\mapsto 1 + 1 \\ 1 + 1 &\mapsto 1\,1\,+. \end{aligned}$$

Thus, we have the following graph with respect to the weak order on $A(2, 1)$; see Figure 1.1 below.

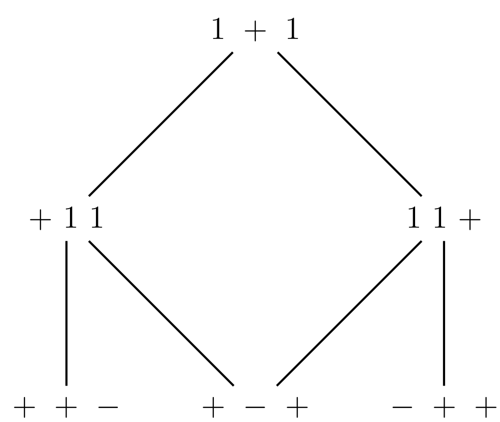


Figure 1.1: Weak order on $A(2, 1)$

Example 1.8. As an another example, consider the set $A(2, 2)$ of all $(2, 2)$ clans. The weak order on this set is given by Figure 1.2:

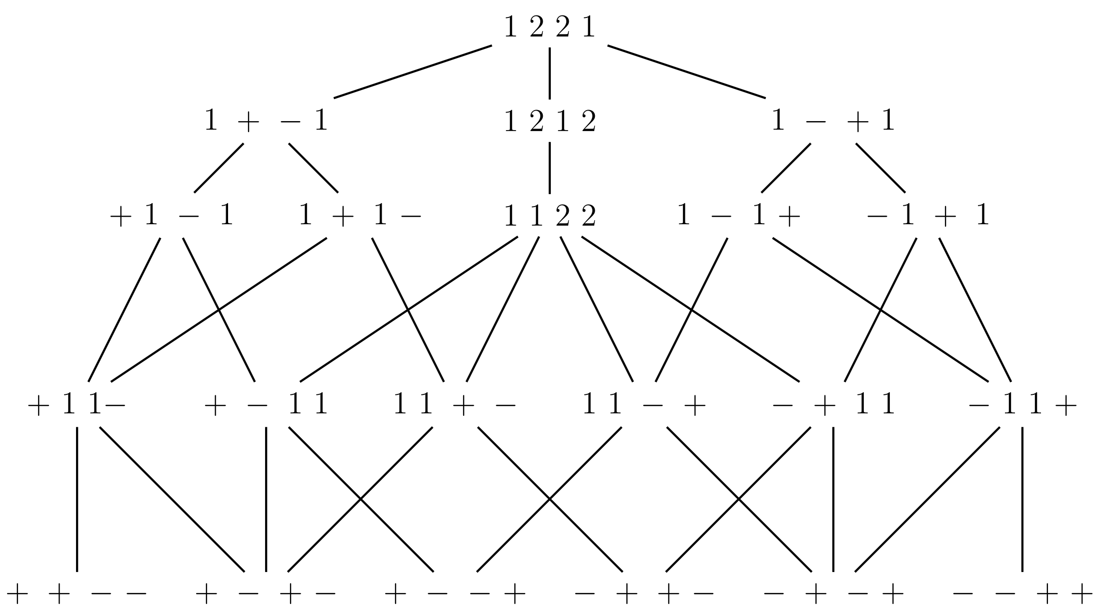


Figure 1.2: Weak order on $A(2,2)$

Finally, we can talk about other types of clans which are used to parametrize K orbits on the other types of flag varieties. The following theorem tells us that the set of K orbits can be parametrized by a subset of $A(p,q)$ for an appropriate choice of p and q.

Theorem 1.3.4 ([8], Theorem 1.5.8)**.** *For symmetric spaces of types B, C, and D, each K orbit on G/B is exactly the intersection of a $K = S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$ orbit on a type A flag variety with a smaller flag variety of an appropriate type, for some appropriate choice of p and q.*

Now, let B be the Borel subgroup consisting of lower triangular elements of G of types B, C and D. Then;

Fact 1.9.

1. $S(O(2p,\mathbb{C}) \times O(2q+1,\mathbb{C}))$ orbits in the flag variety $SO(2n+1,\mathbb{C})/B$ are parametrized by the elements of $B(p,q)$.

2. $GL(n,\mathbb{C})$ orbits in the flag variety $Sp(2n,\mathbb{C})/B$ are parametrized by the elements of $\Delta(n)$.

3. $Sp(2p, \mathbb{C}) \times Sp(2q, \mathbb{C})$ orbits in the flag variety $Sp(2n, \mathbb{C})/B$ are parametrized by the elements of $\Gamma(p, q)$.

4. $S(O(2p, \mathbb{C}) \times O(2q, \mathbb{C}))$ orbits in the flag variety $SO(2n, \mathbb{C})/B$ are parametrized by the elements of $\Theta(p, q)$.

5. $S(O(2p+1, \mathbb{C}) \times O(2q-1, \mathbb{C}))$ orbits in the flag variety $SO(2n, \mathbb{C})/B$ are parametrized by the elements of $M(p, q)$.

6. $GL(n, \mathbb{C})$ orbits in the flag variety $SO(2n, \mathbb{C})/B$ are parametrized by type D (n, n) clans.

Now, let us state the definitions of such clans and give examples. For more details see [3] and [9].

Let γ be a clan of the form $\gamma = c_1 \cdots c_n$. The reverse of γ, denoted by $rev(\gamma)$, is the clan

$$rev(\gamma) = c_n c_{n-1} \cdots c_1.$$

By using this, we can define symmetric clans as follows:

Definition 1.3.5. A (p, q) clan γ is called symmetric if $\gamma = rev(\gamma)$.

Example 1.10. We list all the elements of $B(4, 3)$ as follows;

1 2 3 + 3 2 1, 1 2 + - + 2 1, + + - - - + +, 1 2 3 + 3 1 2, 3 1 2 + 1 2 3,
1 3 2 + 1 3 2, 3 1 1 + 2 2 3, 1 + 2 - 1 + 2, + 1 2 - 2 1 +, 1 + - + - + 1,
1 + 1 - 2 + 2, + 1 2 - 1 2 +, + 1 - + - 1 +, 1 1 3 + 3 2 2, + 1 1 - 2 2 +,
1 - + + + - 1, - 1 + + + 1 -, + - 1 + 1 - +, - + 1 + 1 + -,
1 + 2 - 2 + 1, + - + - + - +, 1 2 + - + 1 2,
1 3 1 + 2 3 2, - + + - + + -, 1 1 + - + 2 2.

Definition 1.3.6. A (p,q) clan γ is called skew-symmetric if $\gamma = -rev(\gamma)$, that is the negative of $rev(\gamma)$.

Example 1.11. All skew-symmetric $(2,2)$ clans are

1 2 1 2, 1 2 2 1, 1 - + 1, - - + +, - 1 1 +, 1 + - 1,

1 1 2 2, + 1 1 -, + + - -, + - + -, - + - +.

Therefore, the cardinality of $\Delta(2)$ is $\Delta_2 = 11$.

Definition 1.3.7. A symmetric (p,q) clan γ is called symp-symmetric if $c_i \neq c_{n+1-i}$ whenever $c_i \in \mathbb{N}$.

Example 1.12. The set of all symp-symmetric $(4,2)$ clans is given by

1 2 + + 1 2, 1 + 2 1 + 2, 1 + 1 2 + 2, + 1 2 1 2 +, + 1 1 2 2 +,

1 1 + + 2 2, - + + + + -, + - + + - +, + + - - + +.

Thus, $\Gamma_{2,1} = 9$.

Remark 1.3.8. It is crucial to observe here that the only difference between symmetric and skew symmetric clans is the positioning of the $\pm$ symbols, not the pairs of natural numbers. For example, the clan $\gamma = 1\ 2\ 1\ 2$ is both a symmetric and a skew symmetric clan.

We end this section, by defining the last type of the clan, even if we leave the details for the future.

Definition 1.3.9. A type D clan is a skew-symmetric (n,n) clan γ such that

1. $c_i \neq c_{2n+1-i}$ whenever $c_i \in \mathbb{N}$,
2. The total number of $-$ signs and the pairs of matching numbers among the first n spots is even.

Example 1.13. There are only 3 type D $(2,2)$ clans, which are

+ + - -, 1 2 1 2, - - + +.

1.4 Lattice Paths

One of our combinatorial results is the beautiful interpretation of clans in terms of lattice paths-more specifically, in terms of the generalized Delannoy numbers.

The Delannoy numbers, denoted by $D(p,q)$ $(p,q \in \mathbb{N})$ are defined by the recurrence relation

$$D(p,q) = D(p-1,q) + D(p,q-1) + D(p-1,q-1) \tag{1.14}$$

and the initial conditions $D(p,0) = D(0,q) = D(0,0) = 1$. Their generating series is

$$\sum_{i,\,j\,\in\mathbb{N}} D(p,q)x^i y^j = \frac{1}{1-x-y-xy}.$$

One of the most appealing properties of Delannoy numbers is that they give the count of lattice paths that move with unit steps $E := (1,0)$, $N := (0,1)$, and $D := (1,1)$ in the plane. More precisely, $D(p,q)$ gives the number of lattice paths that start at the origin $(0,0) \in \mathbb{N}^2$ and end at $(p,q) \in \mathbb{N}^2$ moving with E, N, and D steps only. We will refer to such paths as (p,q) Delannoy paths and denote their collection by $\mathcal{D}(p,q)$. For example, if $(p,q) = (2,2)$, then $D(2,2) = 13$. In Figure 1.3, we depict the elements of $\mathcal{D}(2,2)$.

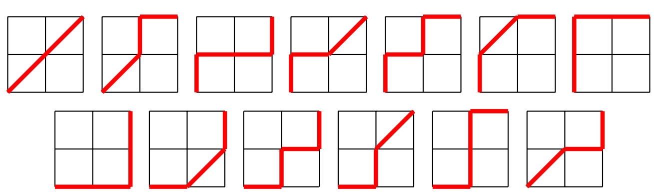


Figure 1.3: Delannoy paths

A lattice path L is a sequence of steps $(L_1,\ldots,L_r)$, where $L_i \in \{N,E,D\}$ for $i = 1,\ldots,r$, and moreover, the second entry of L_i is the first entry of L_{i+1} for $i = 1,\ldots,r-1$. The subset of $\mathcal{D}(p,q)$ which consists of lattice paths with no diagonal step is denoted by $\mathcal{D}^0(p,q)$. More generally, if $a \leq c, b \leq d$ are four nonnegative integers, then we are going to denote by $\mathcal{D}((a,b),(c,d))$ the set of paths that starts at (a,b) and ends at (c,d).

Clearly, the number of elements of $\mathcal{D}(p,q)$ is the Delannoy number $D(p,q)$. Let us denote the number of elements of $\mathcal{D}^0(p,q)$ by $\rho_{p,q}$. Then

$$\rho_{p,q} := \#\mathcal{D}^0(p,q) = \binom{p+q}{q}.$$

Let us finish this chapter by defining the weights for Delannoy paths as follows. This idea plays an important role in our combinatorial results that we will state later.

Now suppose we have 3 sequences of complex numbers $\mathrm{h} = (h_1, h_2, \dots)$, $\mathrm{v} = (v_1, v_2, \dots)$, and $\mathrm{d} = (d_1, d_2, \dots)$, and suppose ζ is a complex number. For a given lattice path $L = (L_1, \dots, L_r)$, put $\mathcal{L} := (\mathrm{h}, \mathrm{v}, \mathrm{d}, \zeta)$. The $\mathcal{L}$-weight of L is defined as the product of weights of the steps of L. More precisely, if $L_i = ((a,b),(c,e))$ is a step in π, then its weight is defined as

$$\omega(L_i) = \begin{cases} v_c & \text{if } L_i \text{ is a vertical step,} \\ h_c\zeta^e & \text{if } L_i \text{ is a horizontal step,} \\ d_c\zeta^{e-1} & \text{if } L_i \text{ is a diagonal step.} \end{cases}$$

The weight of L is

$$\omega(L) := \omega(L_1)\cdots\omega(L_r). \tag{1.15}$$

Chapter 2

Type AIII

We know from the classification of involutions on algebraic groups that there are essentially four different types of involutory automorphisms associated with $SL(n,\mathbb{C})$, which are given as follows:

1. $\sigma_1 : SL(n,\mathbb{C}) \to SL(n,\mathbb{C})$ defined by $\sigma_1(g) = (g^{-1})^\top$ with the fixed point set $SO(n,\mathbb{C})$.

2. $\sigma_2 : SL(2n,\mathbb{C}) \to SL(2n,\mathbb{C})$ defined by $\sigma_2(g) = H_{n,n}(g^{-1})^\top H_{n,n}$, where $H_{n,n}$ as in 1.1. The fixed point subgroup here is $Sp(2n,\mathbb{C})$.

3. $\sigma_3 = \sigma_3(p,q) : SL(n,\mathbb{C}) \to SL(n,\mathbb{C})$ defined by $\sigma_3(g) = J_{p,q}(g^{-1})^\top J_{p,q}$ where $0 < q < p$ are two positive integers such that $n = q + p$. Here, note that the fixed point subgroup is the Levi subgroup $S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$.

4. The final automorphism is the identity $\sigma_4(g) = g$, with the fixed point subgroup $SL(n,\mathbb{C})$. This is the case because $SL(n,\mathbb{C})$ can be viewed as the diagonal in $SL(n,\mathbb{C}) \times SL(n,\mathbb{C})$, σ_4 is induced from the nontrivial involutory automorphism $\widetilde{\sigma_4} : SL(n,\mathbb{C}) \times SL(n,\mathbb{C}) \to SL(n,\mathbb{C}) \times SL(n,\mathbb{C})$ defined by $\widetilde{\sigma_4}((g,h)) = (h,g)$.

One of the most fascinating use of the symmetric group S_n of permutations is parametrization of the homogeneous spaces of $SL(n,\mathbb{C})$. Recall that a Schubert variety, by definition, is the Zariski closure of a Borel orbit in the quotient $SL(n,\mathbb{C})/B$, which is commonly known as the full flag variety of $SL(n,\mathbb{C})$. On one hand, S_n gives a parametrization of all Schubert varieties in $SL(n,\mathbb{C})/B$. On the other hand, each involution induces an involutory automorphism on S_n with the corresponding fixed point sets:

1. $I_n = \{\pi \in S_n : \ \pi^2 = id\}$;

2. FI_n, the set of involutions with no fixed points;

3. $I^{\pm}_{p,q}$, the set of signed (p,q) involutions in I_n, which we define below.

Equivalently, these sets of involutions parametrize the B orbits in the symmetric varieties $SL(n,\mathbb{C})/SL(n,\mathbb{C})^{\sigma_i}$, respectively.

Note that Borel orbit closures form a graded poset, which will be refered as a Bruhat poset, with respect to (set-theoretic) inclusion. These posets are first considered by Richardson and Springer in their seminal paper [4]. The papers [17], [18] give precise descriptions of the covering relations of Bruhat orders on the (fixed point free) involutions in S_n. For more, also see [15] and [4]. Although it is presented by using (p,q) clans in [19], Wyser described the Bruhat order on signed involutions.

Another geometric partial ordering on involutions, namely the weak order, that is extremely useful for studying Bruhat ordering. In particular, the weak order is a ranked poset and its length function agrees with that of the Bruhat order. In general, a convenient way of defining the weak order is via the so called Richardson-Springer monoid action, more can be found in [14], [16] and [12]. Since our goal in this thesis is not studying the poset structure, and since the descriptions of both Bruhat and weak orders are lengthy we skip the details but mention a relevant fact. (See Figure 2.1, where we illustrate the weak order on $I^{\pm}_{2,2}$.)

Now, we are ready to introduce of our main objects signed (p,q) involutions, which we will use to determine the cardinality of $A(p,q)$ and also to construct the algorithms

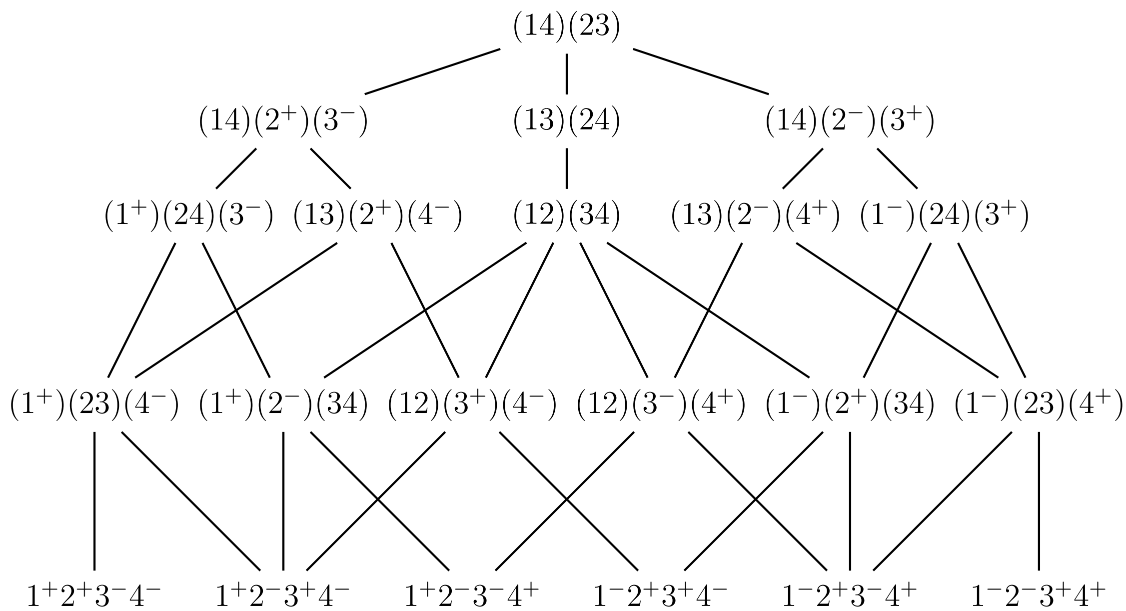


Figure 2.1: Weak order on $I^{\pm}_{2,2}$

for our lattice paths enumeration.

Definition 2.0.1. A signed (p, q) involution $\pi \in S_n$ is an involution with an assignment of $+$ and $-$ signs to the fixed points of π such that there are $p - q$ more $+$'s than $-$'s if $q < p$.

For example, $\pi = (1\ 6)(2\ 3)(5\ 10)(4^-)(7^+)(8^+)(9^+)$ is an element of $I^{\pm}_{6,4}$. Observe here that p is equal to the number of fixed points in π with a $+$ sign attached plus the number of two-cycles in π, while q is equal to the number of fixed points in π with a $-$ sign attached plus the number of two-cycles in π.

We finish this section by explaining the bijection between signed involutions and clans. Although the proof can be found in [20], we give the proof here again due to importance of the statement.

Lemma 2.0.2. There exists a surjective map from the set of clans of order n to the set of involutions in S_n, the symmetric group of permutations on $\{1, \ldots, n\}$.

Proof. Let $\gamma = c_1 \cdots c_n$ be a clan of order n. For each pair of identical numbers

(c_i, c_j) with $i < j$ we have a transposition in S_n which is defined by the indices, that is $(i, j) \in S_n$. Clearly, if (c_i, c_j) and $(c_{i'}, c_{j'})$ are two pairs of identical numbers from γ, then $\{i, j\} \neq \{i', j'\}$. Now we define the involution $\pi = \pi(\gamma)$ corresponding to γ as the product of all transpositions that come from γ. Accordingly, the $\pm$'s in γ correspond to the fixed points of the involution π.

Conversely, if π is an involution from S_n, then we have a (p, q) clan $\gamma = \gamma(\pi)$ that is defined as follows. We start with an empty string $\gamma = c_1 \ldots c_n$ of length n. If $\pi_1 \ldots \pi_n$ is the one-line notation for π, then for each pair of numbers (i, j) such that $1 \leq i < j \leq n$ and $\pi_i = j$, $\pi_j = i$, we put $c_i = c_j = i$. Also, if $i_1, \ldots, i_m$ is the increasing list of indices such that $\pi_{i_j} = i_j$ $(j = 1, \ldots, m)$, then starting from i_1 place a $+$ until the difference between the number of c_{i_j}'s with a $+$ and the number of empty places is $p - q$. At this point place a $-$ in each of the empty places. It is easy to check that γ is a (p, q) clan of order n, hence the proof follows. $\square$

Lemma 2.0.3. There is a bijection between the set of all (p, q) clans and the set of all signed (p, q) involutions.

Proof. Let φ denote the surjection that is constructed in the proof of Lemma 2.0.2. We modify φ as follows. Let $\gamma = c_1 \ldots c_n$ be a (p, q) clan and let $\pi = \varphi(\gamma)$ denote involution that is obtained from γ via φ. If an entry c_i of γ is a $\pm$, then we know that i is a fixed point of π. We label i with $\pm$. Repeating this procedure for each $\pm$ that appear in γ we obtain a signed (p, q) involution $\widetilde{\pi}$. Clearly $\widetilde{\pi}$ is uniquely determined by γ. Therefore, the map defined by $\widetilde{\varphi}(\gamma) = \widetilde{\pi}$ is a bijection. $\square$

2.1 Counting (p, q) clans with k pairs.

Throughout this chapter, let us assume that p and q are two numbers such that $0 \leq q \leq p$ and $p + q = n$. It follows from our Convention 1.3 that making such assumption makes sense.

Let $\gamma = c_1 c_2 \dots c_n$ be a (p,q) clan and let the associated involution π from $I^{\pm}_{p,q}$ be in the form

$$\pi = (i_1, j_1)(i_2, j_2) \cdots (i_k, j_k) d_1 \dots d_{n-2k},$$

such that π has $n-2k$ fixed points, $p-q \leq n-2k$, and there are $p-q$ more $+$'s than $-$'s. It follows that we have the following upper bound for the number of pairs of natural numbers, k.

Remark 2.1.1. Since $q+p-2k = n-2k \geq p-q$, it holds true that $0 \leq k \leq q$.

We denote by $I^{\pm}_{k,p,q}$ the set of all such signed involutions. We denote the cardinality of $I^{\pm}_{k,p,q}$ by $\alpha_{k,p,q}$. Our aim in this section is to give a practical formula for $\alpha_{k,p,q}$.

Theorem 2.1.2. *Let p and q be two numbers such that $0 \leq q \leq p$. The number of all (p,q) clans with k pairs is;*

$$\alpha_{k,p,q} = \binom{q+p}{2k} \frac{(2k)!}{2^k k!} \binom{q+p-2k}{p-k}. \tag{2.1}$$

In particular, we have

$$A_{p,q} = \sum_{k=0}^{q} \alpha_{k,p,q} = \sum_{k=0}^{q} \binom{q+p}{2k} \frac{(2k)!}{2^k k!} \binom{q+p-2k}{p-k}. \tag{2.2}$$

Proof. Observe that once the entries to appear in the transpositions of π are chosen the fixed points, which are ordered in an increasing manner, are uniquely determined. So, the question is equivalent to choosing k transposition from $[n]$ and ordering them to give π. It is not difficult to see that this is indeed given by $\binom{n}{2k}\frac{(2k)!}{2^k k!}$.

Next, we look into ways to place a $+$'s and b -'s on the string $d_1 \dots d_{n-2k}$ so that there are exactly $p-q = a-b$ $+$'s more than -'s. Clearly, this number is equivalent to $\binom{n-2k}{a}$. Since $a+b = n-2k = q+p-2k$ and $a-b = p-q$, we have $a = p-k$. This finishes our proof. $\square$

Alternatively, this formula can be written

$$\alpha_{k,p,q} = \frac{(q+p)!}{(q-k)!(p-k)!}\frac{1}{2^k k!}. \tag{2.3}$$

Observe that the formula (2.3) is defined independently of the inequality $q < p$. From now on, for our combinatorial purposes, we skip mentioning this comparison between p and q and use the equality $\alpha_{k,p,q} = \alpha_{k,q,p}$ whenever it is needed.

Example 2.4. Consider the case when $p = 2$ and $q = 1$. All possible $(2,1)$ clans with 0 pair of matching natural numbers are $+\ +\ -, +\ -\ +, -\ +\ +$. Indeed, it follows from the equation (2.1) that $\alpha_{0,p,q} = 3$. Again, by using the equation (2.1) we can conclude that there are three $(2,1)$ clans with 1 pair of matching natural numbers which are; $1\ 1\ +, 1\ +\ 1, +\ 1\ 1$.

2.2 Recurrences

Let us start this section by recording the following recurrences that we will use for our future computations:

$$\begin{aligned}\alpha_{k,p,q} =& \binom{q+p}{2k}\frac{(2k)!}{2^k k!}\binom{q+p-2k}{p-k} \\ =& \frac{(n-2k+1)(n-2k+2)}{(2k)(2k-1)}\frac{(2k)(2k-1)}{2k}\frac{(q-k+1)(p-k+1)}{(n-2k+2)(n-2k+1)} \\ & \binom{p+q}{2k-2}\frac{(2k-2)!}{2^{k-1}(k-1)!}\binom{p+q-2k+2}{p-k+1} \\ =& \frac{(p-k+1)(q-k+1)}{2k}\alpha_{k-1,p,q}\end{aligned} \tag{2.5}$$

holds for all integers $p, q, k \geq 1$. In a similar manner, it can be also shown that the following

$$\alpha_{k,p,q} = \frac{p+q}{p-k}\alpha_{k,p-1,q} \qquad \text{and} \qquad \alpha_{k,p,q} = \frac{p+q}{q-k}\alpha_{k,p,q-1} \tag{2.6}$$

hold true (whenever they are defined) for all $p, q, k \geq 1$.

We are going to show that $\alpha_{k,p,q}$'s obey a 3-term recurrence and exploit its consequences.

Theorem 2.2.1. *Let p and q be two positive integers. If $k \geq 1$, then the following recurrence relation and its initial condition holds true:*

$$\alpha_{k,p,q} = \alpha_{k,p-1,q} + \alpha_{k,p,q-1} + (q+p-1)\alpha_{k-1,p-1,q-1} \ \text{ and } \ \alpha_{0,p,q} = \binom{p+q}{q}. \tag{2.7}$$

Proof. The proof of the initial condition is straightforward. We are going to construct our proof of the recurrence by analyzing what happens to an involution $\pi \in I^{\pm}_{k,p,q}$ when we remove its largest entry n. Clearly, n appears in π either as a fixed point, or in one of the 2-cycles. Thus we partition $I^{\pm}_{k,p,q}$ into $n+1$ disjoint subsets;

$$I^{\pm}_{k,p,q} = I^{\pm}_{k,p,q}(+) \cup I^{\pm}_{k,p,q}(-) \cup \bigcup_{i=1}^{n-1} I^{\pm}_{k,p,q}(i),$$

where

1. $I^{\pm}_{k,p,q}(+) := \{\pi \in I^{\pm}_{k,p,q} : \text{ n is a fixed point with a + sign}\}$,

2. $I^{\pm}_{k,p,q}(-) := \{\pi \in I^{\pm}_{k,p,q} : \text{ n is a fixed point with a - sign}\}$,

3. $I^{\pm}_{k,p,q}(i) := \{\pi \in I^{\pm}_{k,p,q} : \text{ n appears in the 2-cycle } (i, n)\}$ for $i = 1, \ldots, n-1$.

First, we assume that $\pi \in I^{\pm}_{k,p,q}(+)$, so

$$\pi = (i_1\ j_1)\ldots(i_k\ j_k)d_1\ldots d_{n-2k-1}n^{+}.$$

It follows that by removing n we reduce the total number of + signs by 1. Note that this makes sense because $p-q$ is fixed. Thus, the number of such signed (p, q) involutions on $[n]$ is counted by $\alpha_{k,p-1,q}$. By using a similar argument for the case $\pi \in I^{\pm}_{k,p,q}(-)$, we conclude that there are $\alpha_{k,p,q-1}$ such signed involutions on $[n]$.

Next, and finally, we consider the case where n appears in a 2-cycle (i, j) of π. Then $j = n$ and therefore $\pi \in I^{\pm}_{k,p,q}(i)$. It is obvious that there are $(n-1)$ possibilities for i from n numbers. Removing this 2-cycle from π leaves us with $(n-2)$ elements and $(k-1)$ 2-cycles but it does not change the signs on the fixed points. Of course, once this 2-cycle is removed, we decrease the numbers that are greater than i_r by 1 so that we have valid signed involution whose support lies in I_{n-2}. In particular, since the difference of $+$ and $-$ signs is preserved, we see that the number of such signed (p, q) involutions on $[n]$ is given by $(n-1)\alpha_{k-1,p-1,q-1}$.

Notice that in each of these cases we get an injective map into a set of signed involutions of a smaller size. Indeed, by removing n, in the cases 1. and 2. we get injections into $I^{\pm}_{k,p-1,q}$ and $I^{\pm}_{k,p,q-1}$, respectively. In the case of 3, we get an injection into $I^{\pm}_{k-1,p-1,q-1}$. Conversely, if π' is a signed involution from $I^{\pm}_{k,p-1,q}$ (or, from $I_{k,p,q-1}$), then we append n^+ (resp. n^-) to get an element $\psi_k(+)(\pi') \in I^{\pm}_{k,p,q}$ (resp. $\psi_k(-)(\pi') \in I^{\pm}_{k,p,q}$). If $\pi' \in I^{\pm}_{k-1,p-1,q-1}$, then we pick a number, say $i \in [n-1]$ in $n-1$ different ways; we add 1 to every number j such that $i < j$ and j appears in the standard form of π'; and insert the 2-cycle (i, n) into π'. Let us denote the resulting map by $\psi_k(i)(\pi') \in I^{\pm}_{k,p,q}$ Obviously, these maps, $\psi(\pm)$ and $\psi(i)$'s, are well defined inverses to the procedures that are described in the previous paragraph. Thus, it is now clear that we have built a bijection between $I^{\pm}_{k,p,q}$ and the disjoin union $I^{\pm}_{k,p-1,q} \cup I^{\pm}_{k,p,q-1} \cup \bigsqcup_{i=1}^{n-1} I^{\pm}_{k-1,p-1,q-1}$, proving our claimed recurrence. (We use square-union to indicate that it is a disjoint union of $n-1$ copies of the same set.)

$\square$

After this, it is not hard to see that taking the sum of both sides of the equation

(2.7) over k, where $1 \leq k \leq q-1$, gives us

$$\begin{aligned} A_{p,q} - A_{p-1,q} - A_{p,q-1} - (p+q-1)A_{p-1,q-1} &= \alpha_{0,p,q} - \alpha_{0,p-1,q} - \alpha_{0,p,q-1} \\ &\quad + \alpha_{q,p,q} - \alpha_{q,p,q-1} - (p+q-1)\alpha_{q-1,p-1,q-1} \\ &= 0. \end{aligned}$$

By using the recurrence relation in (2.7) once more, we see that the latter becomes 0. We have just proved the following;

Corollary 2.2.2. Set $A_{p,0} = A_{0,q} = 1$ for all nonnegative integers p and q. Then the numbers $A_{p,q}$ satisfy the following recurrence relation

$$A_{p,q} = A_{p-1,q} + A_{p,q-1} + (p+q-1)A_{p-1,q-1}, \qquad p, q \geq 1. \tag{2.8}$$

2.3 Generating functions

It is well known that the (exponential) generating function for the number of involutions I_n, which we denote by c_n, is given by $e^{t+\frac{t^2}{2}}$. In fact, define polynomials $K_n(x)$ by the formula

$$\sum_{n\geq 0} K_n(x)\frac{t^n}{n!} = e^{xt+\frac{t^2}{2}}.$$

It is well known that $K_n(x) = \sum_{\pi\in I_n} x^{a_1(\pi)}$, where $a_1(\pi)$ denotes the number of 1-cycles (fixed points) of π. See [21, Exercise 5.19]. It easily follows that if $c_{n,r}$ denotes the number of elements of I_n with exactly r 1-cycles, then

$$\sum_{n,r\geq 0} c_{n,r}\frac{t^n}{n!}x^r = e^{xt+\frac{t^2}{2}}. \tag{2.9}$$

Remark 2.3.1. The numbers $c_{n,r}$ appear in our context rather naturally. Suppose we have $q = k \leq p$. Then $\alpha_{k,p,k} = \alpha_{k,k,p}$ is the number of signed involutions on $[p+k]$

such that there are $p-k$ +'s more than -'s on the fixed points. It is not difficult to see in this case that the number of $-$'s is 0. Therefore, $\alpha_{k,p,k}$ is the number of involutions on $[p+k]$ with $p-k$ 1-cycles and whose fixed points have $+$ signs only. In other words, $\alpha_{k,p,k} = c_{p+k,p-k}$.

Corollary 2.3.2. The number of involutions $\pi \in I_n$ with exactly r 1-cycles, $c_{n,r}$, is a special case of $\alpha_{k,p,q}$'s.

Proof. We already know from Remark 2.3.2 that $\alpha_{k,p,k} = c_{p+k,p-k}$. The result follows from the fact that the equations $n = p+k$, $r = p-k$ have a unique solution. □

The polynomial $K_n(x)$ is the sum of the entries of the nth row of Table 2.1. In the sequel, we are going to need the following finite *diagonal* sums of the same table:

$$G_m(x) = \sum_{k=1}^{m} c_{m+k,m-k} x^k. \tag{2.10}$$

$c_{0,0}x^0$	0	0	0	0	$\cdots$
$c_{1,0}x^0$	$c_{1,1}x^1$	0	0	0	$\cdots$
$c_{2,0}x^0$	$c_{2,1}x^1$	$c_{2,2}x^2$	0	0	$\cdots$
$c_{3,0}x^0$	$c_{3,1}x^1$	$c_{3,2}x^2$	$c_{3,3}x^3$	0	$\cdots$

Table 2.1: Dissection of $K_n(x)$'s.

Now, we are ready to describe the generating function $u_p(x) := \sum_{q\geq 0} A_{p,q}x^q$. We first assume that $p \geq 1$. By tabulating a few terms of $u_p(x)$ as in

$$A_{p,0} = \alpha_{0,p,0}$$

$$A_{p,1}x = \alpha_{0,p,1}x + \alpha_{1,p,1}x$$

$$A_{p,2}x^2 = \alpha_{0,p,2}x^2 + \alpha_{1,p,2}x^2 + \alpha_{2,p,2}x^2$$

$$A_{p,3}x^3 = \alpha_{0,p,3}x^3 + \alpha_{1,p,3}x^3 + \alpha_{2,p,3}x^3 + \alpha_{3,p,3}x^3$$

we find that it is going to be useful to determine the following infinite sums first:

$$F_k(x) := \sum_{q \geq k} \alpha_{k,p,q} x^q \qquad (k \geq 1).$$

Indeed, it is clear from the above table that

$$u_p(x) = \frac{1}{1-x} + \sum_{k \geq 1} F_k(x). \tag{2.11}$$

By using recurrences in (2.6) we rewrite $F_k(x)$'s:

$$\begin{aligned}
F_k(x) &= \sum_{q \geq k} \alpha_{k,p,q} x^q = \alpha_{k,p,k} x^k + \sum_{q \geq k+1} \alpha_{k,p,q} x^q \\
&= \alpha_{k,p,k} x^k + \sum_{q \geq k+1} \alpha_{k,p,q-1} \frac{p+q}{q-k} x^q \\
&= \alpha_{k,p,k} x^k + \sum_{q \geq k+1} \left(\alpha_{k,p,q-1} x^q + \alpha_{k,p,q-1} \frac{p+k}{q-k} x^q \right) \\
&= \alpha_{k,p,k} x^k + x \sum_{q \geq k+1} \alpha_{k,p,q-1} x^{q-1} + \sum_{q \geq k+1} \alpha_{k,p,q-1} \frac{p+k}{q-k} x^q \\
&= \alpha_{k,p,k} x^k + x F_k(x) + (p+k) \sum_{q \geq k+1} \alpha_{k,p,q-1} \frac{1}{q-k} x^q \\
&= \alpha_{k,p,k} x^k + x F_k(x) + (p+k) x^k \sum_{q \geq k+1} \alpha_{k,p,q-1} \frac{1}{q-k} x^{q-k} \\
&= \alpha_{k,p,k} x^k + x F_k(x) + (p+k) x^k \int \sum_{q \geq k+1} \alpha_{k,p,q-1} x^{q-k-1} dx.
\end{aligned}$$

We re-organize the last equation as follows:

$$\frac{F_k(x) - x F_k(x)}{x^k} = \alpha_{k,p,k} + (p+k) \int x^{-k} F_k(x) dx.$$

Equivalently,

$$F_k(x)(x^{-k} - x^{-k+1}) = \alpha_{k,p,k} + (p+k)\int F_k(x)x^{-k}dx. \tag{2.12}$$

Taking the derivative of both sides gives us a first order differential equation with variable coefficients:

$$F_k'(x^{-k} - x^{-k+1}) - F_k(kx^{-k-1} + (-k+1)x^{-k} + (p+k)x^{-k}) = 0$$

or

$$F_k' + \frac{k + x(1+p)}{x^2 - x}F_k = 0, \tag{2.13}$$

which is a first order linear separable homogeneous ODE with the initial condition

$$\left.\frac{F_k(x)}{x^k}\right|_{x=0} = \alpha_{k,p,k}.$$

Therefore,

$$F_k(x) = x^k((1-x)^{-(k+p+1)} + \alpha_{k,p,k} - 1).$$

Recall our assumption that $p \geq 1$. Now, taking the sum of both sides over all $k \geq 1$

gives us

$$\begin{aligned}
u_p(x) &= \frac{1}{1-x} + \sum_{k\geq 1} F_k(x) \\
&= \frac{1}{1-x} + \sum_{k\geq 1} x^k(1-x)^{-(k+p+1)} + \sum_{k=1}^{p} \alpha_{k,p,k}x^k - \sum_{k\geq 1} x^k \\
&= \frac{1}{1-x} + \sum_{k\geq 1} x^k(1-x)^{-(k+p+1)} + \sum_{k=1}^{p} \alpha_{k,p,k}x^k - \frac{x}{1-x} \\
&= 1 + \frac{1}{(1-x)^{1+p}} \sum_{k\geq 1} \frac{x^k}{(1-x)^k} + \sum_{k=1}^{p} \alpha_{k,p,k}x^k \\
&= 1 + \frac{1}{(1-x)^{1+p}} \left(\frac{1}{1-\frac{x}{(1-x)}} - 1 \right) + \sum_{k=1}^{p} \alpha_{k,p,k}x^k \\
&= 1 + \frac{1}{(1-x)^{1+p}} \left(\frac{x}{1-2x} \right) + \sum_{k=1}^{p} \alpha_{k,p,k}x^k. \qquad (2.14)
\end{aligned}$$

Recall also that $c_{m,r}$ stands for the number of involutions on $[m]$ with exactly r 1-cycles. Finally, it follows from the equation (2.14) and Remark 2.3.2 that we have the following;

Theorem 2.3.3. *Let p be a nonnegative integer. The generating function u_p is equal to*

$$u_p(x) = \begin{cases} \frac{1}{1-x}; & \text{if } p = 0; \\ 1 + \frac{1}{(1-x)^{1+p}} \left(\frac{x}{1-2x} \right) + \sum_{k=1}^{p} c_{p+k,p-k}x^k; & \text{if } p \geq 1. \end{cases}$$

Now, we are ready to state the generating function for $A_{p,q}$. One of the many options for a bivariate generating function for $A_{p,q}$'s is

$$v(x,y) := \sum_{p,q\geq 0} A_{p,q}x^q \frac{y^p}{p!} = \sum_{p\geq 0} u_p(x) \frac{y^p}{p!}, \qquad (2.15)$$

which is easily seen (by Theorem 2.3.3) to reduce to the calculation of

$$\sum_{p\geq 0} G_p(x)\frac{y^p}{p!}, \tag{2.16}$$

where $G_p(x) = \sum_{k=1}^{p} c_{p+k,p-k}x^k$. Substituting $\alpha_{k,p,k} = c_{p+k,p-k} = \frac{(p+k)!}{(p-k)!2^k k!}$ into (2.16), we have

$$\begin{aligned}
\sum_{p\geq 0} G_p(x)\frac{y^p}{p!} &= \sum_{p\geq 0}\sum_{k=1}^{p}\frac{\alpha_{k,p,k}}{p!}x^k y^p \\
&= \sum_{p\geq 0}\sum_{k=1}^{p}\frac{(p+k)!}{(p-k)!k!p!}\left(\frac{x}{2}\right)^k y^p \\
&= \sum_{p\geq 0}\frac{\sqrt{\frac{2}{\pi}}e^{\frac{1}{x}}\sqrt{\frac{1}{x}}\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x}) - 1}{p!}y^p \\
&= e^{\frac{1}{x}}\sqrt{\frac{2}{\pi x}}\sum_{p\geq 0}\frac{\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!}y^p - \sum_{p\geq 0}\frac{y^p}{p!} \\
&= -e^y + e^{\frac{1}{x}}\sqrt{\frac{2}{\pi x}}\sum_{p\geq 0}\frac{\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!}y^p,
\end{aligned}$$

where $\widetilde{K}_n(x)$ denotes the modified Bessel function of the second kind, which is one of the solutions to the modified Bessel differential equation. Now, the following consequence is immediate from Theorem 2.3.3.

Corollary 2.3.4. The bivariate generating function $\sum_{p,q\geq 0} A_{p,q}x^q\frac{y^p}{p!}$ is given by

$$\frac{e^y}{1-x} + \frac{xe^{\frac{y}{1-x}}}{(1-2x)(1-x)} + e^{\frac{1}{x}}\sqrt{\frac{2}{\pi x}}\sum_{p\geq 0}\frac{\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!}y^p.$$

Another approach for deriving the bivariate generating function (2.15) is by transforming the recurrence relation (2.8) into a partial differential equation as follows. Multiplying both side of the recurrence relation by $\frac{x^q y^p}{p!}$ and taking the sum over all

$p, q \geq 1$ gives us

$$\sum_{p,q\geq 1} \frac{A_{p,q}}{p!} x^q y^p = \sum_{p,q\geq 1} \frac{A_{p-1,q}}{p!} x^q y^p + \sum_{p,q\geq 1} \frac{A_{p,q-1}}{p!} x^q y^q + \sum_{p,q\geq 1} (p+q-1) \frac{A_{p-1,q-1}}{p!} x^q y^p. \tag{2.17}$$

Since

$$\begin{aligned} v(x,y) = \sum_{p,q\geq 0} \frac{A_{p,q}}{p!} x^q y^p &= A_{0,0} + A_{0,1}x + \cdots + A_{0,q}x^q + \dots \\ &+ \frac{A_{1,0}}{1!} y + \cdots + \frac{A_{p,0}}{p!} y^p + \dots \\ &+ \frac{A_{1,1}}{1!} xy + \cdots + \frac{A_{p,1}}{p!} xy^p + \dots \\ &+ \frac{A_{1,2}}{1!} x^2 y + \cdots + \frac{A_{p,2}}{p!} x^2 y^p + \dots \end{aligned}$$

the equation (2.17) combined with the initial conditions $A_{p,0} = A_{0,q} = 1$ gives

$$\begin{aligned} v(x,y) &- \frac{1}{1-x} - e^y + 1 \\ &= \int \sum_{p\geq 1, q\geq 0} \frac{A_{p-1,q}}{(p-1)!} x^q y^{p-1} dy - e^y + x\left(\sum_{p,q\geq 0} \frac{A_{p,q}}{p!} x^q y^p - \frac{1}{1-x} \right) \\ &+ \sum_{p,q\geq 1} p \frac{A_{p-1,q-1}}{p!} x^q y^p + \sum_{p,q\geq 1} q \frac{A_{p-1,q-1}}{p!} x^q y^p - \sum_{p,q\geq 1} \frac{A_{p-1,q-1}}{p!} x^q y^p \\ &= \int \sum_{p\geq 1, q\geq 0} \frac{A_{p-1,q}}{(p-1)!} x^q y^{p-1} dy - e^y + x\left(\sum_{p,q\geq 0} \frac{A_{p,q}}{p!} x^q y^p - \frac{1}{1-x} \right) \\ &+ xy \sum_{p,q\geq 0} \frac{A_{p-1,q-1}}{(p-1)!} x^{q-1} y^{q-1} + \int \sum_{p,q\geq 1} \frac{qA_{p-1,q-1}}{(p-1)!} x^{q-1} y^{p-1} dy \\ &- x \int \sum_{p,q\geq 1} \frac{A_{p-1,q-1}}{(p-1)!} x^{q-1} y^p dy. \end{aligned}$$

Therefore, we have

$$v(x,y)-\frac{1}{1-x}-e^y+1=\int v(x,y)dy-e^y+xv(x,y)-\frac{x}{1-x}+xyv(x,y)$$
$$+x\int\left(\frac{\partial}{\partial x}(xv(x,y))\right)dy-x\int v(x,y)dy,$$

or equivalently,

$$(1-x-xy)v(x,y)=(1-x)\int v(x,y)dy+x\int\left(\frac{\partial}{\partial x}(xv(x,y))\right)dy.$$

Taking the integral of both sides with respect to y yields the following PDE:

$$-xv(x,y)+(1-x-xy)\frac{\partial v(x,y)}{\partial y}=(1-x)v(x,y)+x\left(v(x,y)+x\frac{\partial v(x,y)}{\partial x}\right),$$

or

$$(-x^2)\frac{\partial v(x,y)}{\partial x}+(1-x-xy)\frac{\partial v(x,y)}{\partial y}=(1+x)v(x,y), \tag{2.18}$$

with the initial conditions $v(0,y)=e^y$ and $v(x,0)=\frac{1}{1-x}$.

On one hand, we know that solutions of such PDE's are easily obtained by applying the method of "characteristic curves." Our characteristic curves are $x(r,s), y(r,s)$, and $v(r,s)$. Their tangents are equal to

$$\frac{\partial x}{\partial s}=-x^2, \qquad \frac{\partial y}{\partial s}=1-x-xy, \qquad \frac{\partial v}{\partial s}=(1+x)v, \tag{2.19}$$

with the initial conditions

$$x(r,0)=r, \;\; y(r,0)=0 \;\text{ and }\; v(r,0)=\frac{1}{1-r}.$$

From the first equation given in (2.19) and its initial condition below, we have

$$x(r,s) = \frac{r}{rs+1}. \tag{2.20}$$

Plugging this into the second equation gives us $\frac{\partial y}{\partial s} = 1 - \frac{r}{rs+1}(1+y)$, which is a first order linear ODE. The general solution for this ODE is

$$y(r,s) = \frac{rs^2 - 2rs + 2s}{2(rs+1)}. \tag{2.21}$$

Finally, from the last equation in (2.19) together with its initial condition we conclude that

$$v(r,s) = \frac{e^s(rs+1)}{1-r}.$$

In summary, we outlined the proof of our next result.

Theorem 2.3.5. *Let $v(x,y)$ denote the function that is represented by the series $\sum_{p,q\geq 0} A_{p,q} x^q \frac{y^p}{p!}$ around the origin. If r and s are the variables related to x and y as in equations (2.21) and (2.20), then*

$$v(r,s) = \frac{e^s(rs+1)}{1-r} \tag{2.22}$$

around $(r,s) = (1,0)$.

Unfortunately, the beautiful form in (2.22) of $v(r,s)$ diminishes once the variables r and s are solved in terms of x and y. In fact, this seems to be a nontrivial task due to complicated nature of (2.21). However, one can still recover some information by computing (by brute force of long division) the inverses of power series. We anticipate that this approach will be helpful for understanding special values (at small numbers) of the modified Bessel functions. Such information is useful in number theory.

More precisely, the general solution $S_1(x, y)$ of (2.18) is given as follows:

$$\frac{e^{1/x}F_1(\frac{2xy+2x-1}{2x^2})}{x} \tag{2.23}$$

where $F_1(z)$ is some function in one-variable. We want to choose $F_1(z)$ in such a way that $S_1(x, y) = v(x, y)$ holds true. To do so, first, consider the case where $y = 0$, then by definition $v(x, 0) = \frac{1}{1-x}$. Thus, $F_1(z)$ satisfies the following equation

$$\frac{e^{1/x}F_1(\frac{2x-1}{2x^2})}{x} = \frac{1}{1-x} \quad \text{or} \quad F_1\left(\frac{2x-1}{2x^2}\right) = \frac{xe^{-1/x}}{1-x}. \tag{2.24}$$

Next, we take the inverse of the transformation $z = \frac{2x-1}{2x^2}$ in (2.24) to have $x = \frac{1-\sqrt{1-2z}}{2z}$ and thus,

$$F_1(z) = \frac{e^{-1-\sqrt{1-2z}}(1-\sqrt{1-2z})}{2z-1+\sqrt{1-2z}}. \tag{2.25}$$

Then substitute this into the general form of $S_1(x, y)$ given in (2.23) to conclude that

$$S_1(x,y) = \frac{e^{\frac{1-x}{x}-\frac{\sqrt{x^2-2xy-2x+1}}{x}}\left(\frac{x-\sqrt{x^2-2xy-2x+1}}{x}\right)}{\frac{-x^2+2xy+2x-1+x\sqrt{x^2-2xy-2x+1}}{x}} \tag{2.26}$$

$$= \frac{e^{\frac{1-x-\sqrt{x^2-2xy-2x+1}}{x}}(x-\sqrt{x^2-2xy-2x+1})}{-x^2+2xy+2x-1+x\sqrt{x^2-2xy-2x+1}} \tag{2.27}$$

Finally, we close this section by the following corollary;

Corollary 2.3.6. The bivariate generating series for $A_{p,q}$'s is given by

$$\sum_{p,q\geq 0} A_{p,q}x^q\frac{y^p}{p!} = \frac{e^{\frac{1-x-\sqrt{x^2-2xy-2x+1}}{x}}(x-\sqrt{x^2-2xy-2x+1})}{-x^2+2xy+2x-1+x\sqrt{x^2-2xy-2x+1}}. \tag{2.28}$$

2.4 A combinatorial interpretation

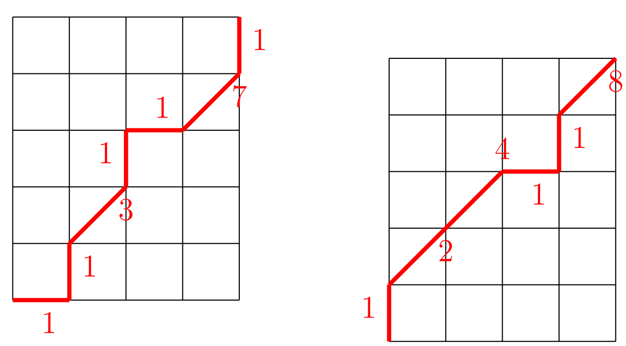
In this section, we re-interpret our $A_{p,q}$'s in terms of weighted lattice paths.

With the notation given in the Section 1.4 in mind, we define the weight of L_i, the i-th step, as follows:

$$\omega(L_i) = \begin{cases} 1 & \text{if } L_i = ((a,b),(a+1,b)); \\ 1 & \text{if } L_i = ((a,b),(a,b+1)); \\ a+b+1 & \text{if } L_i = ((a,b),(a+1,b+1)). \end{cases} \tag{2.29}$$

Then, we define the weight of L, denoted by $\omega(L)$ as the product of the weights of its steps, as defined in Section 1.4.

Example 2.30. Consider the following paths given in $(4,5)$-grid.



The weight of the first path π is $\omega(\pi) = 3 \cdot 7 = 21$ and the weight of the second path π' is $\omega(\pi') = 2 \cdot 4 \cdot 8 = 64$.

Proposition 2.4.1. If p and q are two nonnegative integers, then

$$A_{p,q} = \sum_{L \in \mathcal{D}(p,q)} \omega(L). \tag{2.31}$$

Proof. The following special cases are easy to verify:

$$A_{p,0} = \sum_{L\in\mathcal{D}(p,0)} \omega(L),\ A_{0,q} = \sum_{L\in\mathcal{D}(0,q)} \omega(L),\ \text{ and } A_{1,1} = \sum_{L\in\mathcal{D}(1,1)} \omega(L).$$

Thus, to prove (2.31), it suffices to prove that the right hand side of it satisfies the recurrence in Corollary 2.2.2.

To this end, we partition $\mathcal{D}(p,q)$ into three disjoint subsets, denoted by $\mathcal{D}_N(p,q)$, $\mathcal{D}_E(p,q)$, and $\mathcal{D}_D(p,q)$, where $\mathcal{D}_A(p,q)$ $(A \in \{N,E,D\})$ consists of (p,q) Delannoy paths $L \in \mathcal{D}(p,q)$ whose last step is an A-step. Observe that, the map $\mathcal{D}_N(p,q) \to \mathcal{D}(p,q-1)$ that is defined by omitting the last (north) step is a weight preserving bijection. In a similar way, the map $\mathcal{D}_E(p,q) \to \mathcal{D}(p-1,q)$ that is defined by omitting the last (east) step is a weight preserving bijection. Finally, the map $\psi : \mathcal{D}_D(p,q) \to \mathcal{D}(p-1,q-1)$ that is defined by omitting the last (diagonal) step is a bijection and the weight of an element in its image, say $\psi(L)$, is $p+q-1$ times the weight of L. Putting these observations together we see that

$$\begin{aligned}\sum_{L\in\mathcal{D}(p,q)} \omega(L) &= \sum_{L\in\mathcal{D}_N(p,q)} \omega(L) + \sum_{L\in\mathcal{D}_E(p,q)} \omega(L) + \sum_{L\in\mathcal{D}_D(p,q)} \omega(L)\\ &= \sum_{L\in\mathcal{D}(p,q-1)} \omega(L) + \sum_{L\in\mathcal{D}(p-1,q)} \omega(L) + \sum_{L\in\mathcal{D}(p-1,q-1)} (p+q-1)\omega(L).\end{aligned}$$

But this is the recurrence that we wanted to verify, therefore, the proof is finished. □

Remark 2.4.2. The weights that we use in the equation (2.31) are constant along each antidiagonal in the plane. Indeed, the weight of a D-step which crosses the mth antidiagonal $x + y = m$ is $m - 1$. (In a sense, this gives a "force field" in $\mathbb{R}^2$, and $\alpha_{p,q}$ is the count of paths in $\mathcal{D}(p,q)$ which are weighted against this force field.)

Although Proposition 2.4.1 expresses $A_{p,q}$ as a combinatorial summation it does not give a combinatorial set of objects whose cardinality is given by $A_{p,q}$. It is now desirable to produce an explicit bijection between the set of signed involutions $I^{\pm}_{p,q}$

and the paths with certain labels. To do this, let us first introduce the following new notions.

Definition 2.4.3. A k-diagonal step (in $\mathbb{N}^2$) is a diagonal step L of the form $L = ((a,b),(a+1,b+1))$, where $a,b \in \mathbb{N}$ and $k = a+b+1$. (For an example, see Figure 2.2.)

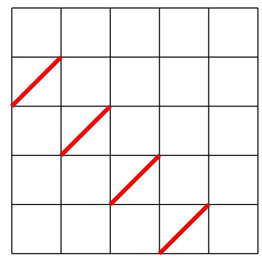
$L =$ 

Figure 2.2: The 4-diagonal steps in $\mathbb{N}^2$.

We proceed to define the weighted Delannoy paths.

Definition 2.4.4. By a labelled step we mean a pair (R,m), where $R \in \{N,E,D\}$ and m is a positive integer such that $m = 1$ if $R = N$ or $R = E$. A weighted (p,q) Delannoy path is a word of the form $R := R_1 \dots R_r$, where R_i's $(i = 1,\dots,r)$ are labeled steps $R_i = (L_i, m_i)$ such that

- $L_1 \dots L_r$ is a Delannoy path from $\mathcal{D}(p,q)$;
- if L_i $(1 \le i \le k)$ is a r-th diagonal step, then $1 \le m_i \le k-1$.

The set of all weighted (p,q) Delannoy paths is denoted by $\mathcal{P}(p,q)$.

Let $\pi = \pi^{(0)} = (i_1,j_1)\dots(i_k,j_k)d_1\dots d_{n-2k}$ be a signed involution from $I^{\pm}_{p,q}$. To construct the corresponding weighted path we proceed algorithmically as in the proof of Theorem 2.2.1.

First we look at where n appears in π. If it appears as a fixed point with a $+$ sign, then we draw a step between (p,q) and $(p-1,q)$. If it appears as a fixed point with a $-$ sign, then we draw a step between (p,q) and $(p,q-1)$. In the these cases, removing n from π results in an involution, that we denote by $\pi^{(1)}$, in either $I^{\pm}_{k,p-1,q}$ or $I^{\pm}_{k,p,q-1}$. If n appears as the second entry of one of the 2-cycles, say $(i_r, j_r) = (i, n)$ (for some r and i), then we draw a D-step between (p, q) and $(p-1, q-1)$, label it with i, and then we remove the two cycle (i, n) from π and reduce every number that is bigger than i in π by -1. Hence we obtain an element $\pi^{(1)}$ of $I^{\pm}_{k-1,p-1,q-1}$. We see from Theorem 2.2.1 that this algorithm results in a bijection. This proves the following theorem;

Theorem 2.4.5. *There is a bijection between the set of weighted (p, q) Delannoy paths and the set of signed (p, q) involutions. In particular, we have*

$$A_{p,q} = \sum_{R \in \mathcal{P}(p,q)} 1.$$

By abusing notation we denote the map that we obtain by ϕ, without giving any reference to the indices p and q. Let us demonstrate this bijection step by step in an example.

Example 2.32. Let $\pi = (1,4)(3,8)2^+5^+6^+7^-$. Then $p+q = 8$ and $p-q = 2$, hence $p = 5, q = 3$. Therefore, $\phi(\pi)$ is the path that is in the last picture of Figure 2.3.

It is interesting, though we postpone its investigation to an upcoming article that the weak order on $I^{\pm}_{p,q}$ is easy to describe using $\mathcal{P}(p, q)$. (The covering relations of the weak order, equivalently the action of the Richardson-Springer monoid on $I^{\pm}_{p,q}$, as described in [12, Figure 2.5] are easy to express in terms of certain simple operations on the paths in $\mathcal{P}(p, q)$.) In this notation, for example, the unique maximal element of $I^{\pm}_{p,q}$ is the path depicted in Figure 2.4. It corresponds to the maximal dimensional Borel orbit in $X = SL(p+q, \mathbb{C})/S(GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$. Closed Borel orbits in X correspond to the elements of $\mathcal{P}^0(p, q)$ in $\mathcal{P}(p, q)$, that is the set of paths with no diagonal step. In other words, the closed Borel orbits in X correspond to the (p, q) Grassmann paths. See [22] for details.

$$\pi = (1,4)(3,8)2^+5^+6^+7^- \longrightarrow \pi^{(1)} = (1,3)2^+4^+5^+6^-$$

$$\pi^{(1)} = (1,3)2^+4^+5^+6^- \longrightarrow \pi^{(2)} = (1,3)2^+4^+5^+$$

$$\pi^{(2)} = (1,3)2^+4^+5^+ \longrightarrow \pi^{(3)} = (1,3)2^+4^+$$

$$\pi^{(3)} = (1,3)2^+4^+ \longrightarrow \pi^{(4)} = (1,3)2^+$$

$$\pi^{(4)} = (1,3)2^+ \longrightarrow \pi^{(5)} = 1^+$$

$$\pi^{(5)} = 1^+ \longrightarrow \pi^{(6)} = \cdot$$

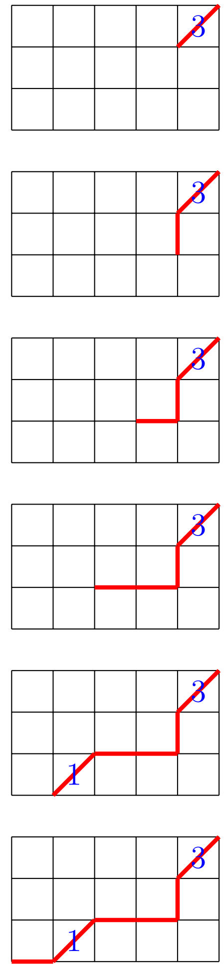


Figure 2.3: Algorithmic construction of ϕ.

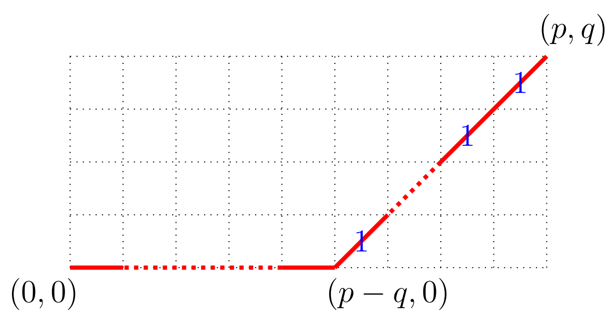


Figure 2.4: The unique maximal element of the weak order on $I^{\pm}_{p,q}$.

By using the combinatorial description of the weak order on clans and the bijection between clans and the lattice paths, we finish this section by the following figure where we illustrate the weak order on the corresponding lattice paths, see Figure 2.5 below.

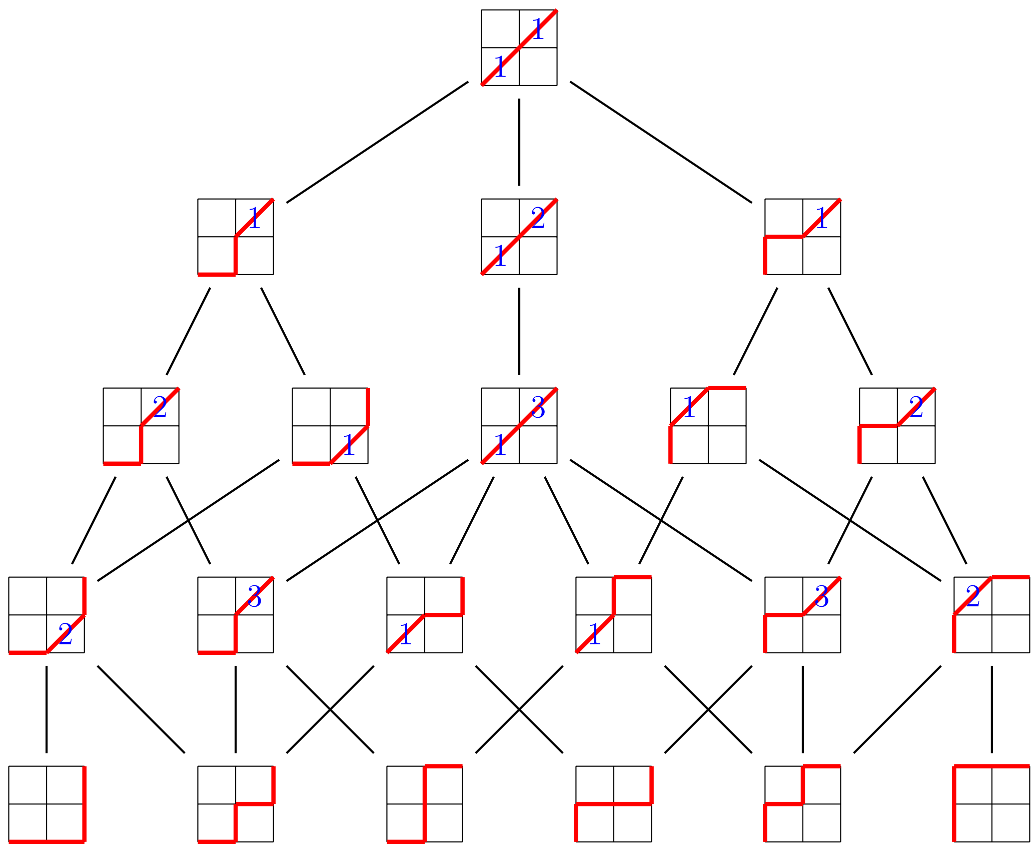


Figure 2.5: Weak order on $P(2,2)$

Remark 2.4.6. At this point, one would ask if it is possible to give the descriptions of the weak order and Bruhat order in terms of lattice paths. The answer to this question is yes. For the Bruhat order, the covering relations are described in [19, Theorem 2.8] and we are able to translate these relations into our language without difficulty. Here we skip the details for a future work.

2.5 Polynomial analogs of $A_{p,q}$'s

In this section, as an application of our combinatorial results, we consider t-analogs of the signed involution numbers $A_{p,q}$, $p, q = 0, 1, \ldots$.

Let us start by defining our first polynomial analog which is defined as follows:

$$D_{p,q}(t) := \frac{1}{t} \sum_{L \in \mathcal{D}(p,q)} t^{\omega(L)}. \tag{2.33}$$

This is the generating function, up to a factor of t, for the weight function ω as defined in (2.29). It follows from definitions that $D_{p,q}(t)$ obeys the recurrence

$$D_{p,q}(t) = D_{p-1,q}(t) + D_{p,q-1}(t) + t^{p+q-2} D_{p-1,q-1}(t^{p+q-1}).$$

Obviously,

$$\frac{\partial}{\partial t}(t D_{p,q}(t))|_{t=1} = A_{p,q}.$$

It is also obvious that $D_{p,q}(1)$ is nothing but the cardinality of the set $\mathcal{D}(p,q)$, the Delannoy number $D(p,q)$. The value at $t = 0$ of $D_{p,q}(t)$ is also easy to find and described below. Evaluating $D_{p,q}(t)$'s at other roots of unities also gives Delannoy numbers. It follows immediately from the definition (2.33) that

$$t(D_{p,q}(t) - D_{p-1,q}(t) - D_{p,q-1}(t)) = \sum_{L \in \mathcal{D}(p-1,q-1)} t^{(p+q-1)\omega(L)}.$$

Therefore, evaluating both sides at ζ and then dividing by ζ gives

$$\frac{1}{\zeta} \sum_{L \in \mathcal{D}(p-1,q-1)} \zeta^{(p+q-1)\omega(L)} = \frac{1}{\zeta} \sum_{L \in \mathcal{D}(p-1,q-1)} 1^{\omega(L)} = D(p-1, q-1)\zeta^{-1}.$$

This proves the following:

Proposition 2.5.1. The value of the difference $D_{p,q}(t) - D_{p-1,q}(t) - D_{p,q-1}(t)$ at a

$(p+q-1)$'th root of unity ζ is equal to $D(p-1,q-1)\zeta^{-1}$.

Next, we are going to have a careful look at the coefficients of $D_{p,q}(t)$. It turns out they are always sums of products of binomial coefficients. Let $n \geq 1$ denote the degree of $D_{p,q}(t)$ and set

$$D_{p,q}(t) = a_0 + a_1 t + \cdots + a_n t^n \qquad (a_i \in \mathbb{N}).$$

We start with the constant term. It is clear from our definition of a path $L \in \mathcal{D}(p,q)$ that $\omega(L) = 1$ if and only if π has at most one diagonal step, which occurs as an initial diagonal step (otherwise the weight would be greater than 1). Consequently,

$$a_0 = \rho_{p,q} + \rho_{p-1,q-1} = \binom{p+q}{q} + \binom{p+q-2}{q-1}. \tag{2.34}$$

Let $0 \leq a_1 < \cdots < a_r \leq p$ and $0 \leq b_1 < \cdots < b_r \leq q$ be two sequences. Next, we are going to focus on the set of paths $\mathcal{D}_{\mathrm{a,b}}(p,q)$ consisting of lattice paths $\pi \in L(p,q)$ with diagonal steps at $L_i = ((a_i, b_i), (a_i+1, b_i+1))$ for $i = 1, \ldots, r$. Clearly each element $L \in \mathcal{D}_{\mathrm{a,b}}(p,q)$ is a concatenation of $r+1$ lattice paths $L^{(1)}, \ldots, L^{(r)}$ each having no diagonal steps. More precisely, $L^{(i)} \in \mathcal{D}((a_i+1, b_i+1), (a_{i+1}, b_{i+1}))$ for $i = 0, \ldots, r$. Here $(a_0, b_0) = (0,0)$ and $(a_{r+1}, b_{r+1}) = (p,q)$. Clearly, the number of lattice paths in $\mathcal{D}_{\mathrm{a,b}}(p,q)$ is then

$$\prod_{i=0}^{r} |\mathcal{D}((a_i+1, b_i+1), (a_{i+1}, b_{i+1}))| = \prod_{i=0}^{r} \binom{a_{i+1}+b_{i+1}-a_i-b_i-2}{a_{i+1}-a_i-1}.$$

Note that the weight of any element $L \in \mathcal{D}_{\mathrm{a,b}}(p,q)$ is equal to

$$\omega(L) = \prod_{i=0}^{r} (a_i + b_i - 1).$$

Thus, by varying the number and choice of diagonal entries we obtain a formula for

$D_{p,q}(t)$:

$$D_{p,q}(t) = \sum_{r=0}^{\min\{q,p\}} \sum_{\substack{0\le a_1<\cdots<a_r\le p\\ 0\le b_1<\cdots<b_r\le q}} \left(\prod_{i=0}^{r} \binom{a_{i+1}+b_{i+1}-a_i-b_i-2}{a_{i+1}-a_i-1} \right) t^{\prod_{i=0}^{r}(a_i+b_i-1)}. \tag{2.35}$$

Remark 2.5.2. The polynomials $D_{p,q}(t)$ are in general not unimodal.

We will occasionally mention the "support of a signed involution" to mean the underlying involution without reference to its signs.

Remark 2.5.3. It is observed in [12] that the length function of the weak order on signed involutions $I^{\pm}_{p,q}$ agrees with that of the weak order on I_n. In other words, the length of $\pi \in I^{\pm}_{p,q}$ is equal to $L(\pi)$, where π is identified with its supporting involution. From now on, by abusing notation we are going to use $L(\cdot)$ for denoting the length function of Bruhat as well as the weak order on $I^{\pm}_{p,q}$.

Moreover, if $\pi \in I^{\pm}_{p,q}$ is the signed involution corresponding to the Borel orbit $\mathcal{O}$, then the dimension of $\mathcal{O}$ is equal to $L(\pi)+c$, where c is the dimension of (any) closed Borel orbit in $X = SL(n,\mathbb{C})/S(GL(p,\mathbb{C})\times GL(q,\mathbb{C}))$. Thus, studying $L(\pi)$ is equivalent to studying dimensions of $S(GL(p,\mathbb{C})\times GL(q,\mathbb{C}))$-orbits.

Now, we are ready to introduce our second t-analog which has algebro-geometric significance. Recall that the inclusion poset on Borel orbit closures in X is the Bruhat order on signed involutions $I^{\pm}_{p,q}$. This is a graded poset and its rank is equal to the length of its maximal element, which is $(1,n)\cdots(q,n+1-q)(q+1)^+\cdots(n-q)^+$. Thus,

$$\begin{aligned}\operatorname{rank}(I^{\pm}_{p,q}) &= \frac{((n-1)+\cdots+(n-q))+(1+\cdots+(q-1)))+q}{2}\\ &= \frac{nq-\frac{q(q+1)}{2}+\frac{q(q-1)}{2}+q}{2} = \frac{pq+q^2}{2}.\end{aligned}$$

Note that the dimension of X is $(n^2-1)-(p^2+q^2-1)=n^2-p^2-q^2=2pq$. Therefore, the smallest possible dimension for a Borel orbit in X is

$$\begin{aligned} f_{min}(p,q) &:= \dim X - \operatorname{rank}(I^{\pm}_{p,q}) \\ &= 2pq - \frac{pq+q^2}{2} = \frac{3pq-q^2}{2}. \end{aligned}$$

If we denote the dimension of the Borel orbit in X attached to $\pi \in I^{\pm}_{p,q}$ by $\dim \pi$, then

$$\dim \pi = L(\pi) + f_{min}(p,q).$$

Since $f_{min}(p,q)$ is constant (relative to p and q), the study of the function $\pi \mapsto \dim \pi$ is equivalent to studying the length function $L(\cdot)$ on $I^{\pm}_{p,q}$, so we consider the following (length) generating function

$$E_{p,q}(t) := \sum_{\pi \in I^{\pm}_{p,q}} t^{L(\pi)}.$$

Our goal is to find a recurrence for $E_{p,q}(t)$. To this end we go back to our ideas in Section 2.3. Indeed, there is an important consequence of the proof of Theorem 2.2.1, where we essentially constructed a bijection $\psi_k = (\psi_k(+), \psi_k(-), \psi_k(1), \dots, \psi_k(n-1))$ from

$$I^{\pm}_{k,p-1,q} \times I^{\pm}_{k,p,q-1} \times \underbrace{I^{\pm}_{k-1,p-1,q-1} \times \cdots \times I^{\pm}_{k-1,p-1,q-1}}_{(n-1\text{-copies})}$$

to $I^{\pm}_{k,p,q}$.

In the light of Corollary 2.4.1, we obtain the bijection

$$\psi : I^{\pm}_{p,q-1} \times I^{\pm}_{p-1,q} \times \underbrace{I^{\pm}_{p-1,q-1} \times \cdots \times I^{\pm}_{p-1,q-1}}_{(n-1\text{-copies})} \longrightarrow I^{\pm}_{p,q} \tag{2.36}$$

defined by $\psi = (\psi(+), \psi(-), \psi(1), \dots, \psi(n-1))$.

Next, we analyze the effect of maps $\psi(\pm)$ and $\psi(i)$, $i = 1, \ldots, n-1$ on the length of $\pi \in I^{\pm}_{p,q}$.

We know from Section 1.1 that $L(\pi)$ is equal to $(\ell(\pi)+k)/2$, where k is the number of 2-cycles in π and $\ell(\pi)$ is the number of inversions in π viewed as a permutation. Thus, if n is a fixed point of π, then removing it from π has no effect on the length:

$$L(\psi(\pm)^{-1}(\pi)) = L(\pi). \tag{2.37}$$

For $\psi(i)$'s, it is more interesting. Suppose π has the standard form $\pi = (i_1, j_1) \cdots (i_k, j_k)c_1 \ldots c_{n-k}$. If n appears in the 2-cycle $(i_r, j_r) = (i, n)$, then by removing (i_r, j_r) from π we loose $n - i$ inversions of the form $n > j$ and we loose $n - i - 1$ inversions of the form $j > i$. Moreover, we loose one 2-cycle. Therefore,

$$\begin{aligned} L(\psi(i)^{-1}(\pi)) &= \frac{\ell(\pi) + k - (2n - 2i - 1) - 1}{2} = \frac{\ell(\pi) + k}{2} - (n - i) \\ &= L(\pi) - (n - i). \end{aligned} \tag{2.38}$$

First we partition $I^{\pm}_{p,q}$ into three disjoint sets $\psi(+)(I^{\pm}_{p-1,q})$, $\psi(-)(I^{\pm}_{p,q-1})$, and $\psi(i)(I^{\pm}_{p-1,q-1})$ $(i = 1, \ldots, n-1)$, then re-organize the sums by using our observations (2.37) and (2.38):

$$\begin{aligned} E_{p,q}(t) &= \sum_{\pi \in \psi(+)(I^{\pm}_{p-1,q})} t^{L(\pi)} + \sum_{\pi \in \psi(-)(I^{\pm}_{p,q-1})} t^{L(\pi)} + \sum_{i=1}^{n-1} \sum_{\pi \in \psi(i)(I^{\pm}_{p-1,q-1})} t^{L(\pi)} \\ &= \sum_{\pi \in I^{\pm}_{p-1,q}} t^{L(\pi)} + \sum_{\pi \in I^{\pm}_{p,q-1}} t^{L(\pi)} + \sum_{i=1}^{n-1} \sum_{\pi \in I^{\pm}_{p-1,q-1}} t^{L(\pi)+(n-i)} \\ &= E_{p-1,q}(t) + E_{p,q-1}(t) + (t + t^2 + \cdots + t^{n-1}) E_{p-1,q-1}(t). \end{aligned}$$

The coefficient of the last term is equal to $[n]_t - 1$, where $[n]_t$ stands for the t-analog of the natural number n:

$$[n]_t := \frac{t^n - 1}{t - 1}.$$

Thus we obtained the proof of the following.

Proposition 2.5.4. The family of polynomials $E_{p,q}(t)$, $p, q \geq 0$ satisfies the following recurrence:

$$E_{p,q} = E_{p-1,q}(t) + E_{p,q-1}(t) + ([q+p]_t - 1)E_{p-1,q-1}(t).$$

Remark 2.5.5. 1. The polynomials $E_{p,q}(t)$ are unimodal.

2. It appears that the sequence $(E_{n,n}(-1))_{n\geq 1}$ is the sequence of number of "grand Motzkin paths" of length n. The sequence $(E_{n,n}(0))_{n\geq 1}$ is the sequence of number of "central binomial coeffients" $\binom{2n}{n}$, $n \geq 1$. (For other interpretations of these sequences, see The On-Line Encyclopedia of Integer Sequences, https://oeis.org.)

3. There is another closely related t-analogue. We define $\widetilde{E}_{p,q}(t)$'s by the recurrence

$$\widetilde{E}_{p,q}(t) = \widetilde{E}_{p-1,q}(t) + \widetilde{E}_{p,q-1}(t) + [q+p-1]_t \widetilde{E}_{p-1,q-1}(t).$$

Similarly to $E_{p,q}(t)$, $\widetilde{E}_{p,q}(t)$ is a unimodal polynomial as well. Both families have interesting specializations.

We end this chapter by defining our third and last t-analog function. The power series $F_k(x) = \sum_{q\geq k} \alpha_{k,p,q} x^q$ played a significant role in our determination of the generating series of $A_{p,q}$'s. We consider a closely related generating function, which, perhaps as an initial choice would be the most natural t-analog; define

$$G_{p,q}(t) := \sum_{k=0}^{q} \alpha_{k,p,q} t^k.$$

By definition, the coefficient of t^k in $G_{p,q}(t)$ is the number of signed involutions from $I^{\pm}_{p,q}$ with k fixed points. Equivalently, $G_{p,q}(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} |I^{\pm}_{k,p,q}| t^k$. The proof follows from the fact that the bijection ϕ maps an element of $I^{\pm}_{k,q,p}$ to a weighted path in $\mathcal{P}(p,q)$ with k diagonal steps.

Proposition 2.5.6. Let p and q be two nonnegative integers. Then

$$G_{p,q}(t) = \sum_{L \in \mathcal{P}(p,q)} t^{\#\text{of diagonal steps in } \pi}.$$

Proposition 2.5.7. The family $\{G_{p,q}(t)\}_{p,q \geq 0}$ satisfies the following recurrence relation:

$$G_{p,q}(t) = G_{p,q-1}(t) + G_{p-1,q}(t) + (p+q-1)tG_{p-1,q-1}(t)$$

with initial conditions $G_{p,0}(t) = G_{0,q}(t) = 1$ for all $p, q \geq 0$.

Proof. The proof is a straightforward application of the recurrences in Theorem 2.2.1. □

As a final remark for this chapter we state the following:

Remark 2.5.8. $G_{p,q}(t)$ seems to be unimodal also.

Chapter 3

Type BI

It follows immediately from our convention 1.3 that throughout this chapter it is essential to assume that $0 \leq q < p$. Due to Cartan's classification of involutions on algebraic groups it is known that there is only one involutory automorphism associated with orthogonal decomposition for $SO(2n+1, \mathbb{C})$, which is given as follows:

$$\sigma(g) = I_{p,2q+1,p} g I_{p,2q+1,p},$$

refer to the section 1.1 for the notation here. See [7] for details.

In this case, it is easy to show that the fixed point subgroup is the Levi subgroup $S(O(2p, \mathbb{C}) \times O(2q+1, \mathbb{C}))$.

Recall that $S(O(2p, \mathbb{C}) \times O(2q+1, \mathbb{C}))$ orbits in the flag variety are parametrized by symmetric $(2p, 2q+1)$ clans. In this chapter, our goal is to find various associated (bivariate) generating functions.

3.1 Counting symmetric $(2p, 2q+1)$ clans with k pairs.

Although we will be dealing with $(2p, 2q+1)$ clans, we still denote $p+q$ by n. Accordingly the number of symmetric $(2p, 2q+1)$ clans is denoted by $B_{p,q}$.

By the proof of Lemma 2.0.3 we know that there is a bijection, denoted by $\widetilde{\varphi}$, between the set of all $(2p, 2q+1)$ clans of order $2n+1$ and the set of all signed $(2p, 2q+1)$ involutions in S_{2n+1}. We have a number of simple observations regarding this bijection.

First of all, we observe that if π is a signed $(2p, 2q+1)$ involution such that $\widetilde{\varphi}(\gamma) = \pi$, where γ is a symmetric $(2p, 2q+1)$ clan, then the following holds true:

- if (i, j) with $1 \leq i < j \leq 2n+1$ is a 2-cycle of π, then $n+1 \notin \{i, j\}$ and $(2n+2-j, 2n+2-i)$ is a transposition of π also.

Secondly, we see from its construction that $\widetilde{\varphi}$ maps a $(2p, 2q+1)$ clan with k pairs to a signed $(2p, 2q+1)$ involution with k transpositions. Let us denote the set of all such involutions by $I^{\mathrm{ort}}_{k,p,q}$ and we define $\beta_{k,p,q}$ as the cardinality

$$\beta_{k,p,q} := |I^{\mathrm{ort}}_{k,p,q}|.$$

Remark 3.1.1. If $\pi \in I^{\mathrm{ort}}_{k,p,q}$, then in the corresponding clan there are $2p-2q-1$ more $+$'s than $-$'s. Notice that the inequality $2p-2q-1 \leq 2p+2q+1-2k$ implies that $0 \leq k \leq 2q+1$.

It follows from the note in Remark 3.1.1 and the fact that $\widetilde{\varphi}$ is a bijection, the number of symmetric $(2p, 2q+1)$ clans is given by

$$B_{p,q} = \sum_{l=0}^{q} (\beta_{2l,p,q} + \beta_{2l+1,p,q}).$$

Our goal in this section is to record a formula for $B_{p,q}$ that depends only on p and q. To this end, first we need to determine the number of $\pm$ symbols in a symmetric $(2p, 2q+1)$ clan.

Lemma 3.1.2. If $\gamma = c_1 \dots c_{2n+1}$ is a symmetric $(2p, 2q+1)$ clan, then either $c_{n+1} = +$ or $c_{n+1} = -$.

Proof. First, assume that γ has even number of pairs. Let k denote this number, $k = 2l$. Let α, β, respectively, denote the number of $+$'s and $-$'s in γ. Then we have

$$\alpha + \beta = 2p + 2q + 1 - 4l \qquad \text{and} \qquad \alpha - \beta = 2p - 2q - 1.$$

It follows that

$$\alpha = 2p - 2l \qquad \text{and} \qquad \beta = 2q - 2l + 1,$$

so, in γ there are odd number of $-$'s and there are even number of $+$'s. As a consequence we see that c_{n+1} is a $-$.

Next, assume that γ has an odd number of pairs, that is $k = 2l + 1$. Arguing as in the previous case we see that there is an odd number of $+$'s, hence c_{n+1} is a $+$. This finishes the proof. $\square$

We learn from the proof of Lemma 3.1.2 that it is important to analyze the parity of pairs, so we record the following corollary of the proof for a future reference.

Corollary 3.1.3. Let k denote the number of pairs in a symmetric $(2p, 2q+1)$ clan γ. Then;

i) If $k = 2l$ $(0 \le l \le q)$, then the number of $+$ symbols in γ is $2(p - l)$.

ii) If $k = 2l + 1$ $(0 \le l \le q)$, then the number of $-$ symbols in γ is $2(q - l)$.

Our next task is determining the number of possible ways of placing k pairs to build from scratch a symmetric $(2p, 2q+1)$ clan

$$\gamma = c_1 \cdots c_n c_{n+1} c_{n+2} \cdots c_{2n+1} \qquad (\text{with } c_{n+1} = \pm).$$

To this end, we start with defining some interrelated sets.

$$\begin{aligned}
PI_{1,1} :=& \{((i,j),(2n+2-j, 2n+2-i)) \mid 1 \le i < j \le n\},\\
PI_{1,2} :=& \{((i,j),(2n+2-j, 2n+2-i)) \mid 1 \le i < n+1 < j \le 2n+1\},\\
PI_1 :=& PI_{1,1} \cup PI_{1,2},\\
PI_2 :=& \{(i,j) \mid 1 \le i < n+1 < j \le 2n+1,\ i+j = 2n+2\}.
\end{aligned}$$

We view PI_1 as the set of placeholders for two distinct pairs that determine each other in γ. The set PI_2 corresponds to the list of stand alone pairs in γ. In other words, if $(i,j) \in PI_2$, then $c_i = c_j$ and $j = 2n+1-i+1$.

Example 3.1. Let us show what PI_1 and PI_2 correspond to with a concrete example. If γ is the symmetric $(8,7)$ clan

$$\gamma = 7\,7 + \,0\,8 + \,9 - \,8 + \,9\,0 + \,2\,2,$$

then $PI_{1,1} = \{((1,2),(14,15))\}$, $PI_{1,2} = \{((5,9),(7,11))\}$, $PI_2 = \{(4,12)\}$.

If (c_i, c_j) is a pair in the symmetric clan γ and if (i,j) is an element of PI_2, then we call (c_i, c_j) a pair of type PI_2. If x is a pair of pairs of the form $((c_i, c_j), (c_{2n+2-j}, c_{2n+2-i}))$ in a symmetric clan γ and if $((i,j),(2n+2-j, 2n+2-i)) \in PI_{1,s}$ $(s \in \{1,2\})$, then we call x a pair of pairs of type $PI_{1,s}$. If there is no need for precision, then we will call x a pair of pairs of type PI_1.

Clearly, if $|PI_1| = b$ and $|PI_2| = a$, then $2b + a = k$ is the total number of pairs in our symmetric clan γ. To see in how many different ways these pairs of indices can be situated in γ, we start with choosing k spots from the first n positions in $\gamma = c_1 \cdots c_{2n+1}$. Obviously this can be done in $\binom{n}{k}$ many different ways. Next, we count different ways of choosing b pairs within the k spots to place the b pairs of pairs of type PI_1. The number of possibilities for this count is $\binom{k}{2b}$. Observe that choosing a pair from PI_1 is equivalent to choosing (i, j) for the pairs of pairs in $PI_{1,1}$ and choosing $(i, 2n+2-j)$ for the pairs of pairs in $PI_{1,2}$. More explicitly, we first choose b pairs among the $2b$ elements and then place them on b spots; this can be done in $\binom{2b}{b} b!$ different ways. Once this is done, finally, the remaining spots will be filled by the a pairs of type PI_2. This can be done in only one way. Therefore, in summary, the number of different ways of placing k pairs to build a symmetric $(2p, 2q+1)$ clan γ is given by

$$\binom{n}{k} \sum_{b=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2b} \binom{2b}{b} b!, \quad \text{or equivalently,} \quad \binom{n}{k} \sum_{b=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2b} \frac{(2b)!}{b!}.$$

In conclusion, we have the following preparatory result.

Theorem 3.1.4. *The number symmetric $(2p, 2q+1)$ clans with k pairs is given by*

$$\beta_{k,p,q} = \begin{cases} \binom{n-2l}{p-l} \binom{n}{2l} a_{2l} & \text{if } k = 2l; \\ \binom{n-(2l+1)}{p-(l+1)} \binom{n}{2l+1} a_{2l+1} & \text{if } k = 2l+1, \end{cases} \tag{3.2}$$

where

$$a_{2l} := \sum_{b=0}^{l} \binom{2l}{2b} \frac{(2b)!}{b!} \quad \text{and} \quad a_{2l+1} := \sum_{b=0}^{l} \binom{2l+1}{2b} \frac{(2b)!}{b!}. \tag{3.3}$$

Consequently, the total number of symmetric $(2p, 2q+1)$ clans (or the cardinality of *the set $B(p,q)$) is given by*

$$B_{p,q} = \sum_{l=0}^{q} \left[\binom{n-2l}{p-l} \binom{n}{2l} a_{2l} + \binom{n-(2l+1)}{p-(l+1)} \binom{n}{2l+1} a_{2l+1} \right].$$

Proof. As clear from the statement of our theorem, we will consider the two cases where k is even and where k is odd separately. We already computed the numbers of possibilities for placing k pairs, which are given by a_{2l} and a_{2l+1}, but we did not finish counting the number of possibilities for placing the signs.

1. $k = 2l$ for $0 \leq l \leq q$. In this case, by Lemma 3.1.2, we see that the number of $+$ signs is $\alpha := 2p - 2l = 2(p-l)$. Notice that because of symmetry condition it is enough to focus on the first n spots to place $\pm$ signs. Thus, there are $\binom{n-2l}{p-l}$ possibilities to place $\pm$ signs.

2. $k = 2l+1$ for $0 \leq l \leq q$. In this case, it follows from Lemma 3.1.2 that the entry in the $(n+1)$-th place is $+$. By using an argument as before, we see that there are $\binom{n-(2l+1)}{p-(l+1)}$ possibilities to place $\pm$ signs.

This finishes the proof. $\square$

Note here that the numbers a_{2l} and a_{2l+1} in Theorem 3.1.4 ($l = 0, 1, \ldots, q$) are special values of certain hypergeometric functions. More precisely,

$$\begin{aligned} a_{2l} &= \left(\frac{-1}{4}\right)^{-l} U\left(-l, \frac{1}{2}, -\frac{1}{4}\right), \\ a_{2l+1} &= \left(\frac{-1}{4}\right)^{-l} U\left(-l, \frac{3}{2}, -\frac{1}{4}\right). \end{aligned}$$

where $U(a,b,z)$ is the confluent hypergeometric function of the second kind. Such functions form one of the two distinct families of hypergeometric functions which solves the Kummer's differential equation

$$zy'' + (c - z)y' - ay = 0 \tag{3.4}$$

for some constants a and c. Kummer's ODE has a regular singular point at the origin and it has an irregular singularity at infinity.

The proof of the following lemma follows from Proposition 3.1.4 and a straightforward calculation.

Lemma 3.1.5. Let $\beta_{k,p,q}$ denote the number of symmetric $(2p, 2q+1)$ clans with k pairs. In this case, the following equality holds true:

$$\beta_{1,p,q} = p\beta_{0,p,q}.$$

The following formula for the number of Borel orbits in $SO(2n+1,\mathbb{C})/S(O(2p,\mathbb{C})\times O(2q+1))$ for $q = 0, 1, 2$ is now a simple consequence of our Theorem 3.1.4.

$$\begin{aligned}
B_{p,0} &= p + 1 \\
B_{p,1} &= (p+1)a_0 + p(p+1)a_1 + \frac{p(p+1)}{2}a_2 + \frac{p(p+1)(p-1)}{6}a_3 \\
&= \frac{7p^3 + 15p^2 + 14p + 6}{6} \\
B_{p,2} &= \binom{p+2}{2}\left(\frac{81p^3 + 22p^2 + 137p + 60}{60}\right) \\
&= \frac{81p^5 + 265p^4 + 365p^3 + 515p^2 + 454p + 120}{120}
\end{aligned}$$

where $a_0 = a_1 = 1, a_2 = 3, a_3 = 7, a_4 = 25, a_5 = 81$, and so on.

Theorem 3.1.4 tells us that, for every fixed q, the integer $B_{p,q}$ can be viewed as a specific value of a polynomial function of p. However, it is already apparent from the case of $q = 1$ that this polynomial may have non-integer coefficients. We conjecture that $q = 0$ is the only case where $p \mapsto B_{p,q}$ is a polynomial function with integral coefficients. We conjecture also that for every nonnegative integer q, as a polynomial in p, $B_{p,q}$ is unimodal.

The formula for $B_{p,q}$ that is derived in Theorem 3.1.4 is not optimal in the sense that it is hard to write down a closed form of its generating function this way. Of course, the complication is due to the form of $\beta_{k,p,q}$, where k is even or odd. Both of the cruces are resolved by considering the recurrences; we will present our results in the next section.

3.2 Recurrences

We start with some easy recurrences. It follows from obvious binomial identities and our formulas in Theorem 3.1.4 that we have

$$\begin{aligned}\beta_{2l,p-1,q} &= \binom{n-1-2l}{p-1-l}\binom{n-1}{2l}a_{2l} \\ &= \frac{p-l}{p+q}\binom{n-2l}{p-l}\binom{n}{2l}a_{2l} = \frac{p-l}{p+q}\beta_{2l,p,q}.\end{aligned}$$

By using the similar argument, we can prove the following Lemma.

Lemma 3.2.1. Let p and q be two positive integers, and l be a nonnegative integer. In this case, whenever both sides of the following equations are defined, they hold true:

$$\beta_{2l,p-1,q} = \frac{p-l}{p+q}\beta_{2l,p,q}, \tag{3.5}$$

$$\beta_{2l,p,q-1} = \frac{q-l}{p+q}\beta_{2l,p,q}, \tag{3.6}$$

$$\beta_{2l+1,p-1,q} = \frac{p-l-1}{p+q}\beta_{2l+1,p,q}, \tag{3.7}$$

$$\beta_{2l+1,p,q-1} = \frac{q-l}{p+q}\beta_{2l+1,p,q}. \tag{3.8}$$

Here, observe that l does not change in them. In the sequel, we will find other recurrences that run over l's. Towards this end, the following lemma, whose proof is simple, will be useful.

Lemma 3.2.2. Let a_k denote the numbers as in (3.3). If $k \geq 2$, then we have

$$a_k = a_{k-1} + 2(k-1)a_{k-2}. \tag{3.9}$$

By using (3.9) we find relations between $\beta_{k,p,q}$'s. Let k be an even number of the form $k = 2l$. Then we find that

$$\begin{aligned}
\beta_{2l,p,q} =& \binom{n-2l}{p-l}\binom{n}{2l} a_{2l} \\
=& \binom{n-2l}{p-l}\binom{n}{2l}(a_{2l-1} + 2(2l-1)a_{2l-2}) \\
=& \binom{n-2l}{p-l}\binom{n}{2l} a_{2l-1} + 2(2l-1)\binom{n-2l}{p-l}\binom{n}{2l} a_{2l-2} \\
=& \frac{n-l-p+1}{2l}\beta_{2l-1,p,q} + 2\frac{(p-l+1)(n-l+1-p)}{2l}\beta_{2l-2,p,q} \\
=& \frac{q-l+1}{2l}\beta_{2l-1,p,q} + 2\frac{(p-l+1)(q-l+1)}{2l}\beta_{2l-2,p,q}.
\end{aligned} \tag{3.10}$$

In a similar manner, for an odd number of the form $k = 2l+1$, we find that

$$\begin{aligned}
\beta_{2l+1,p,q} =& \binom{n-2l-1}{p-l-1}\binom{n}{2l+1} a_{2l+1} \\
=& \binom{n-2l-1}{p-l-1}\binom{n}{2l+1}(a_{2l} + 2(2l)a_{2l-1}) \\
=& \binom{n-2l-1}{p-l-1}\binom{n}{2l+1} a_{2l} + 2(2l)\binom{n-2l-1}{p-l-1}\binom{n}{2l+1} a_{2l-1} \\
=& \frac{p-l}{2l+1}\beta_{2l,p,q} + 2\frac{(p-l)(q-l+1)}{2l+1}\beta_{2l-1,p,q}.
\end{aligned} \tag{3.11}$$

Now we two recurrences (3.10) and (3.11) mixing the terms $\beta_{k,p,q}$ for even and odd k. To separate the parity, we rework on our initial recurrence (3.9).

Lemma 3.2.3. For all $1 \leq l \leq q-1$, the following recurrences:

$$a_{2l+2} = (8l+3)a_{2l} + 4(2l)(2l-1)a_{2l-2} \tag{3.12}$$

$$a_{2l+3} = (8l+7)a_{2l+1} + 4(2l+1)(2l)a_{2l-1} \tag{3.13}$$

with $a_0 = 1, a_1 = 1$ are satisfied.

Proof. We will give a proof for the former equation here. The latter can be proved in a similar way.

We start with splitting the recurrence (3.9) into two recurrences:

$$a_{2l+1} = a_{2l} + 2(2l)a_{2l-1} \tag{3.14}$$

$$a_{2l} = a_{2l-1} + 2(2l-1)a_{2l-2}. \tag{3.15}$$

On one hand it follows from equation (3.15) that we have

$$a_{2l-1} = a_{2l} - 2(2l-1)a_{2l-2}.$$

Plugging this into equation (3.14) yields

$$a_{2l+1} = a_{2l} + 2(2l)(a_{2l} - 2(2l-1)a_{2l-2}) \text{ or}$$

$$a_{2l+1} = (1+2(2l))a_{2l} - 4(2l)(2l-1)a_{2l-2}).$$

On the other hand, we know that

$$a_{2l+2} = a_{2l+1} + 2(2l+1)a_{2l}.$$

If we plug this into the previous equation, then we obtain

$$\begin{aligned}a_{2l+2} =&(1+2(2l))a_{2l} - 4(2l)(2l-1)a_{2l-2} + 2(2l+1)a_{2l}\\ =&(8l+3)a_{2l} - 4(2l)(2l-1)a_{2l-2} \qquad (1 \leq l \leq q-1),\end{aligned}$$

which finishes the proof of our claim. □

Next, by the help of Lemma 3.2.3, we obtain a recurrence relation for $\beta_{k,p,q}$'s where all of k's are even numbers.

$$\begin{aligned}\beta_{2l+2,p,q} =&\binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l+2}\\ =&\binom{n-2l-2}{p-l-1}\binom{n}{2l+2}((8l+3)a_{2l} - 4(2l)(2l-1)a_{2l-2})\\ =&(8l+3)\binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l} + 4(2l)(2l-1)\binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l-2}\\ =&(8l+3)\frac{(p-l)(q-l)}{(n-2l)(n-2l-1)}\binom{n-2l}{p-l}\frac{(n-2l)(n-2l-1)}{(2l+2)(2l+1)}\binom{n}{2l}a_{2l}\\ &-4(2l)(2l-1)\frac{(p-l)(p-l+1)(q-l)(q-l+1)}{(n-2l+2)(n-2l+1)(n-2l)(n-2l-1)}\binom{n-2l}{p-l+1}\\ &\frac{(n-2l+2)(n+1-2l)(n-2l)(n-2l-1)}{(2l+2)(2l+1)(2l)(2l-1)}\binom{n}{2l-2}a_{2l-2}\\ =&(8l+3)\frac{(p-l)(q-l)}{(2l+2)(2l+1)}\beta_{2l,p,q} - 4\frac{(p-l)(p-l+1)(q-l)(q-l+1)}{(2l+2)(2l+1)}\beta_{2l-2,p,q}.\end{aligned} \tag{3.16}$$

The proof of the following recurrence follows from similar arguments.

$$\begin{aligned}\beta_{2l+3,p,q} = (8l+7)&\frac{(q-l)(p-l-1)}{(2l+3)(2l+2)}\beta_{2l+1,p,q}\\ &-4\frac{(p-l)(p-l-1)(q-l)(q-l+1)}{(2l+3)(2l+2)}\beta_{2l-1,p,q}.\end{aligned} \tag{3.17}$$

3.3 Generating Functions

As we mentioned before, we are looking for the closed form of the generating function

$$B(y,z) = \sum_{p\geq 0} B_p(1,y)z^p,$$

where

$$B_{p,q}(x) = \sum_{l=0}^{q} (\beta_{2l,p,q}x^{q-l} + \beta_{2l+1,p,q}x^{q-l}) \quad \text{and} \quad B_p(x,y) = \sum_q B_{p,q}(x)y^q.$$

In particular, we are looking for an expression of $B_{p,q}(1)$ which is simpler than the one that is given in Theorem 3.1.4.

Obviously,

$$B_{p,q-1}(x) = \sum_{l=0}^{q-1} (\beta_{2l,p,q-1}x^{q-l-1} + \beta_{2l+1,p,q-1}x^{q-l-1}).$$

It follows from Lemma 3.2.1 that

$$\begin{aligned}B_{p,q}(x) =& (\beta_{2q,p,q} + \beta_{2q+1,p,q})x^0 + \sum_{l=0}^{q-1} (\beta_{2l,p,q}x^{q-l} + \beta_{2l+1,p,q}x^{q-l}) \\ =& (\beta_{2q,p,q+1} + \beta_{2q+1,p,q+1}) + \sum_{l=0}^{q-1} (p+q)\left(\frac{\beta_{2l,p,q-1}}{q-l}x^{q-l} + \frac{\beta_{2l+1,p,q-1}}{q-l}x^{q-l}\right).\end{aligned}$$

Taking the derivative of both sides of the above equation gives us that

$$B'_{p,q}(x) = \sum_{l=0}^{q-1} (p+q)\left(\beta_{2l,p,q-1}x^{q-l-1} + \beta_{2l+1,p,q-1}x^{q-l-1}\right),$$

or equivalently, gives that

$$B'_{p,q}(x) = (p+q)B_{p,q-1}(x). \tag{3.18}$$

The differential equation (3.18) leads to a PDE for our initial generating function $B_p(x,y)$:

$$
\begin{aligned}
\frac{\partial}{\partial x}(B_p(x,y)) =& \frac{\partial}{\partial x}\left[\sum_{q\geq 0} B_{p,q}(x)y^q\right] = B^{'}_{p,0}y^0 + \sum_{q\geq 1} B^{'}_{p,q}(x)y^q \qquad (B^{'}_{p,0}=0)\\
=& \sum_{q\geq 1}(p+q)B_{p,q-1}(x)y^q = py\sum_{q\geq 1} B_{p,q-1}(x)y^{q-1} + y\sum_{q\geq 1} qB_{p,q-1}(x)y^{q-1}\\
=& pyB_p(x,y) + y\left(\frac{\partial}{\partial y}(y\cdot B_p(x,y))\right)\\
=& y^2\frac{\partial}{\partial y}B_p(x,y) + yB_p(x,y) + pyB_p(x,y).
\end{aligned}
$$

By the last equation we obtain the following PDE:

$$
\frac{\partial}{\partial x}B_p(x,y) - y^2\frac{\partial}{\partial y}B_p(x,y) = y(1+p)B_p(x,y). \tag{3.19}
$$

The general solution $S_2(x,y)$ of (3.19) is given by

$$
S_2(x,y) = \frac{1}{y^{p+1}}F_2\left(\frac{1-xy}{y}\right), \tag{3.20}
$$

where $F_2(z)$ is some function in one-variable. We want to choose $F_2(z)$ in such a way that $S_2(x,y) = B_p(x,y)$ holds true. To do so, first, we look at some special values of $B_p(x,y)$.

If let $x=0$, then $B_p(0,y) = \sum_{q\geq 0} B_{p,q}(0)y^p$ and $B_{p,q}(0) = 2(\beta_{2q,q,p} + \beta_{2q+1,q,p})$ for all $q>0$. Also, recall from the introduction that if $q=0$, then $B_{p,q} = p+1$. Thus, we ask from $F_2(z)$ that it satisfies the following equation

$$
\frac{1}{y^{p+1}}F_2\left(\frac{1}{y}\right) = (p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})y^q,
$$

or that

$$F_2\left(\frac{1}{y}\right) = y^{p+1}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})y^q\right). \tag{3.21}$$

Therefore, we see that our generating function is given by

$$\begin{aligned}
B_p(x,y) &= \frac{1}{y^{p+1}}F_2\left(\frac{1}{y/(1-xy)}\right)\\
&= \frac{1}{y^{p+1}}\left(\frac{y}{1-xy}\right)^{p+1}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})\left(\frac{y}{1-xy}\right)^q\right)\\
&= \left(\frac{1}{1-xy}\right)^{p+1}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})\left(\frac{y}{1-xy}\right)^q\right). \quad (3.22)
\end{aligned}$$

To get a more precise information about $B_{p,q}$'s we substitute $x = 1$ in (3.22):

$$B_p(1,y) = \frac{1}{(1-y)^{p+1}}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})\left(\frac{y}{1-y}\right)^q\right),$$

or

$$(1-y)^{p+1}B_p(1,y) = (p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})\left(\frac{y}{1-y}\right)^q. \tag{3.23}$$

Now we apply the transformation $y \mapsto z = y/(1-y)$ in (3.23):

$$\frac{1}{(1+z)^{p+1}}B_p\left(1, \frac{z}{1+z}\right) = (p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})z^q. \tag{3.24}$$

This finishes the proof of Theorem 3.3.1 since $B_p\left(1, \frac{z}{1+z}\right) = f_p(z)$.

Theorem 3.3.1. *If $f_p(z)$ denotes the polynomial that is obtained from $B_p(1,y)$ by* *the transformation $y \leftrightarrow z/(1-z)$, then we have*

$$f_p(z) = (1+z)^{p+1}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,p,q} + \beta_{2q+1,p,q})z^q\right), \tag{3.25}$$

where a_k's are as in Theorem 3.1.4.

Chapter 4

Type CII

Due to Cartan's classification of involutions on algebraic groups it is known that the involution associated with orthogonal decomposition for $Sp(2n, \mathbb{C})$ is given by

$$\sigma(g) = K_{p,q} g K_{p,q},$$

refer to the section 1.1 for the notation here. See [7], for more details. In this case, it is easy to show that the fixed point subgroup is the Levi subgroup $Sp(2p, \mathbb{C}) \times Sp(2q + 1, \mathbb{C})$.

Recall that $Sp(2p, \mathbb{C}) \times Sp(2q+1, \mathbb{C})$ orbits in the flag variety are parametrized by symp-symmetric $(2p, 2q)$ clans. In this chapter, our goal is to determine the formula for the number of the set of such clans and its relation to lattice path enumeration.

Remark 4.0.1. For this chapter of this present paper, thanks to our Convention 1.3 again, we assume that p and q are nonnegative integers such that $p \geq q$.

4.1 Counting Symp-symmetric $(2p, 2q)$ clans with k-pairs.

Recall that a symp-symmetric $(2p, 2q)$ clan $\gamma = c_1 \dots c_{2n}$ is a symmetric clan such that $c_i \neq c_{2n+1-i}$ whenever c_i is a number. In this chapter, we are going to find various generating functions and combinatorial interpretations for the number $\Gamma_{p,q}$ of symp-symmetric $(2p, 2q)$ clans. We start by stating a simple lemma that tells about the involutions corresponding to symp-symmetric clans.

Lemma 4.1.1. Let $\gamma = c_1 c_2 \dots c_{2n}$ be a symp-symmetric $(2p, 2q)$ clan. If $\pi \in S_{2n}$ is the associated involution with γ, then there are even number of 2-cycles in π.

Proof. First, notice that if for some $1 \leq i < j \leq 2n$ the numbers c_i and c_j form a pair, that is to say a 2-cycle in π, then by symmetry c_{2n+1-i} and c_{2n+1-j} form a pair in π as well. In addition, by the condition that is requiring for all natural c_i's that $c_i \neq c_{2n+1-i}$, c_i and c_{2n+1-j} cannot form a pair in π. Therefore, if we have a pair (c_i, c_j) in π, then we must also have another pair (c_{2n+1-j}, c_{2n+1-i}) which is different from (c_i, c_j). Said differently, the number of 2-cycles in π must be even. □

In the light of Lemma 4.1.1, we will focus on the subset $I^{sp}_{k,p,q} \subset S_{2n}$ consisting of involutions π whose standard cycle decomposition is of the form

$$\pi = (i_1 j_1) \dots (i_{2k} j_{2k}) d_1 \dots d_{2n-4k}.$$

Furthermore, we assume the fixed points of π are labeled by the elements of $\{+, -\}$ in such a way that there are $2p - 2q$ more $+$'s than $-$'s and we want the following conditions be satisfied:

1. $k \leq q$ (this is because there are $2p - 2q$ more $+$'s than $-$'s, hence $2q + 2p - 4k \geq 2p - 2q$);

2. if (i,j) is a 2-cycle such that $1 \leq i < j \leq n$, then $(2n+1-j, 2n+1-i)$ is a 2-cycle also;

3. if (i,j) is a 2-cycle such that $1 \leq i < n+1 \leq j \leq 2n$, then $(2n+1-j, 2n+1-i)$ is a 2-cycle as well.

The (signed) involutions in $I^{sp}_{k,p,q}$ are precisely the involutions that correspond to the symp-symmetric $(2p, 2q)$ clans under the bijection of Lemma 2.0.3, so, $\gamma_{k,p,q}$ stands for the cardinality of $I^{sp}_{k,p,q}$. To find a formula for $\gamma_{k,p,q}$'s we argue similarly to the case of $\beta_{k,p,q}$, by counting the number of possible ways of placing pairs and by counting the number of possible ways of placing $\pm$'s on the fixed points. Also, we make use of the bijection $\widetilde{\varphi}$ of Lemma 2.0.2 to switch between the involution notation and the clan notation.

First of all, an involution π from $I^{sp}_{k,p,q}$ has $2k$ 2-cycles and $2n-4k$ fixed points. The $2k$ 2-cycles, by using numbers from $\{1, \ldots, 2n\}$ can be chosen in $\binom{n}{2k}$; the number of rearrangements of these $2k$ pairs and their entries, to obtain the standard form of an involution, requires $\frac{(2k)!}{k!}$ steps. In other words, the 2-cycles of π are found and placed in the standard ordering in $\binom{n}{2k}\frac{(2k)!}{k!}$ possible ways. Once we have the 2-cycles of the involution, we easily see that the numbers and their positions in the corresponding symp-symmetric clan are uniquely determined.

Next, we determine the number of ways to place $\pm$ symbols. This amounts to finding the number of ways of placing 2α +'s and 2β -'s on the string $d_1 \ldots d_{2n-4k}$ so that there are exactly $2p - 2q = 2\alpha - 2\beta$ +'s more than -'s. By applying the inverse of the bijection $\widetilde{\varphi}$ of Lemma 2.0.2, we will use the symmetry condition on the corresponding clan. Thus, we observe that it is enough to focus on the first n places of the clan only. Now, the number of +'s in the first n places can be chosen in $\binom{n-2k}{\alpha}$ different ways. Once we place the +'s, the remaining entries will be filled with $-$'s. Clearly there is now only one way of doing this since we placed the numbers and the + signs already. Therefore, to finish our counting, we need to find what that α is.

Since $\alpha+\beta=n-2k=q+p-2k$ and since $\alpha-\beta=p-q$, we see that $\alpha=p-k$.

In summary, the number of possible ways of constructing a signed involution corresponding to a symp-symmetric $(2p,2q)$ clan is given by

$$\gamma_{k,p,q}=\binom{q+p}{2k}\frac{(2k)!}{k!}\binom{q+p-2k}{p-k}. \tag{4.1}$$

Note here that we are using $n=p+q$. The right-hand side of (4.1) can be expressed more symmetrically as follows:

$$\gamma_{k,p,q}=\frac{(q+p)!}{(q-k)!(p-k)!k!}. \tag{4.2}$$

4.2 Recurrences

Observe that the formula (4.2) is defined independently of the inequality $q\le p$. From now on, for our combinatorial purposes, we skip mentioning this comparison between p and q and use the equality $\gamma_{k,p,q}=\gamma_{k,q,p}$ whenever it is needed. Also, we record the following obvious recurrences for future reference:

$$\gamma_{k,p,q}=\frac{(p-k+1)(q-k+1)}{k}\gamma_{k-1,p,q}, \tag{4.3}$$

$$\gamma_{k,p-1,q}=\frac{p-k}{p+q}\gamma_{k,p,q}, \tag{4.4}$$

$$\gamma_{k,p,q-1}=\frac{q-k}{p+q}\gamma_{k,p,q}. \tag{4.5}$$

These recurrences hold true whenever both sides of the equations are defined. Notice that in (4.3)–(4.5) the parity, namely k does not change. Next, we will show that $\gamma_{k,p,q}$'s obey a 3-term recurrence once we allow change in all three numbers p,q, and k.

Lemma 4.2.1. Let p and q be two positive integers. If $k \geq 1$, then we have

$$\gamma_{k,p,q} = \gamma_{k,p-1,q} + \gamma_{k,p,q-1} + 2(q+p-1)\gamma_{k-1,p-1,q-1} \text{ and } \gamma_{0,p,q} = \binom{p+q}{p}. \tag{4.6}$$

Proof. Instead of proving our result directly, we will make use of a similar result that we proved before. Let $\widetilde{\gamma}_{k,p,q}$ denote the number

$$\widetilde{\gamma}_{k,p,q} = \frac{(q+p)!}{2^k(q-k)!(p-k)!k!}. \tag{4.7}$$

In [22], it is proven that

$$\widetilde{\gamma}_{k,p,q} = \widetilde{\gamma}_{k,p-1,q} + \widetilde{\gamma}_{k,p,q-1} + (p+q-1)\widetilde{\gamma}_{k-1,p-1,q-1} \tag{4.8}$$

holds true for all $p, q, k \geq 1$. Note that $\widetilde{\gamma}_{0,p,q} = \binom{p+q}{p}$, which is our initial condition for $\gamma_{k,p,q}$'s. Therefore, combining (4.8) with the fact that $\gamma_{k,p,q} = 2^k\widetilde{\gamma}_{k,p,q}$ finishes our proof. □

Observe that here $\widetilde{\gamma}_{k,p,q}$ is nothing but $\alpha_{k,p,q}$ which is given in Section 2.1.

Remark 4.2.2. Recall that $\Gamma_{p,q} = \sum_k \gamma_{k,p,q}$. From now on we will assume that $\Gamma_{p,q} = 1$ whenever one or both of p and q are zero.

Proposition 4.2.3. For all positive integers p and q, the following recurrence relation holds true:

$$\Gamma_{p,q} = \Gamma_{p-1,q} + \Gamma_{p,q-1} + 2(p+q-1)\Gamma_{p-1,q-1}. \tag{4.9}$$

Proof. Let us begin by summing both sides of equation (4.6) over k with $1 \leq k \leq q-1$.

It follows that

$$\begin{aligned}\Gamma_{p,q} - \Gamma_{p-1,q} - \Gamma_{p,q-1} - 2(p+q-1)\Gamma_{p-1,q-1} &= \gamma_{0,p,q} - \gamma_{0,p-1,q} - \gamma_{0,p,q-1} \\ &+ \gamma_{q,p,q} - \gamma_{q,p,q-1} - 2(p+q-1)\gamma_{q-1,p-1,q-1} \\ &= 0.\end{aligned}$$

□

4.3 Generating functions

One of the many options for a bivariate generating function for $\Gamma_{p,q}$'s is the following

$$v(x,y) := \sum_{p,q\geq 0} \Gamma_{p,q} \frac{(2x)^q y^p}{p!}. \tag{4.10}$$

Let us tabulate first few terms of $v(x,y)$:

$$\begin{aligned}\sum_{p,q\geq 0} \Gamma_{p,q} \frac{(2x)^q y^p}{p!} &= \Gamma_{0,0} + \Gamma_{0,1} 2x + \cdots + \Gamma_{0,q}(2x)^q + \dots \\ &+ \frac{\Gamma_{1,0}}{1!} y + \cdots + \frac{\Gamma_{p,0}}{p!} y^p + \dots \\ &+ \frac{\Gamma_{1,1}}{1!}(2x)y + \cdots + \frac{\Gamma_{p,1}}{p!}(2x)y^p + \dots \\ &+ \frac{\Gamma_{1,2}}{1!}(2x)^2 y + \cdots + \frac{\Gamma_{p,2}}{p!}(2x)^2 y^p + \dots.\end{aligned} \tag{4.11}$$

It follows from our Remark 4.2.2 and equation (4.11) that

$$v(x,y) = \frac{1}{1-2x} + e^y - 1 + \sum_{p,q\geq 1} \Gamma_{p,q} \frac{(2x)^q y^p}{p!}. \tag{4.12}$$

We feed this observation into our recurrence (4.9) and use similar arguments for the right hand side of it:

$$\begin{aligned}
v(x,y) - \frac{1}{1-2x} - e^y + 1 &= \int \sum_{p\geq 1, q\geq 0} \frac{\Gamma_{p-1,q}}{(p-1)!}(2x)^q y^{p-1} dy - e^y \\
&+ 2x\left(\sum_{p,q\geq 0} \frac{\Gamma_{p,q}}{p!}(2x)^q y^p - \frac{1}{1-2x}\right) \\
&+ 2\sum_{p,q\geq 1} p\frac{\Gamma_{p-1,q-1}}{p!}(2x)^q y^p + 2\sum_{p,q\geq 1} q\frac{\Gamma_{p-1,q-1}}{p!}(2x)^q y^p \\
&- 2\sum_{p,q\geq 1} \frac{\Gamma_{p-1,q-1}}{p!}(2x)^q y^p \\
&= \int \sum_{p\geq 1, q\geq 0} \frac{\Gamma_{p-1,q}}{(p-1)!}(2x)^q y^{p-1} dy - e^y \\
&+ 2x\left(\sum_{p,q\geq 0} \frac{\Gamma_{p,q}}{p!}(2x)^q y^p - \frac{1}{1-2x}\right) \\
&+ 4xy\sum_{p,q\geq 0} \frac{\Gamma_{p-1,q-1}}{(p-1)!}(2x)^{q-1} y^{q-1} \\
&+ 4x\int \sum_{p,q\geq 1} \frac{q\Gamma_{p-1,q-1}}{(p-1)!}(2x)^{q-1} y^{p-1} dy \\
&- 4x\int \sum_{p,q\geq 1} \frac{\Gamma_{p-1,q-1}}{(p-1)!}(2x)^{q-1} y^{p-1} dy.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
v(x,y) - \frac{1}{1-2x} - e^y + 1 &= \int v(x,y)dy - e^y + 2x\ v(x,y) - \frac{2x}{1-2x} + 4xy\ v(x,y) \\
&+ x\int \left(\frac{\partial}{\partial x}(2x\ v(x,y))\right)dy - 4x\int v(x,y)dy,
\end{aligned}$$

or equivalently,

$$(1-2x-4xy)v(x,y) = (1-4x)\int v(x,y)dy + x\int \left(\frac{\partial}{\partial x}(2xv(x,y))\right)dy.$$

Now differentiating with respect to y gives us a PDE:

$$-4x\ v(x,y) + (1-2x-4xy)\frac{\partial v(x,y)}{\partial y} = (1-4x)\ v(x,y) + x\left(2v(x,y) + 2x\frac{\partial v(x,y)}{\partial x}\right),$$

which we reorganize as in

$$(-2x^2)\frac{\partial v(x,y)}{\partial x} + (1-2x-4xy)\frac{\partial v(x,y)}{\partial y} = (1+2x)\ v(x,y). \tag{4.13}$$

Here, we have the obvious initial conditions

$$v(0,y) = e^y \quad \text{and} \quad v(x,0) = \frac{1}{1-2x}.$$

Solutions of such PDE's are easily obtained by applying the method of "characteristic curves." Our characteristic curves are $x(r,s), y(r,s)$, and $v(r,s)$. Their tangents are equal to

$$\frac{\partial x}{\partial s} = -2x^2, \qquad \frac{\partial y}{\partial s} = 1-2x-4xy, \qquad \frac{\partial v}{\partial s} = (1+2x)v, \tag{4.14}$$

with the initial conditions

$$x(r,0) = r, \quad y(r,0) = 0 \quad \text{and} \quad v(r,0) = \frac{1}{1-2r}.$$

From the first equation given in (4.14) and its initial condition underneath, we have

$$x(r,s) = \frac{r}{2rs+1}. \tag{4.15}$$

Plugging this into the second equation gives us $\frac{\partial y}{\partial r} = 1 - \frac{2}{2rs+1}(1+2y)$, which is a first order linear ODE. The general solution for this ODE is

$$y(r,s) = \frac{3s + 4r^2s^3 - 6r^2s^2 + 6rs^2 - 6rs}{3(2rs+1)^2}. \tag{4.16}$$

Finally, from the last equation in (4.14) together with its initial condition we conclude that

$$v(r,s) = \frac{e^s(2rs+1)}{1-2r}.$$

In summary, we outlined the proof of our next result.

Theorem 4.3.1. *Let $v(x,y)$ denote the power series $\sum_{p,q\geq 0} \Gamma_{p,q}(2x)^q \frac{y^p}{p!}$. If r and s are the variables related to x and y as in equations (4.16) and (4.15), then we*

$$v(r,s) = \frac{e^s(2rs+1)}{1-2r}. \tag{4.17}$$

We finish this section with a remark.

Remark 4.3.2. Although we solved our PDE by using the useful method of characteristic curves, the answer is given as a function of transformed coordinates r and s. Actually, we can find the solution in x and y. Indeed, it is clear from the outset that the general solution $S_3(x,y)$ of (4.13) is given by

$$S_3(x,y) = \frac{e^{1/(2x)} F_3(\frac{6xy+3x-1}{6x^3})}{x}, \tag{4.18}$$

where $F_3(z)$ is some function in one-variable. (This can easily be verified by substituting $S_3(x,y)$ into the PDE.) Let us find a concrete expression for $F_3(z)$ here so that the initial condition $S_3(x,y) = v(x,y)$ holds true. To this end, we set $y = 0$. In this case, we know that $v(x,0) = \frac{1}{1-2x}$. Therefore, $F_3(z)$ satisfies the following equation:

$$\frac{e^{1/(2x)} F_3(\frac{3x-1}{6x^3})}{x} = \frac{1}{1-2x} \quad \text{or} \quad F_3\left(\frac{3x-1}{6x^3}\right) = \frac{xe^{-1/(2x)}}{1-2x}. \tag{4.19}$$

The inverse of the transformation $z = \frac{3x-1}{6x^3}$ which appears in (4.19) is given by

$$x = \frac{1}{6^{1/3}(-3z^2 + \sqrt{3}\sqrt{-2z^3 + 3z^4})^{1/3}} + \frac{(-3z^2 + \sqrt{3}\sqrt{-2z^3 + 3z^4})^{1/3}}{6^{2/3}z}. \tag{4.20}$$

By back substitution of (4.20) into (4.19), we find an expression for $F_3(z)$, which in turn will be evaluated at $\frac{6xy+3x-1}{6x^3}$ (as in (4.18)). Obviously the resulting expression is very complicated, however, this way we can write the solution of our PDE in x and y only.

4.4 A combinatorial interpretation

In this section, our aim is to present the lattice interpretation for $\Gamma_{p,q}$. In order to do that, we first define the weighted Delannoy paths, similar to the one defined in the Section 2.4, as follows: Let L be a Delannoy path that ends at the lattice point $(p, q) \in \mathbb{N}^2$. Recall that we agree to represent L as a word $L_1 L_2 \dots L_r$, where each L_i $(i = 1, \dots, r)$ is a pair of lattice points, say $L_i = ((a, b), (c, d))$, and $(c - a, d - b) \in \{N, E, D\}$. In this notation, we define the weight of the i-th step as

$$\omega(L_i) = \begin{cases} 1 & \text{if } L_i = ((a,b),(a+1,b)); \\ 1 & \text{if } L_i = ((a,b),(a,b+1)); \\ 2(a+b+1) & \text{if } L_i = ((a,b),(a+1,b+1)). \end{cases}$$

Finally, we define the weight of L, denoted by $\omega(L)$ as the product of the weights of its steps.

Example 4.21. Let L denote the Delannoy path that is depicted in Figure 4.1. In this case, the weight of L is $\omega(L) = 6 \cdot 12 \cdot 16 = 1152$.

Proposition 4.4.1. Let p and q be two nonnegative integers and let $\mathcal{D}(p, q)$ denote

$L =$

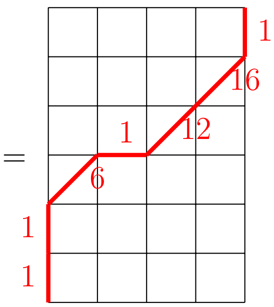


Figure 4.1: An example of weighted Delannoy path.

the corresponding set of Delannoy paths. In this case, we have

$$\Gamma_{p,q} = \sum_{L \in \mathcal{D}(p,q)} \omega(L).$$

Proof. Let $\Gamma'_{p,q}$ denote the sum $\sum_{L \in \mathcal{D}(p,q)} \omega(L)$. As a convention we define $\Gamma'_{0,0} = 1$. Recall that n stands for $p+q$. We prove our claim $\Gamma'_{p,q} = \Gamma_{p,q}$ by induction on n. Obviously, if $n = 1$, then (p,q) is either $(0,1)$ or $(1,0)$, and in both of these cases, there is only one step which either N or E. Therefore, $\Gamma'_{p,q} = 1$ in this case. Now, let n be a positive integer and we assume that our claim is true for all (p,q). We will prove that $\Gamma_{p,q} = \Gamma'_{p,q}$, whenever $p+q = n+1$. To this end, we look at the possibilities for the ending step of a Delannoy path $L = L_1 \dots L_r \in \mathcal{D}(p,q)$. If L_r is a diagonal step, then

$$\omega(L) = (2(p+q)-1)\omega(L_1 \dots L_{r-1}).$$

In particular, $L_1 \dots L_{r-1} \in \mathcal{D}(p-1, q-1)$. If L_r is from $\{N, E\}$, then

$$\omega(L) = \omega(L_1 \dots L_{r-1}).$$

In particular, $L_1 \dots L_{r-1} \in \mathcal{D}(p-1, q)$ or $L_1 \dots L_{r-1} \in \mathcal{D}(p, q-1)$. depending on

$L_r = E$ or $L_r = N$. We conclude from these observations that

$$\begin{aligned}
\Gamma'_{p,q} &= \Gamma'_{p-1,q} + \Gamma'_{p,q-1} + 2(p+q-1)\Gamma'_{p-1,q-1} \\
&= \Gamma_{p-1,q} + \Gamma_{p,q-1} + 2(p+q-1)\Gamma_{p-1,q-1} \qquad \text{(by induction hypothesis)} \\
&= \Gamma_{p,q}.
\end{aligned}$$

This finishes the proof of our claim. □

Although Proposition 4.4.1 expresses $\Gamma_{p,q}$ as a combinatorial summation it does not give a combinatorial set of objects whose cardinality is given by $\Gamma_{p,q}$. The last result of this section offers such an interpretation.

Definition 4.4.2. A k-diagonal step (in $\mathbb{N}^2$) is a diagonal step L of the form $L = ((a,b),(a+1,b+1))$, where $a,b \in \mathbb{N}$ and $k = a+b+1$.

As an example, in Figure 4.2 we depict all 4-diagonal steps in $\mathbb{N}^2$.

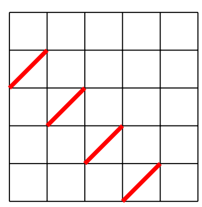
$L =$ 

Figure 4.2: 4-diagonal steps in $\mathbb{N}^2$.

Next, we define the weighted Delannoy paths.

Definition 4.4.3. By a labelled step we mean a pair (R,m), where $R \in \{N,E,D\}$ and m is a positive integer such that $m = 1$ if $R = N$ or $R = E$. A weighted (p,q) Delannoy path is a word of the form $R := R_1 \dots R_r$, where R_i's $(i = 1,\dots,r)$ are labeled steps $R_i = (L_i, m_i)$ such that

- $L_1 \dots L_r$ is a Delannoy path from $\mathcal{D}(p,q)$;
- if L_i $(1 \le i \le r)$ is a k-th diagonal step, then $2 \le m_i \le 2k-1$.

The set of all weighted (p,q) Delannoy paths is denoted by $\mathcal{D}^w(p,q)$.

Theorem 4.4.4. *There is a bijection between the set of weighted (p,q) Delannoy paths and the set of symp-symmetric $(2p,2q)$ clans. In particular, we have*

$$\Gamma_{p,q} = \sum_{W \in \mathcal{D}^w(p,q)} 1.$$

The proof of Theorem 4.4.4 is based on the same idea however it requires more attention in some of the constructions that are involved.

Proof. Let $d_{p,q}$ denote the cardinality of $\mathcal{D}^w(p,q)$. We will prove that $d_{p,q}$ obeys the same recurrence as $\Gamma_{p,q}$'s and it satisfies the same initial conditions.

Let $\gamma = c_1 \dots c_{2n}$ be a symp-symmetric $(2p,2q)$ clan and let $\pi = \pi_\gamma$ denote the signed involution

$$\pi = (i_1, j_1) \dots (i_{2k}, j_{2k}) d_1^{s_1} \dots d_{2n-4k}^{s_{2n-4k}}, \quad \text{where } s_1, \dots, s_{2n-4k} \in \{+,-\},$$

which is given by $\pi = \widetilde{\varphi}(\gamma)$ (Here, $\widetilde{\varphi}$ is the map that is constructed in the proof of Lemma 2.0.3.) We will construct a weighted (p,q) Delannoy path $W = W_\gamma$ which is uniquely determined by π.

First, we look at the position of $2n$ in π. If it appears as a fixed point with a $+$ sign, then we draw a step between (p,q) and $(p-1,q)$. If it appears as a fixed point with a $-$ sign, then we draw a a step between (p,q) and $(p,q-1)$. We label both of these steps by 1 to turn them into labeled steps. Next, we remove the fixed points 1 and $2n$ from π and then subtract 1 from each remaining entry. The result is a either signed $(2(p-1),2q)$ involution or a signed $(2p,2(q-1))$ involution. Now, by our induction hypothesis, in the first case, there are $d_{p-1,q}$ possible ways of extending this path to a weighted (p,q) Delannoy path. In a similar manner, in the latter case there are $d_{p,q-1}$ possible ways of extending it to a weighted (p,q) Delannoy path.

Now we assume that $2n$ appears in a 2-cycle in π, say (i_s, j_s), where $1 \leq s \leq k$. Then $(i_r, j_r) = (i, 2n)$, for some $i \in \{2, \ldots, 2n-1\}$. Then by the symmetry condition, there is a partnering 2-cycle, which is necessarily of the form $(1, i')$ for some i'. In this case, we draw a D-step between (p, q) and $(p-1, q-1)$ and we label this step by i. Then we remove the two cycle $(i, 2n)$ as well as its partner $(1, i')$ from π. Let us denote the resulting object by $\pi_0^{(1)}$. To get rid of the gaps created by the removal of two 2-cycles, we renormalize the remaining entries by appropriately subtracting numbers so that the resulting object, which we denote by $\pi^{(1)}$ has every number from $\{1, \ldots, 2n-4\}$ appears in it exactly once. It is easy to see that we have a signed $(2(p-1), 2(q-1))$ involution which corresponds to a symp-symmetric $(2(p-1), 2(q-1))$ clan under $\widetilde{\varphi}^{-1}$. Now, the label of this diagonal step can be chosen as one the $2(p+q-1)$ numbers from $\{2, \ldots, 2n-1\}$. Finally, let us note that there are $d_{p-1,q-1}$ possible ways to extend this labeled diagaonal step to a weighted (p, q) Delannoy path.

Combining our observations we see that, starting with a random symp-symmetric $(2p, 2q)$ clan, there are exactly

$$d_{p-1,q} + d_{p,q-1} + 2(p+q-1)d_{p-1,q-1} \tag{4.22}$$

possible weight Delannoy paths that we can construct. By induction hypothesis the number (4.22) is equal to $\Gamma_{p,q}$. This finishes our proof.

□

Let us illustrate our construction by an example.

Example 4.23. Let γ denote the symp-symmetric $(10, 6)$ clan

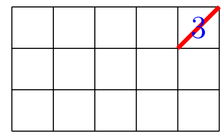
$$\gamma = 4\ +\ 6\ -\ +\ 1\ 1\ +\ +\ 2\ 2\ +\ -\ 4\ +\ 6,$$

and let π denote the corresponding signed involution

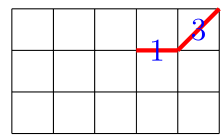
$$\pi = (1,14)(3,16)(6,7)(10,11)2^+4^-5^+8^+9^+12^+13^-15^+.$$

The steps of our constructions are shown in Figure 4.3.

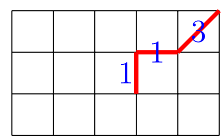
$$\pi$$
$$\downarrow$$
$$\pi^{(1)} = (4,5)(8,9)1^+2^-3^+6^+7^+10^+11^-12^+$$



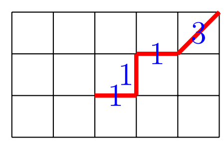
$$\pi^{(1)} = (4,5)(8,9)1^+2^-3^+6^+7^+10^+11^-12^+$$
$$\downarrow$$
$$\pi^{(2)} = (3,4)(7,8)1^-2^+5^+6^+9^+10^-$$



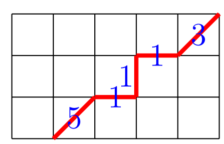
$$\pi^{(2)} = (3,4)(7,8)1^-2^+5^+6^+9^+10^-$$
$$\downarrow$$
$$\pi^{(3)} = (2,3)(6,7)1^+4^+5^+8^+$$



$$\pi^{(3)} = (2,3)(6,7)1^+4^+5^+8^+$$
$$\downarrow$$
$$\pi^{(4)} = (1,2)(5,6)3^+4^+$$



$$\pi^{(4)} = (1,2)(5,6)3^+4^+$$
$$\downarrow$$
$$\pi^{(5)} = 1^+2^+$$



$$\pi^{(5)} = 1^+2^+$$
$$\downarrow$$
$$\pi^{(6)} = \cdot$$

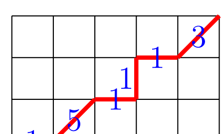


Figure 4.3: Algorithmic construction of the bijection onto weighted Delannoy paths.

Chapter 5

Type CI

In this chapter, we focus on the enumeration of Borel orbits in the classical symmetric variety $Sp(2n, \mathbb{C})/GL(n, \mathbb{C})$, also known as skew symmetric bilinear form - Type CI. Due to Cartan's classification of involutions on algebraic groups it is known that the involution associated with polarization for $Sp(2n, \mathbb{C})$ is given by

$$\sigma(g) = -S_{n,n} g S_{n,n},$$

refer to the section 1.1 for the notation here. See [7], for details. In this case, it is easy to show that the fixed point subgroup is $GL(n, \mathbb{C})$.

Since these orbits are parametrized by skew symmetric (n, n) clans, in this part of the thesis, we are going to find various generating functions and combinatorial interpretations for the number Δ_n of skew symmetric (n, n) clans.

5.1 Counting skew symmetric (n, n) clans with k pairs.

It comes from the definition that for a given skew symmetric (n, n) clan the number of $+$ signs among the first half (first n-spot) must equal to the number of $-$ signs among second half. Note also that the total number of $+$ and $-$ signs is denoted by $\alpha + \beta$ for the first half.

Now, we will focus on the subset of S_{2n} consisting of involutions π whose standard cycle decomposition is of the form

$$\pi = (i_1 j_1) \dots (i_k j_k) d_1 \dots d_{2n-2k}.$$

Remark 5.1.1. Observe that the number of transposition can not exceed n because the inequality $0 \leq 2n - 2k$ implies that $0 \leq k \leq n$.

Our next task is determining the number of possible ways of placing k pairs to build from scratch a skew symmetric (n, n) clan$\gamma = c_1 \cdots c_n c_{n+1} c_{n+2} \cdots c_{2n}$. In order to do that let us first state the following lemma:

Lemma 5.1.2. There are 2^{n-k} ways of placing $\pm$ symbols among $2n$ spots to obtain skew symmetric (n, n) clans.

Proof. By skew symmetry condition, it is enough to focus on the first half of the $2n$ string. Assume that there are j many $+$ signs in the first half. Since there are k many pairs of natural numbers there are $2n - 2k$ fixed points in the involution. Then, the inequality $2(\alpha + \beta) = 2n - 2k$ yields us that $\alpha + \beta = n - k$. Therefore, one can conclude that there are

$$\sum_{j=0}^{n-k} \binom{n-k}{j} = 2^{n-k} \tag{5.1}$$

possible ways to place + symbols among those remaining $n-k$ spots, which completes the proof. $\square$

In [20], the number of different ways of placing k pairs to build any symmetric (p,q) clan γ is determined. Here, thanks to Remark 1.3.8, it is enough to mimic the proof given for counting skew symmetric (n,n) clans. Now, we are ready to state our main theorem for this section:

Theorem 5.1.3. *For every nonnegative integer k with $k \leq n$, we have*

$$\delta_{k,n} = 2^{n-k}\binom{n}{k}a_k; \quad \textit{where} \quad a_k := \sum_{b=0}^{\lfloor k/2 \rfloor}\binom{k}{2b}\frac{(2b)!}{b!}. \tag{5.2}$$

In particular, we have

$$\Delta_n = \sum_{k=0}^{n}\delta_{k,n} = \sum_{k=0}^{n}2^{n-k}\binom{n}{k}a_k.$$

Proof. Let us first define the following interrelated sets:

$$\begin{aligned}
PI_{1,1} :=& \{((i,j),(2n+1-j,2n+1-i)) \mid 1 \leq i < j \leq n\},\\
PI_{1,2} :=& \{((i,j),(2n+1-j,2n+1-i)) \mid 1 \leq i \leq n < j \leq 2n\},\\
PI_1 :=& PI_{1,1} \cup PI_{1,2},\\
PI_2 :=& \{(i,j) \mid 1 \leq i \leq n < j \leq 2n,\ i+j = 2n+1\}.
\end{aligned}$$

We view PI_1 as the set of placeholders for two distinct pairs that determine each other in γ. The set PI_2 corresponds to the list of stand alone pairs in γ. In other words, if $(i,j) \in PI_2$, then $c_i = c_j$ and $j = 2n-i+1$.

If (c_i,c_j) is a pair in the skew symmetric clan γ and if (i,j) is an element of PI_2, then we call (c_i,c_j) a pair of type PI_2. If x is a pair of pairs of the form $((c_i,c_j),(c_{2n+1-j},c_{2n+1-i}))$ in a skew symmetric clan γ and if $((i,j),(2n+1-j,2n+$

$1-i)) \in PI_{1,s}$ ($s \in \{1,2\}$), then we call x a pair of pairs of type $PI_{1,s}$. If there is no need for precision, then we will call x a pair of pairs of type PI_1.

Clearly, if $|PI_1| = b$ and $|PI_2| = a$, then $2b + a = k$ is the total number of pairs in our skew symmetric clan γ. To see in how many different ways these pairs of indices can be situated in γ, we start with choosing k spots from the first n positions in $\gamma = c_1 \cdots c_{2n+1}$. Obviously this can be done in $\binom{n}{k}$ many different ways. Next, we count different ways of choosing b pairs within the k spots to place the b pairs of pairs of type PI_1. This number of possibilities for this count is $\binom{k}{2b}$. Observe that choosing a pair from PI_1 is equivalent to choosing (i,j) for the pairs of pairs in $PI_{1,1}$ and choosing $(i, 2n+1-j)$ for the pairs of pairs in $PI_{1,2}$. More explicitly, we first choose b pairs among the $2b$ elements and then place them on b spots; this can be done in $\binom{2b}{b} b!$ different ways. Once this is done, finally, the remaining spots will be filled by the a pairs of type PI_2. This can be done in only one way. Therefore, in summary, the number of different ways of placing k pairs to build a symmetric (n,n) clan γ is given by

$$\binom{n}{k} \sum_{b=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2b} \binom{2b}{b} b!, \quad \text{or equivalently,} \quad \binom{n}{k} \sum_{b=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2b} \frac{(2b)!}{b!}.$$

$\square$

Remark 5.1.4. By a straightforward calculation, it is easy to check that the following recurrence

$$a_k = a_{k-1} + 2(k-1)a_{k-2},$$

with $a_0 = a_1 = 1$ for all $k \geq 2$ holds.

Example 5.3. It is easy to show that there are exactly 11 skew symmetric $(2,2)$ clans. The graph below is obtained by using the combinatorial description of the weak order on skew symmetric clans, details can be found in [16].

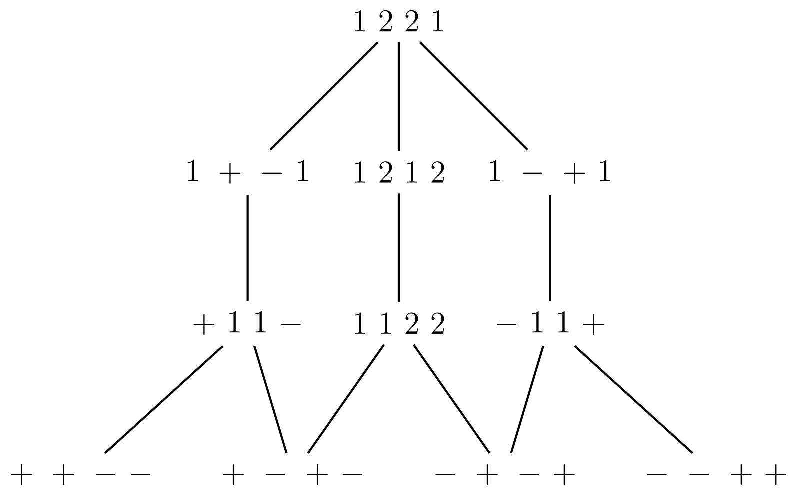


Figure 5.1: Weak order on $\Delta(2)$

Example 5.4. All the possible skew symmetric $(3, 3)$ clans are:

- - - + + +, + + + - - -, + + - + - -, + - + - + -, - + + - - +, - - + - + +,
+ - + - +, + - - + + -, 1 + + - - 1, + 1 + - 1 -, + + 1 1 - -, 1 - - + + 1,
- 1 - + 1 +, - - 1 1 + +, 1 + - + - 1 , 1 - + - + 1, - 1 + - 1 +, + 1 - + 1 -,
+ - 1 1 + -, - + 1 1 - +, + 1 1 2 2 -, 1 1 - + 2 2, 1 - 1 2 + 2, - 1 1 2 2 +, 1 2 + - 1 2,
- 1 2 1 2 +, 1 2 - + 1 2, 1 + 2 1 - 2, + 1 2 1 2 -, 1 + 1 2 - 2, 1 1 + - 2 2, 1 2 + - 2 1,
1 + 2 2 - 1, + 1 2 2 1 -, 1 2 - + 2 1, 1 - 2 2 + 1, - 1 2 2 1 +, 1 - 2 1 + 2, 1 2 3 3 2 1,
1 2 3 1 2 3, 1 2 1 3 2 3, 1 2 3 2 3 1, 1 1 2 2 3 3, 1 2 3 3 1 2, 1 2 2 3 3 1.

Thus, there are 45 skew symmetric $(3, 3)$ clans.

Note here that Δ_n is the coefficient of $\frac{x^n}{n!}$ in

$$\sum \frac{(2x)^i}{i!} \cdot \sum \frac{a_j x^j}{j!}$$

whose generating function is $e^{2x} \cdot e^{x^2+x} = e^{x^2+3x}$. Furthermore, note also that the first few values of Δ_n are $1, 3, 11, 45, 201, 963, 4899, 26253, 147345, 862083, 5238459,$ which is [OEIS] A083886 with exponential generating function $\Delta_n(x) = e^{x^2+3x} = \sum_n^\infty \Delta_n \frac{x^n}{n!}$. To justify this generating function, we need to verify that Δ_n satisfies the following recurrence relation:

$$\Delta_n = 3\Delta_{n-1} + 2(n-1)\Delta_{n-2} \quad \text{with} \quad \Delta_0 = 1, \Delta_1 = 3. \tag{5.5}$$

5.2 Recurrences

We start by recording the following obvious recurrence for future reference:

$$\delta_{k,n} = \frac{2n}{n-k}\delta_{k,n-1}. \tag{5.6}$$

This recurrence holds true whenever both sides of the equation is defined. It follows from the recurrence relation (5.5) given in previous Section 5.1 and straightforward calculation that we have the following lemma:

Lemma 5.2.1. Let n be a positive integers. If $k \geq 2$, then we have

$$\delta_{k,n} = \frac{n-k-1}{2k}\delta_{k-1,n} + \frac{(n-k+1)(n-k+2)}{2k}\delta_{k-2,n}, \tag{5.7}$$

with the initial values $\delta_{0,n} = 2^n$ and $\delta_{1,n} = n2^{n-1}$.

Here note that in this recurrence relation n does not change. In the sequel, we now ready to determine other recurrence that runs over both n and k. It follows from the Lemma 5.2.1 and equation 5.6 that we have

$$\begin{aligned}
\delta_{k,n} &= \frac{n-k-1}{2k}\delta_{k-1,n} + \frac{(n-k+1)(n-k+2)}{2k}\delta_{k-2,n} \\
&= \frac{n-k-1}{2k}\frac{2n}{n-k+1}\delta_{k-1,n-1} + \frac{(n-k+1)(n-k+2)}{2k}\frac{2n}{n-k+2}\delta_{k-2,n} \\
&= \frac{n}{k}\delta_{k-1,n-1} + \frac{2n(n-1)}{k}\delta_{k-2,n-2}.
\end{aligned}$$

We have just proved the following lemma:

Lemma 5.2.2. For all positive integers $n, k \geq 2$, the following recurrence relation holds true:

$$\delta_{k,n} = \frac{n}{k}\delta_{k-1,n-1} + \frac{2n(n-1)}{k}\delta_{k-2,n-2}. \tag{5.8}$$

5.3 Generating Functions

One of the many other options for a bivariate generating function for Δ_n's is given as follows. Let $\Delta(x,y)$ denote the following generating function,

$$\Delta(x,y) := \sum_{n\geq 0} \Delta_n(x) y^n. \tag{5.9}$$

where

$$\Delta_n(x) = \sum_{k=0}^{n} \delta_{k,n} x^{n-k}.$$

Let us tabulate first few terms of $\Delta(x,y)$:

$$\begin{aligned}\sum_{0\leq k\leq n} \delta_{k,n} x^{n-k} y^n &= \delta_{0,0} + \delta_{0,1}xy + \delta_{0,2}x^2y^2 + \delta_{0,3}x^3y^3 \ldots \\ &+ \delta_{1,1}y + \cdots + \delta_{1,2}xy^2 + \delta_{1,3}x^2y^3 \ldots \\ &+ \delta_{2,2}y^2 + \cdots + \delta_{2,3}xy^3 + \delta_{2,4}x^2y^4 \ldots .\end{aligned} \tag{5.10}$$

It follows from the equation (5.6) that we have

$$\begin{aligned}\Delta_n(x) &= \sum_{k=0}^{n-1} \frac{2n}{n-k}\delta_{k,n-1}x^{n-k} + \delta_{n,n}\\ &= \delta_{n,n} + 2n\sum_{k=0}^{n-1}\delta_{k,n-1}\int x^{n-1-k}dx\\ &= \delta_{n,n} + 2n\int\sum_{k=0}^{n-1}\delta_{k,n-1}x^{n-1-k}dx\\ &= a_n + 2n\Delta_{n-1}(x).\end{aligned}$$

Taking the derivative of both sides of the above equation gives us that

$$\Delta_n^{'}(x) = 2n\Delta_{n-1}(x). \tag{5.11}$$

The differential equation (5.11) leads to a PDE for our initial generating function $\Delta(x, y)$:

$$\begin{aligned}\frac{\partial}{\partial x}(\Delta(x,y)) =& \frac{\partial}{\partial x}\left[\sum_{n\geq 0}\Delta_n^{'}(x)y^n\right] = \sum_{n\geq 0}2n\Delta_{n-1}(x)y^n\\ =& \sum_{n\geq 1}2n\Delta_{n-1}(x)y^n = 2(\Delta_0(x)y + 2\Delta_1(x)y^2 + 3\Delta_3(x)y^3 + \dots)\\ =& 2y\frac{\partial}{\partial y}(\Delta_0(x)y + \Delta_1(x)y^2 + \Delta_3(x)y^3 + \dots)\\ =& 2y\frac{\partial}{\partial y}\Big(y\Delta(x,y)\Big).\end{aligned}$$

By the last equation we obtain the following PDE:

$$\frac{\partial}{\partial x}\Delta(x,y) - 2y^2\frac{\partial}{\partial y}\Delta(x,y) = 2y\Delta(x,y). \tag{5.12}$$

The general solution $S_4(x,y)$ of (5.12) is given by

$$S_4(x,y) = \frac{1}{y} F_4\left(\frac{1-2xy}{y}\right), \tag{5.13}$$

where $F_4(z)$ is some function in one-variable. We want to choose $F_4(z)$ in such a way that $S(x,y) = \Delta(x,y)$ holds true. To do so, first, we look at some special values of $\Delta(x,y)$.

If let $x = 0$, then by the equation (5.10) we have $\Delta(0,y) = \sum_{n\geq 0} \delta_{n,n} y^n$ and $\delta_{n,n} = a_n$ for all $n \geq 0$. Thus, we ask from $F_4(z)$ that it satisfies the following equation

$$\frac{1}{y} F_4\left(\frac{1}{y}\right) = \sum_{n\geq 0} a_n y^n \quad \text{or that} \quad F_4\left(\frac{1}{y}\right) = y \sum_{n\geq 0} a_n y^n. \tag{5.14}$$

Therefore, we see that our generating function is given by

$$\begin{aligned} \Delta(x,y) &= \frac{1}{y} F_4\left(\frac{1}{y/(1-2xy)}\right) \\ &= \frac{1}{y}\left(\frac{y}{1-2xy}\right) \sum_{n\geq 0} a_n \left(\frac{y}{1-2xy}\right)^n \\ &= \frac{1}{1-2xy} \sum_{n\geq 0} a_n \left(\frac{y}{1-2xy}\right)^n. \end{aligned} \tag{5.15}$$

In order to get a more precise information about p, q's we substitute $x = 1$ in (5.15):

$$\Delta(1,y) = \frac{1}{(1-2y)} \sum_{n\geq 0} a_n \left(\frac{y}{1-2y}\right)^n,$$

or

$$(1-2y)\Delta(1,y) = \sum_{n\geq 0} a_n \left(\frac{y}{1-2y}\right)^n. \tag{5.16}$$

Now we apply the transformation $y \mapsto z = y/(1-2y)$ in (5.16):

$$\frac{1}{(1+2z)}\Delta\left(1, \frac{z}{1+2z}\right) = \sum_{n\geq 0} a_n z^n. \tag{5.17}$$

This proves the following theorem.

Theorem 5.3.1. *If $f(z)$ denotes the polynomial that is obtained from $\Delta(1,y)$ by the transformation $y \leftrightarrow z/(1+2z)$, then we have*

$$f(z) = (1+2z)\sum_{n\geq 0} a_n z^n. \tag{5.18}$$

5.4 A combinatorial interpretation

In this section, our aim is to give a combinatorial set of objects whose cardinality is given by Δ_n. More explicitly, we produce an explicit bijection between the set of skew symmetric (n,n) clans and the Delannoy paths with certain labels, is denoted by $\mathcal{D}^\omega(n,n)$.

Theorem 5.4.1. *There is a bijection between the set of weighted (n,n) Delannoy paths and the set of skew symmetric (n,n) clans. In particular, we have*

$$\Delta_n = \sum_{W\in\mathcal{D}^\omega(n,n)} 1.$$

Proof. Let e_n denote the cardinality of $\mathcal{D}^\omega(n,n)$. We will prove that e_n obeys the same recurrence as Δ_n's and it satisfies the same initial conditions. Let $\gamma = c_1 \dots c_{2n}$ be a skew symmetric (n,n) clan and let $\pi = \pi_\gamma$ denote the signed involution

$$\pi = (i_1, j_1)\dots(i_{2k}, j_{2k})d_1^{s_1}\dots d_{2n-4k}^{s_{2n-4k}}, \quad \text{where } s_1,\dots,s_{2n-4k} \in \{+,-\},$$

which is given by $\pi = \varphi(\gamma)$ (Here, φ is the bijection between clans and signed involution, see [20] for detail). We will construct a weighted (n, n) Delannoy path $W = W_\gamma$ which is uniquely determined by π.

First, we look at the positions of $2n$ and its partnering 1 in π. The reason why we do is that the skew symmetry condition. If $2n$ appears as a fixed point with a $+$ sign, 1 must appear as a fixed point with a $-$ sign. Then we draw first a W-step between (n, n) and $(n-1, n)$ and a S-step between $(n-1, n)$ and $(n-1, n-1)$. We label both of these steps by 1 to turn them into labeled steps. Next, we remove the fixed points 1 and $2n$ from π. The result is a skew symmetric $(n-1, n-1)$ involution. Now, by our induction hypothesis, in the first case, there are e_{n-1} possible ways of extending this path to a weighted (n, n) Delannoy path. In a similar manner, one can show that for the case $2n$ appears as a fixed point with a $-$ sign, 1 must appear as a fixed point with a $+$ sign there are again e_{n-1} possible ways of extending this path to a weighted (n, n) Delannoy path.

Next consider the case where $2n$ appears in a transposition. Then by the symmetry condition, there is either a partnering 2-cycle, which is necessarily of the form $(1, i')$ for some i' or there is not.

In the case when $2n$ has partnering, we draw a D-step between (n, n) and $(n-2, n-2)$ and we label this step by i. Then we remove the two cycle $(i, 2n)$ as well as its partner $(1, i')$ from π. Let us denote the resulting object by $\pi^{(1)}$. To get rid of the gaps created by the removal of two 2-cycles, we renormalize the remaining entries by appropriately subtracting numbers so that the resulting object, which we denote by $\pi^{(1)}$ has every number from $\{1, \dots, 2n-4\}$ appears in it exactly once. It is easy to see that we have a signed $(n-2, n-2)$ involution which corresponds to a skew symmetric $(n-2, n-2)$ clan under $\widetilde{\varphi}^{-1}$. Now, the label of this diagonal step can be chosen as one the $2(n-1)$ numbers from $\{2, \dots, 2n-1\}$. Finally, let us note that there are e_{n-2} possible ways to extend this labeled diagonal step to a weighted (n, n) Delannoy path.

As a final step, consider the case when $2n$ does not have a partnering, we draw a D-step between (n, n) and $(n-1, n-1)$ and we label this step by i. Then we remove the two cycle $(i, 2n)$. To get rid of the gaps created by the removal of two 2-cycles, we re-normalize the remaining entries by appropriately subtracting numbers so that the resulting object has every number from $\{1, \ldots, 2n-4\}$ appears in it exactly once. Finally, let us note that there are e_{n-1} possible ways to extend this labeled diagaonal step to a weighted (n, n) Delannoy path.

Combining our observations we see that, starting with a random skew symmetric (n, n) clan, there are exactly

$$3e_{n-1} + 2(n-1)e_{n-2} \tag{5.19}$$

possible weight Delannoy paths that we can construct. By induction hypothesis the number (5.19) is equal to e_n. This finishes our proof. □

Let us illustrate our construction by an example.

Example 5.20. Let γ denote the skew symmetric $(5, 5)$ clan

$$\gamma = 4\ +\ 6\ -\ 2\ 2\ +\ 4\ -\ 6,$$

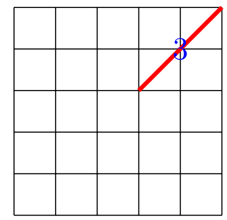
and let π denote the corresponding signed involution

$$\pi = (1, 8)(3, 10)(5, 6)2^+4^-7^+9^-.$$

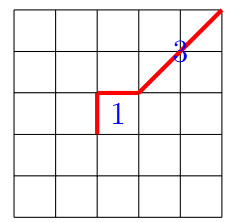
The steps of our constructions are shown in Figure 5.2.

The combinatorial description of the weak order on the set of all skew symmetric (n, n) clans is given in [8]. By using that description of the weak order and the bijection between clans and the lattice paths, we finish this section by the following

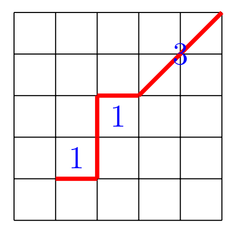
$$\pi = (1,8)(3,10)(5,6)2^+4^-5^+7^+9^-$$
$$\downarrow$$
$$\pi^{(1)} = 1^+2^-(3,4)5^-6^+$$



$$\pi^{(1)} = 1^+2^-(3,4)5^-6^+$$
$$\downarrow$$
$$\pi^{(2)} = \pi^{(1)} = 1^-(2,3)4^+$$



$$\pi^{(2)} = 1^-(2,3)4^+$$
$$\downarrow$$
$$\pi^{(3)} = (1,2)$$



$$\pi^{(3)} = (1,2)$$
$$\downarrow$$
$$\pi^{(4)} = \cdot$$

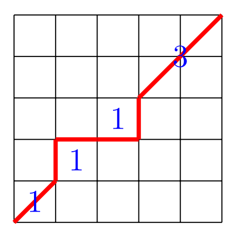


Figure 5.2: Algorithmic construction of the bijection onto weighted Delannoy paths.

figure where we illustrate the weak order on the corresponding lattice paths, see Figure (5.3) below.

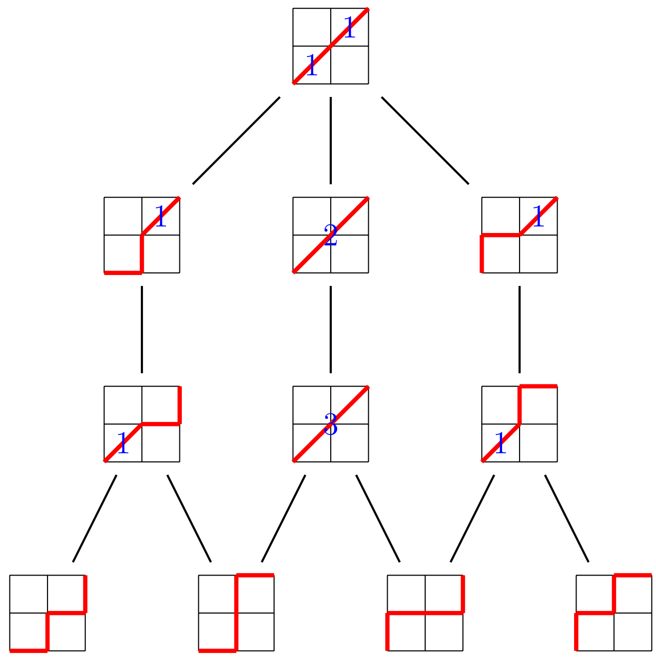


Figure 5.3: Weak order on $\mathcal{D}^{\omega}(2,2)$

At this point, one would ask the same question which was:

Question 5.21. Is it possible to give the descriptions of the weak order (or Bruhat order) in terms of lattice paths?

For the weak order, the covering relations are described in [8, Subsection 3.1.2] and we it is possible to translate these relations into our language. Here we again skip the details for a future work.

Chapter 6

Type DI(i)

It follows immediately from our convention 1.3 that throughout this chapter it is essential to assume that $0 \leq q \leq p$. Again details can be found in [7]. In this case it is easy to check that the involution on $SO(2n, \mathbb{C})$ defined by

$$\sigma(g) = I_{p,2q,p} g I_{p,2q,p}^{-1},$$

gives us the fixed point subgroup is $S(O(2p, \mathbb{C}) \times O(2q, \mathbb{C}))$.

Recall that $S(O(2p, \mathbb{C}) \times O(2q, \mathbb{C}))$ orbits in the flag variety are parametrized by symmetric $(2p, 2q)$ clans. In this chapter, our goal is to find various associated (bivariate) generating functions.

6.1 Counting symmetric $(2p, 2q)$ clans with k-pairs.

Although our aim is to count all symmetric $(2p, 2q)$ clans, we will denote the number of symmetric $(2p, 2q)$ clans by $\Theta_{p,q}$. We start by stating a simple lemma that tells about the involutions corresponding to symmetric clans.

Lemma 6.1.1. Let $\gamma = c_1 c_2 \dots c_{2n}$ be a symmetric $(2p, 2q)$ clan. Then there are exactly even number of pairs of natural numbers in γ.

Proof. Suppose to the contrary that the number of transpositions k is odd, that is $k = 2l + 1$ for some positive integer l. It follows from the symmetry that number of $+$ signs, say α, among the first n spots must be equal to number of $+$ signs among the second n spots. The same argument works for $-$ signs as well, which will be denoted by β. On one hand, this tells us that

$$2\alpha + 2\beta = 2n - 4l - 2 = 2p + 2q - 4l - 2.$$

On the other hand, by the definition of the symmetric clan we have

$$2\alpha - 2\beta = 2p - 2q.$$

Combining these two gives us the following:

$$2\alpha = 2\beta - 2l - 1 = 2(p - l) - 1,$$

which is a contradiction. □

Before we move to our main result of this section, let us state the following remark that is about the upper bound for k, and therefore for l.

Remark 6.1.2. Because of our Convention 1.3, there must be $2p - 2q$ more $+$'s than $-$'s, hence $2q + 2p - 4k \geq 2p - 2q$. Thus, $k \leq 2q$ and $l \leq q$.

Our next aim is to determine the number of possible ways of placing k pairs to build from scratch a symmetric $(2p, 2q)$ clan $\gamma = c_1 \cdots c_n c_{n+1} c_{n+2} \cdots c_{2n}$, which will be denoted by $\theta_{l,p,q}$.

Theorem 6.1.3. *The number of symmetric $(2p, 2q)$ clans with $k = 2l$ pairs is given* *by*

$$\theta_{l,p,q} = \binom{n-2l}{p-l}\binom{n}{2l} a_{2l} \quad \textit{where} \quad a_{2l} := \sum_{b=0}^{l} \binom{2l}{2b} \frac{(2b)!}{b!} \tag{6.1}$$

Consequently, the total number of symmetric $(2p, 2q)$ *clans is given by*

$$\Theta_{p,q} = \sum_{l=0}^{q} \binom{n-2l}{p-l}\binom{n}{2l} a_{2l}.$$

Proof. We have already computed the numbers of possibilities for placing $k = 2l$ pairs, which is given by a_{2l}, but we did not finish counting the number of possibilities for placing the $\pm$ symbols.

Since $k = 2l$ for $0 \leq l \leq q$ the number of $+$ signs is $\alpha := 2p - 2l = 2(p - l)$. Notice that because of symmetry condition it is enough to focus on the first n spots to place $\pm$ symbols. Thus, there are $\binom{n-2l}{p-l}$ possibilities to place $\pm$ signs. This finishes the proof. □

We finish this section with a couple of simple observation and an example.

Example 6.2. Consider the case when $p = 2$ and $q = 1$.

- There are 3 symmetric $(4, 2)$ clans with 0 transposition, which are explicitly

+ + - - + +, + - + + - +, - + + + + -.

- There are 9 symmetric $(4, 2)$ clans with 1 transposition, which are explicitly

2 + + 1 2, 1 + 2 2 + 1, 1 2 + + 1 2, + 1 2 2 1 +, 1 + 1 2 + 2,
1 + 2 1 + 2, + 1 1 2 2 +, + 1 2 1 2 +, 1 1 + + 2 2.

Therefore, in total there are 12 symmetric $(4, 2)$ clans.

Remark 6.1.4. The first few values of $\Theta_{p,q}$ for all $q = 0, 1, 2, \ldots$ are

$$\begin{aligned}
\Theta_{p,0} &= 1 \\
\Theta_{p,1} &= \frac{3p^2 + 5p + 2}{2} \\
\Theta_{p,2} &= \frac{25p^4 + 86p^3 + 95p^2 + 58p + 24}{24},
\end{aligned}$$

and so on. Observe here that for every fixed q, the integer $\Theta_{p,q}$ can be viewed as a specific value of a polynomial function of p. However, it is already apparent from the case of $q = 1$ that this polynomial may have non-integer coefficients. We conjecture that $q = 0$ is the only case where $p \mapsto \Theta_{p,q}$ is a polynomial function with integral coefficients. We conjecture also that for every nonnegative integer q, as a polynomial in p, $\Theta_{p,q}$ is unimodal.

6.2 Recurrences

We start with some easy recurrences. In this case, whenever both sides of the following equations are defined, they hold true:

$$\theta_{l,p-1,q} = \frac{p-l}{p+q}\theta_{l,p,q}, \tag{6.3}$$

$$\theta_{l,p,q-1} = \frac{q-l}{p+q}\theta_{l,p,q}, \tag{6.4}$$

Note that l does not change in them. In the sequel, we will find other recurrences that run over l's. Towards this end, the following lemma whose proof is simple, will be useful.

Let us first recall that the following recurrence relation for a_{2l}. The proof can be found in [20].

Lemma 6.2.1. For all $1 \leq l \leq q-1$, the following recurrences:

$$a_{2l+2} = (8l+3)a_{2l} + 4(2l)(2l-1)a_{2l-2} \tag{6.5}$$

with $a_0 = 1, a_2 = 3$ are satisfied.

It follows from the Lemma 6.2.1 that we obtain a recurrence relation for $\theta_{k,p,q}$'s where all of k's are even numbers.

$$\begin{aligned}
\theta_{l+1,p,q} =& \binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l+2} \\
=& \binom{n-2l-2}{p-l-1}\binom{n}{2l+2}((8l+3)a_{2l} - 4(2l)(2l-1)a_{2l-2}) \\
=& (8l+3)\binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l} + 4(2l)(2l-1)\binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l-2} \\
=& (8l+3)\frac{(p-l)(q-l)}{(2l+2)(2l+1)}\theta_{l,p,q} - 4\frac{(p-l)(p-l+1)(q-l)(q-l+1)}{(2l+2)(2l+1)}\theta_{l-1,p,q}.
\end{aligned} \tag{6.6}$$

6.3 Generating Function

In this section, we are looking for the closed form of the generating function

$$\Theta(y,z) = \sum_{p \geq 0} \Theta_p(1,y)z^p,$$

where

$$\Theta_{p,q}(x) = \sum_{l=0}^{q} \theta_{l,p,q}x^{q-l} \quad \text{and} \quad \Theta_p(x,y) = \sum_q \Theta_{p,q}(x)y^q.$$

In particular, we are looking for an expression of $\Theta_{p,q}(1)$. Obviously,

$$\Theta_{p,q-1}(x) = \sum_{l=0}^{q-1} \theta_{l,p,q-1}x^{q-l-1}.$$

It follows from the relation (6.4) that

$$\Theta_{p,q}(x) = \sum_{l=0}^{q-1} \theta_{l,p,q} x^{q-l} + \theta_{q,p,q}.$$

Taking the derivative of both sides of the above equation gives us that

$$\Theta'_{p,q}(x) = \sum_{l=0}^{q-1} (p+q)\theta_{l,p,q-1} x^{q-l-1},$$

or equivalently gives that

$$\Theta'_{p,q}(x) = (p+q)\Theta_{p,q-1}(x). \tag{6.7}$$

The differential equation (6.7) leads to a PDE for our initial generating function $\Theta_p(x, y)$:

$$\begin{aligned}
\frac{\partial}{\partial x}(\Theta_p(x,y)) =& \frac{\partial}{\partial x}\left[\sum_{q\geq 0} \Theta_{p,q}(x) y^q\right] = \sum_{q\geq 1} \Theta'_{p,q}(x) y^q = \sum_{q\geq 1} (p+q)\Theta_{p,q-1}(x) y^q \\
=& py \sum_{q\geq 1} \Theta_{p,q-1}(x) y^{q-1} + y \sum_{q\geq 1} q\Theta_{p,q-1}(x) y^{q-1} \\
=& py\Theta_p(x,y) + y\left(\frac{\partial}{\partial y}(y \cdot \Theta_p(x,y))\right) \\
=& y^2 \frac{\partial}{\partial y}\Theta_p(x,y) + y\Theta_p(x,y) + py\Theta_p(x,y).
\end{aligned}$$

with the initial conditions $\Theta_p(0, y) = \sum_{q\geq 0} \theta_{q,p,q} y^q$. By the last equation we obtain the PDE that we mentioned in the introduction:

$$\frac{\partial}{\partial x}\Theta_p(x,y) - y^2 \frac{\partial}{\partial y}\Theta_p(x,y) = y(1+p)\Theta_p(x,y). \tag{6.8}$$

The general solution $S_5(x,y)$ of (6.8) is given by

$$S_5(x,y) = \frac{1}{y^{p+1}} F_5\left(\frac{1-xy}{y}\right), \tag{6.9}$$

where $F_5(z)$ is some function in one-variable. We want to choose $F_5(z)$ in such a way that $S_5(x,y) = \Theta_p(x,y)$ holds true. To do so, first, we look at some special values of $\Theta_p(x,y)$. It is not surprising at all that we have the PDE as in the one shown in Chapter 3.

If let $x = 0$, then $\Theta_p(0,y) = \sum_{q\geq 0} \theta_{q,p,q} y^q$ for all $q > 0$. Thus, we ask from $F_5(z)$ that it satisfies the following equation

$$F_5\left(\frac{1}{y}\right) = y^{p+1} \sum_{q\geq 0} \theta_{q,p,q} y^q.$$

Therefore, we can conclude that our generating function is given by

$$\begin{aligned}
\Theta_p(x,y) &= \frac{1}{y^{p+1}} F_5\left(\frac{1}{y/(1-xy)}\right) \\
&= \frac{1}{y^{p+1}} \left(\frac{y}{1-xy}\right)^{p+1} \sum_{q\geq 0} \theta_{q,p,q} \left(\frac{y}{1-xy}\right)^q \\
&= \left(\frac{1}{1-xy}\right)^{p+1} \sum_{q\geq 0} \theta_{q,p,q} \left(\frac{y}{1-xy}\right)^q .
\end{aligned} \tag{6.10}$$

To get a more precise information about $\Theta_{p,q}$'s we substitute $x = 1$ in (6.10):

$$\Theta_p(1,y) = \frac{1}{(1-y)^{p+1}} \sum_{q\geq 0} \theta_{q,p,q} \left(\frac{y}{1-y}\right)^q .$$

Now, we apply the transformation $y \mapsto z = y/(1-y)$ in (6.10):

$$\Theta_p\left(1, \frac{z}{1+z}\right) = \frac{1}{(1+z)^{p+1}} \sum_{q\geq 0} \theta_{q,p,q} \left(\frac{z}{1+z}\right)^q . \tag{6.11}$$

This finishes the proof of the following theorem since $\Theta_p\left(1, \frac{z}{1+z}\right) = f_p(z)$.

Theorem 6.3.1. *If $f_p(z)$ denotes the polynomial that is obtained from $\Theta_p(1, y)$ by the transformation $y \leftrightarrow z/(1+z)$, then we have*

$$f_p(z) = \frac{1}{(1+z)^{p+1}} \sum_{q \geq 0} \theta_{q,p,q} \left(\frac{z}{1+z}\right)^q. \tag{6.12}$$

Chapter 7

Type DI(ii)

It follows immediately from our convention 1.3 that throughout this chapter it is essential to assume that $1 \leq q \leq p+1$. Due to Cartan's classification of involutions on algebraic groups it is known that there is only one involutory automorphism associated with orthogonal decomposition for $SO(2n+1, \mathbb{C})$, which is given as follows:

$$\sigma(g) = \begin{pmatrix} id_{2p+1} & 0 \\ 0 & -id_{2q-1} \end{pmatrix} g \begin{pmatrix} id_{2p+1} & 0 \\ 0 & -id_{2q-1} \end{pmatrix},$$

refer to the section 1.1 for the notation here. See [7], for details.

In this case, one can show that the fixed point subgroup is $S(O(2p+1, \mathbb{C}) \times O(2q-1, \mathbb{C}))$.

Recall that $S(O(2p+1, \mathbb{C}) \times O(2q-1, \mathbb{C}))$ orbits in the flag variety are parametrized by symmetric $(2p+1, 2q-1)$ clans. In this chapter, our goal is to find various associated (bivariate) generating functions.

7.1 Counting symmetric $(2p+1, 2q-1)$ clans with k-pairs.

Although our aim is to count all symmetric $(2p+1, 2q-1)$ clans, we will denote the number of symmetric $(2p+1, 2q-1)$ clans by $M_{p,q}$. We start by stating a simple lemma that tells about the involutions corresponding to symmetric clans.

Lemma 7.1.1. Let $\gamma = c_1 c_2 \dots c_{2n}$ be a symmetric $(2p+1, 2q-1)$ clan. Then there are exactly odd number of pairs of natural numbers in γ.

Proof. Suppose to the contrary that the number of transpositions k is even, that is $k = 2l$ for some positive integer l. It follows from the symmetry that number of $+$ signs, say α, among the first n spots must be equal to number of $+$ signs among the second n spots. The same argument works for $-$ signs as well, which will be denoted by β. On one hand, this tells us that

$$2\alpha + 2\beta = 2n - 4l = 2p + 2q - 4l.$$

On the other hand, by the definition of the symmetric clan we have

$$2\alpha - 2\beta = 2p - 2q + 2.$$

Combining these two gives us the following:

$$2\alpha = 2(p - l) + 1,$$

which is a contradiction. $\square$

Before we move to our main result for this section, let us state the following remark that is about the upper bound for k, and therefore for l.

Remark 7.1.2. Because of our Convention 1.3, there must be $2p-2q+2$ more $+$'s than $-$'s, hence $2q+2p-2k \geq 2p-2q+2$. Thus, $k \leq 2q-1$ and $l \leq q$.

Our next aim is to determine the number of possible ways of placing k pairs to build from scratch a symmetric $(2p+1, 2q-1)$ clan $\gamma = c_1 \cdots c_{2n}$, which will be denoted by $\mu_{l,p,q}$.

Theorem 7.1.3. *The number of symmetric* $(2p+1, 2q-1)$ *clans with* $k = 2l-1$ *pairs is given by*

$$\mu_{l,p,q} = \binom{n-2l+1}{p-l+1}\binom{n}{2l-1} a_{2l-1} \quad \textit{where} \quad a_{2l-1} := \sum_{b=0}^{l-1} \binom{2l-1}{2b} \frac{(2b)!}{b!} \tag{7.1}$$

Consequently, the total number of symmetric $(2p+1, 2q-1)$ *clans is given by*

$$M_{p,q} = \sum_{l=1}^{q} \binom{n-2l+1}{p-l+1}\binom{n}{2l-1} a_{2l-1}.$$

Proof. We have already computed the numbers of possibilities for placing $k = 2l-1$ pairs, which is given by a_{2l-1}, but we did not finish counting the number of possibilities for placing the $\pm$ signs.

Since $k = 2l-1$ for $1 \leq l \leq q$. In this case, by using an argument as before, we see that there are $\binom{n-(2l-1)}{p-(l-1)}$ possibilities to place $\pm$ signs. This finishes the proof. $\square$

We finish this section with a simple observation and an example.

Example 7.2. Consider the case when $p = 1$ and $q = 2$.

- There are 6 symmetric $(3,3)$ clans with 1 transposition, which are explicitly

  +-11-+, -+11+-, +1–1+, -1++1-, 1+–+1, 1-++-1.

- There are 7 symmetric $(3,3)$ clans with 3 transposition, which are explicitly

  112233, 123123, 121323, 123231, 122331, 123312, 123321.

Therefore, in total there are 13 symmetric $(3,3)$ clans.

Remark 7.1.4. The first few values of $M_{p,q}$ for all $q = 1, 2, \ldots$ are

$$\begin{aligned} M_{p,1} &= \mu_{1,p,1} = p + 1 \\ M_{p,2} &= \frac{7p^3 + 27p^2 + 32p + 12}{6} \\ M_{p,3} &= \frac{25p^4 + 86p^3 + 95p^2 + 58p + 24}{24}, \end{aligned}$$

and so on. Observe here that for every fixed q, the integer $M_{p,q}$ can be viewed as a specific value of a polynomial function of p. However, it is already apparent from the case of $q = 2$ that this polynomial may have non-integer coefficients. We conjecture that $q = 1$ is the only case where $p \mapsto M_{p,q}$ is a polynomial function with integral coefficients. We conjecture also that for every nonnegative integer q, as a polynomial in p, $M_{p,q}$ is unimodal.

7.2 Recurrences

We start with some easy recurrences. In this case, whenever both sides of the following equations are defined, they hold true:

$$\mu_{l,p-1,q} = \frac{p-l}{p+q}\mu_{l,p,q}, \tag{7.3}$$

$$\mu_{l,p,q-1} = \frac{q-l}{p+q}\mu_{l,p,q}, \tag{7.4}$$

Note here that l does not change in them. In the sequel, we will find other recurrences that run over l's. Towards this end, the following lemma, whose proof is simple, will be useful.

Let us first recall that the following recurrence relation for a_{2l-1}. The proof can be found in [20].

Lemma 7.2.1. For all $3 \leq l \leq q$, the following recurrences:

$$a_{2l-1} = (8l-9)a_{2l-3} - 4(2l-3)(2l-4)a_{2l-5} \tag{7.5}$$

with $a_1 = 1, a_3 = 7$ are satisfied.

It follows from Lemma 7.2.1 that we obtain a recurrence relation for $\mu_{l,p,q}$'s;

$$\mu_{l+1,p,q} = (8l-1)\frac{(p-l)(q-l)}{(2l)(2l+1)}\mu_{l,p,q} - 4\frac{(p-l)(p-l+1)(q-l)(q-l+1)}{(2l)(2l+1)}\mu_{l-1,p,q}. \tag{7.6}$$

7.3 Generating Functions

In this section, we are looking for the closed form of the generating function

$$M(y,z) = \sum_{p\geq 0} M_p(1,y)z^p,$$

where

$$M_{p,q}(x) = \sum_{l=1}^{q} \mu_{l,p,q}x^{q-l} \quad \text{and} \quad M_p(x,y) = \sum_{q} M_{p,q}(x)y^q.$$

In particular, we are looking for an expression of $M_{p,q}(1)$. Obviously,

$$M_{p,q-1}(x) = \sum_{l=0}^{q-1} \mu_{l,p,q-1}x^{q-l-1}.$$

It follows from the equation (7.3) that

$$M_{p,q}(x) = \sum_{l=0}^{q-1} \mu_{l,p,q}x^{q-l} + \mu_{q,p,q}$$

Taking the derivative of both sides of the above equation gives us that

$$M_{p,q}^{'}(x) = \sum_{l=0}^{q-1}(p+q)\mu_{l,p,q-1}x^{q-l-1},$$

or, equivalently, gives that

$$M_{p,q}^{'}(x) = (p+q)M_{p,q-1}(x). \tag{7.7}$$

The differential equation (7.7) leads to a PDE for our initial generating function $M_p(x,y)$:

$$\begin{aligned}\frac{\partial}{\partial x}(M_p(x,y)) =& \frac{\partial}{\partial x}\left[\sum_{q\geq 1} M_{p,q}(x)y^q\right] = \sum_{q\geq 1} M_{p,q}^{'}(x)y^q \\ =& M_{p,1}^{'}(x)y\sum_{q\geq 2}(p+q)M_{p,q-1}(x)y^q \\ =& p\sum_{q\geq 2} M_{p,q-1}(x)y^q + y\sum_{q\geq 2} qM_{p,q-1}(x)y^{q-1} \\ =& pyM_p(x,y) + y\left(\frac{\partial}{\partial y}(y\cdot M_p(x,y))\right) \\ =& y^2\frac{\partial}{\partial y}M_p(x,y) + yM_p(x,y) + pyM_p(x,y).\end{aligned}$$

with the initial conditions $M_p(0,y) = \sum_{q\geq 0}\mu_{q,p,q}y^q$. By the last equation we obtain the PDE that we mentioned in the introduction:

$$\frac{\partial}{\partial x}M_p(x,y) - y^2\frac{\partial}{\partial y}M_p(x,y) = y(1+p)M_p(x,y). \tag{7.8}$$

The general solution $S_7(x,y)$ of (7.8) as we know from the previous section is given by

$$S_6(x,y) = \frac{1}{y^{p+1}}F_6\left(\frac{1-xy}{y}\right), \tag{7.9}$$

where $F_6(z)$ is some function in one-variable. We want to choose $F_6(z)$ in such a way that $S_6(x,y) = M_p(x,y)$ holds true. To do so, first, we look at some special values of $M_p(x,y)$.

If let $x = 0$, then $M_p(0,y) = \sum_{q\geq 1} \mu_{q,p,q} y^q$ for all $q \geq 1$. Thus, we ask from $F_6(z)$ that it satisfies the following equation

$$F_6\left(\frac{1}{y}\right) = y^{p+1} \sum_{q\geq 1} \mu_{q,p,q} y^q.$$

Therefore, we can conclude that our generating function is given by

$$\begin{aligned}
M_p(x,y) &= \frac{1}{y^{p+1}} F_6\left(\frac{1}{y/(1-xy)}\right) \\
&= \frac{1}{y^{p+1}} \left(\frac{y}{1-xy}\right)^{p+1} \sum_{q\geq 1} \mu_{q,p,q} \left(\frac{y}{1-xy}\right)^q \\
&= \left(\frac{1}{1-xy}\right)^{p+1} \sum_{q\geq 1} \mu_{q,p,q} \left(\frac{y}{1-xy}\right)^q. \qquad (7.10)
\end{aligned}$$

To get a more precise information about $M_{p,q}$'s we substitute $x = 1$ in (7.10):

$$M_p(1,y) = \frac{1}{(1-y)^{p+1}} \sum_{q\geq 1} \mu_{q,p,q} \left(\frac{y}{1-y}\right)^q.$$

Now, we apply the transformation $y \mapsto z = y/(1-y)$ in (7.10):

$$M_p\left(1, \frac{z}{1+z}\right) = \frac{1}{(1+z)^{p+1}} \sum_{q\geq 1} \mu_{q,p,q} \left(\frac{z}{1+z}\right)^q. \qquad (7.11)$$

This finishes the proof of Theorem 7.3.1 since $M_p\left(1, \frac{z}{1+z}\right) = f_p(z)$.

Theorem 7.3.1. *If $f_p(z)$ denotes the polynomial that is obtained from $M_p(1,y)$ by* *the transformation* $y \leftrightarrow z/(1+z)$, *then we have*

$$f_p(z) = \frac{1}{(1+z)^{p+1}} \sum_{q \geq 1} \mu_{q,p,q} \left(\frac{z}{1+z} \right)^q . \tag{7.12}$$

Appendix A

Another approach for the generating functions

A.1 Modified Bessel Function of the Second Kind

The bivariate generating function $\sum_{p,q\geq 0} A_{p,q} x^q \frac{y^p}{p!}$ is given by

$$v(x,y) = \frac{e^y}{1-x} + \frac{xe^{\frac{y}{1-x}}}{(1-2x)(1-x)} + \sum_{p\geq 0} \frac{e^{\frac{1}{x}}\sqrt{\frac{2}{\pi x}}\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!} y^p.$$

In order to get a better expression for $v(x,y)$ we should re-arrange the latter by using some identities of Bessel function. Recall that a modified spherical Bessel function of the second kind is the second solution to the modified spherical Bessel differential equation, given by

$$k_p(z) = \sqrt{\frac{2}{\pi z}}\widetilde{K}_{p+\frac{1}{2}}(z).$$

Writing $k_p(z) = e^{-z}\nu_p(z)$, where $\nu_p(z)$ are given by the recurrence relation

$$\nu_p(z) = \nu_{p-2}(z) + \frac{(2p-1)}{z}\nu_{p-1}(z); \;\; \text{with} \;\; \nu_0(z) = \frac{1}{z}, \;\; \nu_1(z) = \frac{z+1}{z^2}.$$

Therefore, our sum becomes

$$\sum_{p\geq 0}\frac{e^{\frac{1}{x}}\sqrt{\frac{2}{\pi x}}\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!}y^p=\frac{1}{x}\sum_{p\geq 0}\frac{\nu_p(\frac{1}{x})}{p!}y^p.$$

Then, writing $\nu_p(\frac{1}{x})=\frac{N_p(\frac{1}{x})}{(\frac{1}{x})^{p+1}}$ where $N_p(\frac{1}{x})$ are given by the following recurrence relation

$$N_p\left(\frac{1}{x}\right)=\left(\frac{1}{x}\right)^2 N_{p-2}\left(\frac{1}{x}\right)+(2p-1)N_{p-1}\left(\frac{1}{x}\right);\ \text{ with } N_0\left(\frac{1}{x}\right)=1,\ \ N_1\left(\frac{1}{x}\right)=1+\frac{1}{x},$$

Moreover by using the given identities above, we can compute the first few values for small nonnegative integer indices and re-arrange the terms

$$\begin{aligned}
\widetilde{K}_{0+\frac{1}{2}}(z) &= \sqrt{\frac{\pi z}{2}}e^{-z}\frac{1}{z}=\sqrt{\frac{\pi z}{2}}e^{-z}\frac{N_0(z)}{z}\\
\widetilde{K}_{1+\frac{1}{2}}(z) &= \sqrt{\frac{\pi z}{2}}e^{-z}\frac{z+1}{z^2}=\sqrt{\frac{\pi z}{2}}e^{-z}\frac{N_1(z)}{z^2}\\
\widetilde{K}_{2+\frac{1}{2}}(z) &= \sqrt{\frac{\pi z}{2}}e^{-z}\frac{z^2+3z+3}{z^3}=\sqrt{\frac{\pi z}{2}}e^{-z}\frac{N_2(z)}{z^3}\\
\widetilde{K}_{3+\frac{1}{2}}(z) &= \sqrt{\frac{\pi z}{2}}e^{-z}\frac{z^3+6z^2+15z+15}{z^4}=\sqrt{\frac{\pi z}{2}}e^{-z}\frac{N_3(z)}{z^4}\\
\widetilde{K}_{4+\frac{1}{2}}(z) &= \sqrt{\frac{\pi z}{2}}e^{-z}\frac{z^4+10z^3+45z^2+105z+105}{z^5}=\sqrt{\frac{\pi z}{2}}e^{-z}\frac{N_4(z)}{z^5}
\end{aligned}$$

gives us

$$\sum_{p\geq 0}\frac{e^{\frac{1}{x}}\sqrt{\frac{2}{\pi x}}\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!}y^p=\frac{1}{x}\sum_{p\geq 0}\frac{\nu_p(\frac{1}{x})}{p!}y^p=\sum_{p\geq 0}\frac{AN_p(\frac{1}{x})}{p!}(xy)^p.$$

Therefore,

$$v(x,y)=\frac{e^y}{1-x}+\frac{xe^{\frac{y}{1-x}}}{(1-2x)(1-x)}+\sum_{p\geq 0}\frac{N_p(\frac{1}{x})}{p!}(xy)^p.$$

A.2 Another Approach for the Generating Function for Type BI

In this appendix, as we promised in the introduction, we outline a method for approximating the number of symmetric $(2p, 2q+1)$ clans with k pairs, $\beta_{k,p,q}$. Recall our notation that $A_e(x) = \sum_{l=0}^{q} \beta_{2l,p,q} x^{2l}$ and $A_o(x) = \sum_{l=0}^{q} \beta_{2l+1,p,q} x^{2l+1}$.

First of all, by multiplying both sides of the recurrence relation (3.10) by x^{2l} and summing over l lead us to the following integral/differential equation

$$\begin{aligned} A_e(x) - \beta_{0,p,q} =& (q+1)\int (A_o(x) - \beta_{2q+1,p,q}x^{2q+1})dx - \frac{x}{2}(A_o(x) - \beta_{2q+1,p,q}x^{2q+1}) \\ &+2(pq+p+q+1)\int x(A_e(x) - \beta_{2q,p,q}x^{2q})dx \\ &-(p+q+1)x^2(A_e(x) - \beta_{2q,p,q}x^{2q}) + \frac{x^3}{2}A_e^{'}(x) - q\beta_{2q,p,q}x^{2q+2}. \end{aligned}$$

We get rid of the integrals by taking the derivative with respect to x and then we reorganize our equation which is now a second order ODE as in

$$x^3 A_e^{''}(x) - \Big((2p+2q-1)x^2 + 2\Big)A_e^{'}(x) + 4pqxA_e(x) - xA_o^{'}(x) + (2q+1)A_0(x) = 0.$$

By applying a similar procedure to the recurrence relation (3.11) and also by using the fact that $\beta_{1,p,q} = p\beta_{0,p,q}$, we obtain our second order ODE:

$$\begin{aligned} x^3 A_o^{''}(x) - \Big((2p+2q-1)x^2 + 2\Big)A_o^{'}(x) + (4pq+2p-2q-1)xA_o(x) \\ - xA_e^{'}(x) + 2pA_e(x) = 0. \end{aligned}$$

Note that we the following initial conditions that follow from the definitions of $A_e(x)$

and $A_o(x)$:

$$\begin{aligned}A_e(0) =&\beta_{0,p,q} \text{ and } A_o(0) = 0,\\ A_e^{'}(0) =&0 \text{ and } A_o^{'}(0) = p\beta_{0,p,q} = \beta_{1,p,q}.\end{aligned}$$

We will reduce our second order system to a first order ODE by setting $u(x) := A_e^{'}(x)$ and $v(x) := A_o^{'}(x)$. Then

$$\begin{aligned}x^3u'(x) =&((2p+2q-1)x^2+2)u(x) - 4pqxA_e(x) - (2q+1)A_0(x) + xv(x)\\ x^3v'(x) =&((2p+2q-1)x^2+2)v(x) - (4pq+2p-2q-1)xA_o(x) - 2pA_e(x) + xu(x)\\ x^3A_e^{'}(x) =&x^3u(x)\\ x^3A_o^{'}(x) =&x^3v(x).\end{aligned}$$

We write this system in matrix form

$$x^3X' = A(x)X,$$

where $A(x)$ the 4×4 matrix as defined below. Note that our initial conditions become

$$X(0) = \begin{bmatrix} u(0)\\ v(0)\\ A(0)\\ A(0)\end{bmatrix} = \begin{bmatrix} 0\\ p\beta_{0,p,q}\\ \beta_{0,p,q}\\ 0\end{bmatrix}. \tag{A.1}$$

Once a system of first order ordinary differential equations of this type is given, formal series solutions can always be obtained by carrying out the computational procedure, which is outlined in [23]. We will use those techniques to solve the above system of first order ordinary differential equations.

Before proceeding any further let us define the matrices $A_0, A_1, \ldots$ by decomposing the coefficient matrix $A(x)$:

$$A(x) = \sum_{k=0}^{\infty} A_k x^k = \begin{bmatrix} 2 & 0 & 0 & -(2q+1) \\ 0 & 2 & -2p & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^0$$

$$+ \begin{bmatrix} 0 & 1 & -4pq & 0 \\ 1 & 0 & 0 & -(4pq+2q-2q-1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^1$$

$$+ \begin{bmatrix} (2q+2q-1) & 0 & 0 & 0 \\ 0 & (2p+2q-1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^2 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x^3 + \mathbf{0}x^4 + \ldots.$$

Since the eigenvalues of the leading matrix A_0 fall into two groups, namely $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = 2$, there exists a normalizing transformation matrix P obtained from the Jordan canonical form of A_0. More precisely, since

$$\begin{aligned} B_0 =& \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \\ =& \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -p & 0 \\ 1 & 0 & 0 & -\frac{2q+1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & -(2q+1) \\ 0 & 2 & -2p & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2q+1}{2} & 0 & 0 & 1 \\ 0 & p & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ =& P^{-1}A_0P. \end{aligned}$$

the normalizing transformation $X = PY$ turns our system into

$$x^3Y' = B(x)Y; \quad \text{with} \quad Y(0) = \begin{bmatrix} 0 \\ \beta_{0,p,q} \\ 0 \\ 0 \end{bmatrix} \tag{A.2}$$

where

$$
\begin{aligned}
B(x) =& P^{-1}A(x)P \\
=& \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{(2q+1)(4p-3)}{2} & 0 & 0 & 1 \\ 0 & p(1-4q) & 1 & 0 \end{bmatrix} x \\
&+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & p(2p+2q-1) & 2p+2q-1 & 0 \\ \frac{(2q+1)(2p+2q-1)}{2} & 0 & 0 & 2p+2q-1 \end{bmatrix} x^2 \\
&+ \begin{bmatrix} 0 & p & 1 & 0 \\ \frac{2q+1}{2} & 0 & 0 & 1 \\ -\frac{p(2q+1)}{2} & 0 & 0 & -p \\ 0 & -\frac{p(2q+1)}{2} & -\frac{2q+1}{2} & 0 \end{bmatrix} x^3 + \mathbf{0}x^4 + \mathbf{0}x^5 + \dots
\end{aligned}
$$

We denote the coefficient matrix of x^i $(i = 0, 1, 2, \dots)$ in $B(x)$ by B_i. Thus,

$$B(x) = B_0 + B_1 x + B_2 x^2 + B_3 x^3.$$

We will work with a system that is obtained from $B(x)$ by a "shearing" transformation. Let Q be a formal power series of the form $Q = \sum Q_r x^r$ with Q_r's are some constant matrices of order 4. We assume that our desired solution $Y = Y(x)$ for $x^3 Y' = BY$ is of the form $Y = QZ$ for some 4×1 column matrix $Z = Z(x)$. Formally substituting QZ into $x^3 Y' = B(x)Y$ will give us a new ODE:

$$x^3(QZ)' = BQZ \Rightarrow x^3(Q'Z + QZ') = BQZ \quad \text{or} \quad x^3 Z' = (Q^{-1}BQ + x^3 Q^{-1}Q')Z.$$

Let C denote the formal power series $\sum C_r x^r$ that is defined by

$$Q^{-1}BQ + x^3 Q^{-1}Q' = C = \sum C_r x^r, \tag{A.3}$$

hence our ODE is equivalent to

$$x^3 Z' = CZ. \tag{A.4}$$

By multiplying both sides of (A.3) with Q and rearranging we obtain a new ODE whose solution will lead to a solution of (A.5):

$$x^3 Q' = QC - BQ. \tag{A.5}$$

To solve (A.5) we simply substitute $B = \sum B_r x^r$, $Q = \sum Q_r x^r$ and $C = \sum C_r x^r$ and the equate coefficients. Then we get the following relations which we call as our fundamental recurrences.

(i) $0 = Q_0 C_0 - B_0 Q_0$;

(ii) $0 = (Q_0 C_1 - B_1 Q_0) + (Q_1 C_0 - B_0 Q_1)$;

(iii) $(r-2)Q_{r-2} = \sum_{i=0}^{r} (Q_i C_{r-i} - B_{r-i} Q_i)$ for $r \geq 2$.

We will recursively assign specific values to the matrices Q_i, $i = 0, 1, 2, \ldots$ which will allow us to solve (A.5). Along the way we will determine the series $C(x) = \sum x^i C_i$, which is what we want to solve in the first place. Indeed, our goal is to choose Q_i's in such a way that C_i's become block diagonal. To this end, we assume that Q_i ($i = 0, 1, 2, \ldots$) is a block anti-diagonal matrix:

$$Q_i = \begin{bmatrix} 0 & Q_i^{12} \\ Q_i^{21} & 0 \end{bmatrix}$$

for some 2×2 matrices Q_i^{12}, Q_i^{21} $(i = 0, 1, 2, \dots)$.

Step 1. We choose $Q_0 = I_4$, the 4×4 identity matrix. It follows from (i) that

$$C_0 = B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

We have a remark in order.

Remark A.2.1. Let us point out that, since

$$Q_i B_0 - C_0 C_i = \begin{bmatrix} 0 & 2Q_i^{12} \\ -2Q_i^{21} & 0 \end{bmatrix} \quad \text{for } i = 1, 2, \dots \tag{A.6}$$

by using the fundamental recurrences (ii) and (iii) we will always be able to choose Q_i^{12} and Q_i^{21} so that C_i is of the form

$$C_i = \begin{bmatrix} C_i^{11} & 0 \\ 0 & C_i^{22} \end{bmatrix},$$

where C_i^{11} and C_i^{22} are 2×2 matrices.

Step 2. By (ii) and Step 1, $C_1 = B_1 - Q_1 C_0 + B_0 Q_1$, so we set

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{(2q+1)(4p-1)}{4} & 0 & 0 & 0 \\ 0 & \frac{p(4q-1)}{2} & 0 & 0 \end{bmatrix} \Longrightarrow C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Step 3. By (iii) and Steps 1,2, $C_2 = B_2 - Q_1C_1 + B_1Q_1 - Q_2C_0 + B_0Q_2$, so we set

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{p(4p+8q-3)}{4} & 0 & 0 \\ -\frac{(2q+1)(4q-1)}{8} & 0 & 0 & 0 \end{bmatrix} \Longrightarrow$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2p+2q-1 & 0 \\ 0 & 0 & 0 & 2p+2q-1 \end{bmatrix}.$$

In a similar manner, we put

$$Q_3 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{(2q+1)(16p^2+16pq-1)}{16} & 0 & 0 & 0 \\ 0 & \frac{-p(16pq+16q^2-8p-16q+1)}{8} & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \underset{3}{C} = \begin{bmatrix} 0 & p & 0 & 0 \\ \frac{2q+1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & -\frac{2q+1}{2} & 0 \end{bmatrix}.$$

The above computations are in some sense are our initial conditions. To get a better understanding of the general case we make a few more preliminary observations and formal computations.

$$C_i^{jj} = B_i^{jj} \quad \text{for } i = 0,1,2,3 \quad \text{and} \quad j = 1,2. \tag{A.7}$$

$$Q_iC_1 = \begin{bmatrix} 0 & Q_i^{12}C_1^{22} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_1Q_i = \begin{bmatrix} 0 & 0 \\ B_1^{22}Q_i^{21} & B_1^{21}Q_i^{12} \end{bmatrix} \tag{A.8}$$

$$Q_iC_2 = \begin{bmatrix} 0 & Q_i^{12}C_2^{22} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2Q_i = \begin{bmatrix} 0 & 0 \\ B_2^{22}Q_i^{21} & B_2^{21}Q_i^{12} \end{bmatrix} \tag{A.9}$$

$$Q_iC_3 = \begin{bmatrix} 0 & Q_i^{12}C_3^{22} \\ Q_i^{21}C_3^{11} & 0 \end{bmatrix} \quad \text{and} \quad B_3Q_i = \begin{bmatrix} B_3^{12}Q_i^{21} & B_3^{11}Q_i^{12} \\ B_3^{22}Q_i^{21} & B_3^{21}Q_i^{12} \end{bmatrix} \tag{A.10}$$

$$Q_iC_j = \begin{bmatrix} 0 & Q_i^{12}C_j^{22} \\ Q_i^{21}C_j^{11} & 0 \end{bmatrix} \tag{A.11}$$

Finally, since $B_r = 0$, the fundamental recurrence (iii) simplifies to

$$C_r = (r-2)Q_{r-2} - \left(\sum_{i=0}^{3}(Q_{r-i}C_i - B_iQ_{r-i})\right) - \left(\sum_{i=4}^{r-1} Q_{r-i}C_i\right). \tag{A.12}$$

Recall that we started with the system $x^3X' = A(x)X$ which is transformed into $x^3Y' = B(x)Y$ by conjugating with a constant matrix, and the latter system is transformed into $x^3Z' = C(x)Z$ by the shearing transformation $Y = Q(x)Z$.

Proposition A.2.2. Let $C(x) = \sum_r C_r x^r$ and $Q(x) = \sum_r Q_r x^r$ be as in the previous paragraph. If $r \geq 4$, then we have

$$C_r = \begin{bmatrix} Q_{r-3}^{21} & 0 \\ 0 & B_1^{21}Q_{r-1}^{12} + B_2^{21}Q_{r-2}^{12} + B_3^{21}Q_{r-3}^{12} \end{bmatrix}.$$

In particular, the system $x^3Z' = C(x)Z$ decomposes into two 2×2 systems of ODE's

$$x^3K' = R(x)K \tag{A.13}$$

$$x^3L' = T(x)L \tag{A.14}$$

where

$$R(x) = C_3^{11}x^3 + \sum_{r\geq 4} Q_{r-3}^{21}x^r;$$
$$T(x) = \sum_{i=0}^{3} C_i^{22}x^3 + \sum_{r\geq 4}(B_1^{21}Q_{r-1}^{12} + B_2^{21}Q_{r-2}^{12} + B_3^{21}Q_{r-3}^{12})x^r.$$

Proof. Since C_r is a block diagonal matrix, recurrence (A.12) combined with equations (A.7)–(A.11) gives us the desired result. □

What remains is to solving the systems (A.13) and (A.14). The former ODE is relatively easy since it does not have a singularity anymore. However, the second ODE (A.14) does have a singularity. Moreover, we still do not know the exact forms of neither $Q^{12}(x)$ nor $Q^{21}(x)$. On the positive side, by taking advantage of the particular structure of $B(x)$'s we are able to find recurrences for $R(x)$ and $T(x)$.

To find a recurrence for the blocks of Q_r's, we use Proposition A.2.2 as well as the simplified fundamental recurrence (A.12) as follows:

$$
\begin{aligned}
C_r &= \begin{bmatrix} Q^{21}_{r-3} & 0 \\ 0 & B^{21}_1 Q^{12}_{r-1} + B^{21}_2 Q^{12}_{r-2} + B^{21}_3 Q^{12}_{r-3} \end{bmatrix} \\
&= \begin{bmatrix} 0 & (r-2)Q^{12}_{r-2} \\ (r-2)Q^{21}_{r-2} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2Q^{12}_r \\ -2Q^{21}_r & 0 \end{bmatrix} - \begin{bmatrix} 0 & Q^{12}_{r-1}B^{22}_1 \\ -B^{22}_1 Q^{21}_{r-1} & -B^{21}_1 Q^{12}_{r-1} \end{bmatrix} \\
&- \begin{bmatrix} 0 & Q^{12}_{r-2}B^{22}_2 \\ -B^{22}_2 Q^{21}_{r-2} & -B^{21}_2 Q^{12}_{r-2} \end{bmatrix} - \begin{bmatrix} -B^{12}_3 Q^{21}_{r-3} & Q^{12}_{r-3}B^{22}_3 - B^{11}_3 Q^{12}_{r-3} \\ Q^{21}_{r-3}B^{11}_3 - B^{22}_3 Q^{21}_{r-3} & -B^{21}_3 Q^{12}_{r-3} \end{bmatrix} \\
&- \sum_{i=4} \begin{bmatrix} 0 & Q^{12}_{r-i} \\ Q^{21}_{r-i} & 0 \end{bmatrix} \begin{bmatrix} Q^{21}_{i-3} & 0 \\ 0 & B^{21}_1 Q^{12}_{i-1} + B^{21}_2 Q^{12}_{i-2} + B^{21}_3 Q^{12}_{i-3} \end{bmatrix}.
\end{aligned}
$$

Then, the entries of C_r are

$$
\begin{aligned}
C^{11}_r &= Q^{21}_{i-3} \\
C^{12}_r &= (r-2)Q^{12}_{r-2} - 2Q^{12}_r - Q^{12}_{r-1}B^{22}_1 \\
&\quad - Q^{12}_{r-2}B^{22}_2 - Q^{12}_{r-3}B^{22}_3 + B^{11}_3 Q^{12}_{r-3} \\
&\quad - \sum_{i=4} Q^{12}_{r-i}(B^{21}_1 Q^{12}_{i-1} + B^{21}_2 Q^{12}_{i-2} + B^{21}_3 Q^{12}_{i-3}) \\
C^{21}_r &= (r-2)Q^{21}_{r-2} + 2Q^{21}_r + B^{22}_1 Q^{21}_{r-1} + B^{22}_2 Q^{21}_{r-2} \\
&\quad - Q^{21}_{r-3}B^{11}_3 + B^{22}_3 Q^{21}_{r-3} - \sum_{i=4} Q^{21}_{r-i} Q^{21}_{i-3} \\
C^{22}_r &= B^{21}_1 Q^{12}_{i-3} + B^{21}_2 Q^{12}_{i-2} + B^{21}_3 Q^{12}_{i-3}.
\end{aligned}
$$

Observe that the diagonal blocks do not give us any new information, however, the anti-diagonal blocks do. By the equality of the bottom left blocks, we have

$$2Q_r^{21} = -(r-2)Q_{r-2}^{21} - B_1^{22}Q_{r-1}^{21} - B_2^{22}Q_{r-2}^{21} + Q_{r-3}^{21}B_3^{11} - B_3^{22}Q_{r-3}^{21} + \sum_{i=4}^{r-1} Q_{r-i}^{21}Q_{i-3}^{21} \tag{A.15}$$

Similarly, the equality of the top right blocks give

$$\begin{aligned}2Q_r^{12} &= (r-2)Q_{r-2}^{12} - Q_{r-1}^{12}B_1^{22} - Q_{r-2}^{12}B_2^{22} - Q_{r-3}^{12}B_3^{22} + B_3^{11}Q_{r-3}^{12} \\ &\quad - \sum_{i=4}^{r-1} Q_{r-i}^{12}(B_1^{21}Q_{i-1}^{12} + B_2^{21}Q_{i-2}^{12} + B_3^{21}Q_{i-3}^{12}).\end{aligned} \tag{A.16}$$

Obviously, these recurrences enable us to write the precise forms of the ODE's (A.13) and (A.14). Both of these ODE's can now be solved by applying suitable shearing transformations leading to a solution of our original equation $x^3X' = A(x)X$. However, due to its high computational cost the result is still not better than the expressions for $\beta_{k,p,q}$'s that we recorded in Theorem 3.1.4.

A.3 Another Approach for the Generating Function for Type CI

Let us start this section by defining our main generating polynomial for our first approach as follows:

$$P(x) := \sum_{k=0}^{n} \delta_{k,n} x^k.$$

Multiplying by x^k both sides of the equation (5.7) and taking the sum over k yield us the following equation:

$$
\begin{aligned}
\sum_{k=2}^{n} \delta_{k,n} x^k &= \sum_{k=2}^{n} \frac{n-k+1}{2k} \delta_{k-1,n} x^k + \sum_{k=2}^{n} \frac{(n-k+1)(n-k+2)}{2k} \delta_{k-2,n} x^k \\
&= \frac{n+1}{2} \sum_{k=2}^{n} \delta_{k-1,n} \frac{x^k}{k} - \frac{1}{2} \sum_{k=2}^{n} \delta_{k-1,n} x^k + \frac{n^2+3n+2}{2} \sum_{k=2}^{n} \delta_{k-2,n} \frac{x^k}{k} \\
&- \frac{2n+3}{2} \sum_{k=2}^{n} \delta_{k-2,n} x^k + \frac{1}{2} \sum_{k=2}^{n} k \delta_{k-2,n} x^k \\
&= \frac{n+1}{2} \int \sum_{k=2}^{n} \delta_{k-1,n} x^{k-1} dx - \frac{x}{2} \sum_{k=2}^{n} \delta_{k-1,n} x^{k-1} \\
&+ \frac{n^2+3n+2}{2} \int \sum_{k=2}^{n} \delta_{k-2,n} x^{k-1} dx \\
&- \frac{(2n+3)x^2}{2} \sum_{k=2}^{n} \delta_{k-2,n} x^{k-2} + \frac{x}{2} \sum_{k=2}^{n} k \delta_{k-2,n} x^{k-1}.
\end{aligned}
$$

If we rewrite this equation in terms of $P(x)$, then we have the following integral equation:

$$
\begin{aligned}
P(x) - \delta_{0,n} - \delta_{1,n} x = \frac{n+1}{2} \int (P(x) - \delta_{0,n} - \delta_{n,n} x^n) dx - \frac{xP(x)}{2} + \frac{\delta_{0,n} x}{2} \\
+ \frac{\delta_{n,n} x^{n+1}}{2} + \frac{n^2+3n+2}{2} \int (xP(x) - \delta_{n-1,n} x^n - \delta_{n,n} x^{n+1}) dx \\
- \frac{(2n+3)x^2}{2} P(x) + \frac{2n+3}{2} \delta_{n-1,n} x^{n+1} + \frac{2n+3}{2} \delta_{n,n} x^{n+2} \\
+ x^2 P(x) + \frac{x^3}{2} P^{'}(x) - \frac{n+1}{2} \delta_{n-1,n} x^{n+1} - \frac{n+2}{2} \delta_{n,n} x^{n+2}.
\end{aligned}
$$

Finally, differentiating the both sides gives us second order ODE with irregular singular point at $x = 0$ which is written as follows:

$$
x^3 P^{''}(x) - ((2n-2)x^2 + x + 2) P^{'}(x) + ((n^2-n)x + n) P(x) = 0. \tag{A.17}
$$

This can also be written as in the following form:

$$\frac{d}{dx}\left(e^{\frac{1+x}{x^2}}x^{2-2n}P^{'}(x)\right)+e^{\frac{1+x}{x^2}}x^{-1-2n}(n+(-1+n)nx)P(x)=0$$

which is a Sturm-Liouville equation. Unfortunately, due to its high computational cost like in the previous case the result is still not better than the expressions for $\delta_{k,n}$'s that we recorded in Theorem 6.1.

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Biography

The author was born in Adana, Turkey in 1986. She entered the 9 Eylul University, Izmir in 2005 and graduated with the first place in Mathematic in 2010. She received her MSc degree in Mathematics from the same university in 2012. The author started her PhD program at the Tulane University in the Department of Mathematics in August 2013, eventually completing the program in May 2018.