TOPOLOGICAL SYMMETRIES OF $\mathbb{R}^3$

A DISSERTATION SUBMITTED ON APRIL 2018

TO THE DEPARTMENT OF MATHEMATICS

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS OF THE SCHOOL

OF SCIENCE AND ENGINEERING OF TULANE UNIVERSITY

FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

BY

Fang Sun

FANG SUN

APPROVED: S Kwasik

SLAWOMIR KWASIK, PH.D., CHAIRMAN

MAHIR CAN, PH.D.

RAFAL KOMENDARCZYK, PH.D.

Al Vitter

AL VITTER, PH.D.

DAGANG YANG, PH.D.

Abstract

We shall prove that if a finite group G acts topologically and faithfully on $\mathbb{R}^3$, then G is isomorphic to a subgroup of $O(3)$.

TOPOLOGICAL SYMMETRIES OF $\mathbb{R}^3$

A DISSERTATION SUBMITTED ON APRIL 2018

TO THE DEPARTMENT OF MATHEMATICS

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS OF THE SCHOOL

OF SCIENCE AND ENGINEERING OF TULANE UNIVERSITY

FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

BY

FANG SUN

APPROVED: ____________________

SLAWOMIR KWASIK, PH.D., CHAIRMAN

____

MAHIR CAN, PH.D.

____

RAFAL KOMENDARCZYK, PH.D.

____

AL VITTER, PH.D.

____

DAGANG YANG, PH.D.

Acknowledgments

I wish to pay my gratitude to many people without whom this work could not have been finished. I am deeply indebted to Professor Slawomir Kwasik for his constant encouragement and enlightening guidance. It is his generous help that make this work and many others possible. The lessons of mathematics and life I learned from him will always be precious.

I would like to thank our committee members Rafal Komendarczyk, Mahir Can, Al vitter, Dagang Yang for their valuable time and suggestions.

I want to thank Yuanzhen Cheng, Lin Jiu, Stafa Mentor, Tong Wu, Zhuolin Qu and many others for their help and company during my stay at Tulane University.

My appreciation also goes to the chairman of our department, Professor Morris Kalka, our staff Mark Brown and Arquette Guidry, for providing vital support.

Finally, I am sincerely grateful to my family and loved ones for their understanding and help through all these years. I wish to thank my love Xue Ye. The longing of our reunion lighted my way in the darkest of times. This thesis is dedicated to them.

Contents

1 Introduction

1.1 Backgrounds

Let G be a finite group acting faithfully on a smooth manifold $\mathcal{M}^n$. In the theory of transformation groups one usually considers two types of actions: topological actions and smooth actions. It was realized very early (cf. [2],[3]) that there are topological actions which can not be smoothed out, i.e. actions which are not conjugate (topologically) to smooth actions.

In the case of an Euclidean space $\mathbb{R}^n$ there is a particularly nice class of actions, namely the orthogonal actions, i.e., actions on $\mathbb{R}^n$ such that each element of the group is an orthogonal transformation. It is an elementary result in the theory of transformation groups that a linear action by a finite group on $\mathbb{R}^n$ is conjugate to a orthogonal action. A natural question is: Is an arbitrary action conjugate to an orthogonal one?

The answer turns out to depend on both the category of discussion (topological or smooth) and the dimension n.

Topological actions: For dimension $n \leqslant 2$, the answer is positive: every topological finite group action on $\mathbb{R}^n(n \leqslant 2)$ is conjugate to an orthogonal action (cf. [5],[13],[20]). In dimension $n = 3$ we have a negative result: there are topological nonsmoothable actions of cyclic groups on $\mathbb{R}^3$ (cf. [2],[3]). In fact, the existence of uncountably many equivalence class of such actions dashes any hope for classification. In dimension $n = 4$, there are even actions where the group G is not isomorphic to a subgroup of $O(4)$ (cf. Remark 4.2 in this paper).

Smooth actions: Smooth actions behave slightly better (as one would expect). Every such action on $\mathbb{R}^n$ ($n \leqslant 3$) is conjugate to an orthogonal one (a highly nontrivial result, cf. [23]). However there exist actions that are not conjugate to linear ones for every $n \geqslant 4$ (cf. [16],[17],[26]).

Yet despite the failure of geometric rigidity, for smooth action on $\mathbb{R}^4$ the algebraic rigidity still holds. Namely, any finite group G acting smoothly and faithfully on $\mathbb{R}^4$ has to be a subgroup of $O(4)$ (cf. Remark 4.1 in this paper).

Given the above algebraic rigidity, our goal is to complete the picture by establishing an analogous algebraic rigidity for topological action in dimension $n = 3$, where the geometric rigidity is lost (this is clearly the best one can hope for, see Remark 4.2). To be more specific, the following is the center of our attention:

Problem: *Let G be a finite group acting topologically on $\mathbb{R}^3$. Is G isomorphic to a subgroup of $O(3)$?*

This paper provides the positive answer to this problem. Namely:

Main Theorem: *Let G be a finite group acting topologically and faithfully on $\mathbb{R}^3$. Then G is isomorphic to a finite subgroup of $O(3)$. Moreover if the action is orientation preserving, then G is isomorphic to a subgroup of $SO(3)$.*

There are two main difficulties in dealing with the above problem.

The first difficulty is the lack of a systematic study of topological actions of finite groups on 3-manifolds due to the existence of "very exotic" actions on $\mathbb{R}^3(S^3)$. The most striking results in this area being constructions of R.H.Bing ([2], [3]) of cyclic group actions on $\mathbb{R}^3(S^3)$ with wildly embedded (and knotted) fixed point sets.

Note that such phenomena can not happen in the smooth setting (i.e., the Smith Conjecture).

These constructions indicate serious problems (impossibility?) with a development of equivariant versions of some powerful tools in 3-dimensional topology which could help to analyze the above problem. Note that in the smooth (locally linear) setting such tools were successfully developed (cf. [12], [19], [24]).

The second difficulty in dealing with the above problem is the low dimension of $\mathbb{R}^n$ i.e. $n = 3$. This low dimension prevents the use of higher dimensional techniques (for example the surgery theory) which could be helpful in purely topological setting (cf. [21], [22] for the case of topological actions in dimension four).

As a consequence our proof of the Main Theorem uses heavily techniques of homological algebra and finite group theory.

Moreover, it appears that the general strategy and specific methods used in our proof could be of an independent interest. It seems that they are very well suited to study the following (cf. [8]. Problem 3):

Conjecture: Let G be a finite group acting topologically on S^3. Then G is isomorphic to a subgroup of $O(4)$.

Before providing the details we outline the general strategy of the proof.

Throughout this paper, G is assumed to be a finite group acting topologically on $\mathbb{R}^3$. All group actions in this paper are assumed to be faithful. Also, for groups H, G, $H \lhd G$ means H is a normal subgroup of G.

1.2 Outline of Proof

Our consideration are divided into two cases:

Case P: The action of G on $\mathbb{R}^3$ is orientation preserving.

Case R: The action of G on $\mathbb{R}^3$ is (possibly) orientation reversing.

It should be pointed out that Case R does NOT reduce to the Case P! To be more specific, even assuming a positive result for the Case P, when dealing with the Case R one still has to employ the general strategy and specific methods used in analyzing the Case P. Consequently we provide all the details leading to the proof of the orientation preserving case and then shows how specific modifications of these considerations lead to the proof of the Main Theorem.

To treat the Case P we split our proof into:

Case PS: G is solvable.

Case PN: G is not solvable.

Suppose now that G is solvable. We shall prove that G is isomorphic to a subgroup of $SO(3)$ by induction. Solvability guarantees the existence of a subnormal series (ending with G) whose factors are cyclic groups of prime order.

For the induction step, it suffices to prove the following:

Assume that we have a group extension $0 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0$, *where* $\mathbb{Z}_p$ *stands for cyclic group of prime order* p. *Suppose that* G *acts faithfully and orientation preservingly on* $\mathbb{R}^3$ *and* H *is isomorphic to a subgroup of* $SO(3)$, *then* G *is isomorphic to a subgroup of* $SO(3)$.

To this end, we need the following:

A.1. $\{0 \longrightarrow \mathbb{Z}_n \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0,\ n \in \mathbb{Z}^+,\ p \text{ is odd}\} \Longrightarrow G$ is cyclic.

A.2. $\{0 \longrightarrow \mathbb{Z}_n \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0,\ n \in \mathbb{Z}^+\} \Longrightarrow G$ is cyclic or dihedral.

B.1. $\{0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0,\ n \geqslant 2,\ p \text{ is odd}\} \Longrightarrow G$ is the alternating group A_4

B.2. $\{0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0,\ n \geqslant 2\} \Longrightarrow G$ is dihedral

C. $\{0 \longrightarrow A_4 \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0,\ p \text{ prime}\} \Longrightarrow p = 2$ and $G \cong S_4$.

D. There is no extension of of the form $0 \longrightarrow S_4 \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0$, p prime, assuming G acts faithfully and orientation preservingly on $\mathbb{R}^3$.

The above results will complete the induction step and finish Case PS.

Now suppose G is not solvable.

From the above discussions we know that the Sylow p-subgroup of G, $\mathrm{Syl}_p(G)$, is cyclic for p-odd. For the 2-subgroups, it turns out $\mathrm{Syl}_2(G)$ is either cyclic or dihedral.

If $\mathrm{Syl}_2(G)$ is cyclic, then G is metacyclic, in particular solvable (cf. [18]), thus by Case PS, G is isomorphic to a subgroup of $SO(3)$.

If $\mathrm{Syl}_2(G)$ is dihedral, a theorem of Suzuki (cf. [27]) gives a normal subgroup $G_1 \lhd G$ with $[G : G_1] \leqslant 2$ and $G \cong Z \times L$, where Z is solvable and $L \cong PSL(2, p)$, p-prime. It will be shown that in this case G is solvable or $G \cong A_5$. In either case G is isomorphic to a subgroup of $SO(3)$.

This concludes the outline of the proof of the Main Theorem in the orientation preserving case.

The **Case R**: We shall start with some preliminary results. Then, mimicking the strategy employed in Case P, we define six types of Obstruction Kernels, from Type A to Type F. Now given the orientation preserving case of the Main Theorem it is clear that the group G is an extension of a finite subgroup of $SO(3)$ by $\mathbb{Z}_2$. The Obstruction Kernels will allow us to exclude all possibilities where the group G is not contained in $O(3)$.

Now we present the necessary details of the above process.

2 Proof of the orientation preserving case

This section and the next contain the proof of the Main Theorem.

2.1 Preliminaries

Since G acts (orientation preservingly) on $\mathbb{R}^3$, there is an induced orientation preserving action on S^3, the one point compactification. In what follows, the action of G on S^3 will be assumed to be this one.

The following results will be crucial in our proof.

Proposition 1 . *If G is cyclic and nontrivial, then $(S^3)^G = S^1$ (= stands for homeomorphism).*

Proof . See [25] Theorem on p.162 and [4] p.145 Theorem 7.11. □

The above proposition has the following corollary:

Corollary 1 . *If $(S^3)^G = S^1$, then G is cyclic.*

Proof . Let $H \subset G$ be a nontrivial subgroup. Take any nontrivial element $a \in H$, then $\langle a \rangle \subset H \subset G$, where $\langle a \rangle$ is the cyclic subgroup generated by a. Thus $(S^3)^{\langle a \rangle} \supset (S^3)^H \supset (S^3)^G$. The two ends of this sequence are homeomorphic to S^1 by Proposition 1 and the assumption. Since any embedding of an S^1 in another is surjective, $(S^3)^H = S^1 = (S^3)^G$. Now H is taken arbitrarily, which implies that the action of G on $S^3 - (S^3)^G = S^3 - S^1$ is free. It is a well known fact that $S^3 - S^1$ is a cohomological 1-sphere. Thus the Tate cohomology of G has period 2, i.e., G is cyclic (cf. [6] p.159). □

Consequently for a finite group G acting orientation preservingly on R^3, $(S^3)^G = S^1$ if and only if G is cyclic and nontrivial.

Another result which plays an important role in our considerations is the following:

Theorem 1 . *If G is a 2-group acting orientation preservingly on $\mathbb{R}^3$, then G is cyclic or dihedral.*

Proof . By the result of Dotzel and Hamrick (cf. [11]), there is an orthogonal action of G on $\mathbb{R}^4$ and a map $\phi : \Sigma^k S^3 \longrightarrow \Sigma^k S^3 (k \geqslant 1)$, where Σ^k denotes k-fold suspension, such that

(i) ϕ is equivariant with respect to the original action on the domain and the action induced by the orthogonal action on $S^3 \subset \mathbb{R}^4$ on the codomain.
(ii) ϕ induces $\mathbb{Z}_p$-homology isomorphisms on (suspension of) fixed point sets of non-trivial subgroups.

These fixed point sets are spheres, thus their dimension are preserved by ϕ.

Since $(S^3)^G \neq \varnothing$ in the original action, so the same holds for the orthogonal action. Let $v \in S^0$ be a fixed point. Let W be the orthogonal complement of v in $\mathbb{R}^4$, identified with $\mathbb{R}^3$. G acts orthogonally on W. Checking the fixed point set, one sees that this action is faithful, thus $G \subset O(3)$.

Now suppose G is neither cyclic or dihedral, then as a subgroup of $O(3)$, G contains $-I_3$. Let $A = -I_3$

Since $\langle A \rangle$ is cyclic, then $(S^3)^{\langle A \rangle} = S^1$ in the original action by Proposition 1, and hence the same is true for the orthogonal action on $S^3 \subset \mathbb{R}^4$. This implies that $(\mathbb{R}^4)^{\langle A \rangle} = \mathbb{R}^2$. This is impossible. Thus G has to be cyclic or dihedral. □

Remark 1 . The first two paragraphs do not need the assumption on the action to be orientation preserving, thus we have shown that any 2-group acting on $\mathbb{R}^3$ is a subgroup of $O(3)$.

2.2 Obstruction Kernels

We start with the following lemma:

Lemma 1 . *If G acts orientation preservingly on $\mathbb{R}^3$, with H, H' cyclic subgroups of G, $H \cap H' \neq \{0\}$, $\langle H \cup H' \rangle = G$ where $\langle H \cup H' \rangle$ denotes the subgroup generated by $H \cup H'$. Then G is cyclic.*

Proof . Consider the inclusions $(S^3)^H \subset (S^3)^{H \cap H'} \supset (S^3)^{H'}$. Since these subgroups are all nontrivial and cyclic, Proposition 1 implies that $(S^3)^H = (S^3)^{H'} = S^1$. Now $G = \langle H \cup H' \rangle$ implies $(S^3)^G = S^1$, thus G is cyclic by Corollary 1. □

Now we provide a list of seven types of finite, nontrivial groups and show that these groups cannot act faithfully and orientation preservingly on $\mathbb{R}^3$. These groups will be called obstruction kernels (abbreviated as O.K.) in what follows. Later we will show in an essence that for a finite group G which can act faithfully on S^3, either G is isomorphic to a subgroup of $SO(3)$, or G would contain one of these O.K. groups. Hence G must be isomorphic to a subgroup of $SO(3)$.

Obstruction Kernel of Type 0: Let p, q be distinct odd primes. Let φ be an action of $\mathbb{Z}_2$ on $\mathbb{Z}_p \oplus \mathbb{Z}_q$ which is trivial on the $\mathbb{Z}_p$ factor and multiplication by -1 on $\mathbb{Z}_q$. Then $G = (\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes_\varphi \mathbb{Z}_2$ does not admit a faithful and orientation preserving action on $\mathbb{R}^3$

Proof . There is a canonical subgroup $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2 \subset (\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes_\varphi \mathbb{Z}_2 = G$. Assume such action of G on $\mathbb{R}^3$ exists. Since the action on $\mathbb{Z}_p$ is trivial, $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2 \cong \mathbb{Z}_{2p}$. Letting $H = \mathbb{Z}_{2p}, H' = \mathbb{Z}_p \oplus \mathbb{Z}_q = \mathbb{Z}_{pq}$, we get $H \cap H' = \mathbb{Z}_p$, $\langle H \cup H' \rangle = G$. So by Lemma 1 G is cyclic. But this implies that the subgroup $\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_2$ is cyclic, contradicting to the assumption. □

Obstruction Kernel of Type 1: Let $G = \mathbb{Z}_{2m} \times \mathbb{Z}_2, m \geqslant 2$. Then G does not admit a faithful and orientation preserving action on $\mathbb{R}^3$.

Proof . Assume the action exists. Let $H = \mathbb{Z}_{2m} \subset G$, $H' = \langle (1, 1) \rangle \subset G$. Then H, H' satisfy the assumption of Lemma 1, thus $G = \mathbb{Z}_{2m} \times \mathbb{Z}_2$ is cyclic, which is impossible. □

Obstruction Kernel of Type 2: Let q be an odd prime and $k \geqslant 1$. Let φ be an action of $\mathbb{Z}_{2^{k+1}}$ on $\mathbb{Z}_q$ such that $\varphi(1)$ is multiplication by (-1).. Then $G = \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{2^{k+1}}$ does not admit a faithful and orientation preserving action on $\mathbb{R}^3$.

Proof . Assume the action exists. Let $H = \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{2^k} \subset \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{2^{k+1}}$, where $\mathbb{Z}_{2^k}$ is generated by $2 \in \mathbb{Z}_{2^{k+1}}$. Then H is cyclic since the action φ restricts to a trivial one on $\mathbb{Z}_{2^k}$. Let H' be the subgroup $\mathbb{Z}_{2^{k+1}}$ of G. Then H, H' satisfy the condition of Lemma 1, and hence G is cyclic, which can not be the case. □

Obstruction Kernel of Type 3: Generalized quaternion groups $Q_{4m}, (m \geqslant 2)$ do not admit faithful and orientation preserving actions on $\mathbb{R}^3$.

Proof . Assume the action exists. Let q be a prime factor of m. Note that there is a canonical subgroup $Q_{4q} \subset Q_{4m}$.

If q is odd, $Q_{4q} = \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_4$, $\varphi(1)$ is multiplication by (-1). This is O.K. of Type 2.

If $q = 2$, then $Q_{4q} = Q_8$ is a 2-group. This case is excluded by Theorem 1. □

Obstruction Kernel of Type 4: Let φ be an action of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^k}, k \geqslant 3$ such that $\varphi(1)$ is multiplication by $2^{k-1} \pm 1$. Then $G = \mathbb{Z}_{2^k} \rtimes_\varphi \mathbb{Z}_2$ does not admit a faithful and orientation preserving action on $\mathbb{R}^3$.

Proof . This is a consequence of Theorem 1. □

Obstruction Kernel of Type 5: Let p be a prime such that $p \equiv 1 (\mathrm{mod}\, 4)$ and φ be an action of $\mathbb{Z}_4$ on $\mathbb{Z}_p$ such that $\varphi(1) = n \in \mathbb{Z}_p^* = \mathrm{Aut}\, \mathbb{Z}_p$ for some n satisfying $n^2 = -1 (\mathrm{mod}\, p)$. Then $G = \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_4$ does not admit a faithful and orientation preserving action on $\mathbb{R}^3$.

Proof . Let Φ be a faithful and orientation preserving action of G on $\mathbb{R}^3$. Then $\mathbb{Z}_p$ is a normal subgroup of G, and $\mathbb{Z}_4$ acts on the fixed point set $(S^3)^{\mathbb{Z}_p} = S^1$ (Proposition 1). Let f be the restriction of $\Phi(0, 1)$ on $(S^3)^{\mathbb{Z}_p} = S^1$. Note that f^4, being the restriction of $\Phi(0, 4)$, is the identity, and a homeomorphism f of S^1 satisfying this must also satisfy $f^2 = \mathrm{id}$. That is, $(0, 2) \in \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_4$ acts on $(S^3)^{\mathbb{Z}_p} = S^1$ by the identity. Thus $(S^3)^{\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2} = (S^3)^{\mathbb{Z}_p} = S^1$, where $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2 \subset \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_4$ is the canonical subgroup. Hence $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2$ is cyclic. This contradicts the definition of φ. □

Before presenting the O.K. of Type 6, we need to address the convention regarding the automorphism group of the dihedral group D_{2n}. It is a well known fact that $\mathrm{Aut}\, D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ where the semi-direct product is with respect to the canonical identification $\mathrm{Aut}\, \mathbb{Z}_n \cong \mathbb{Z}_n^*$. An element $(t, s) \in \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ maps $a^i b$ to $a^{si+t} b$ and a^i to a^{si}. Here a, b are standard generators of D_{2n} ($a^n = b^2 = \mathrm{id}, aba = b$).

Obstruction Kernel of Type 6: Let n, t be even integers and $n \geqslant 4$. Let φ be an action of $\mathbb{Z}_2$ on $\mathbb{Z}_n$ such that $\varphi(1)$ is $(t, -1) \in \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$. Then $G = D_{2n} \rtimes_\varphi \mathbb{Z}_2$ does not admit a faithful and orientation preserving action on $\mathbb{R}^3$.

Proof . Assume the action exists. Let $i = \frac{t+2}{2}$, then $2i - t = 2$. Now $(a^i b, 1)^2 = (a^{2i-t}, 0) = (a^2, 0)$. Define $H = \langle (a^i b, 1) \rangle$, and note that $|H| = n$. Define $H' = \langle (a, 0) \rangle$ with $|H'| = n$. Then $H \cap H' = \langle (a^2, 0) \rangle \neq \{0\}$. Let $G' = \langle H \cup H' \rangle$, then Lemma 1 can be applied to G', H, H'. Thus G' is cyclic, of order$> n$. By examining the elements of $G = D_{2n} \rtimes_\varphi \mathbb{Z}_2$, one can see that none has order exceeding n, a contradiction. □

2.3 The Solvable Case

Now we consider the case of solvable G. We will determine the possibilities at each extension step, then an induction will produce the desired result. In the rest of Section 2, G **is a finite group that acts faithfully and orientation preservingly on** $\mathbb{R}^3$**.**

2.3.1 Extension of Cyclic Groups

We are going to prove A.1 and A.2 of the Section 1.2. Let us start with A.1.

Proposition 2 . *Suppose there is a short exact sequence* $0 \to \mathbb{Z}_n \to G \to \mathbb{Z}_p \to 0$, *$p$ an odd prime. Then G is cyclic.*

Proof . Proposition 1 implies that $(S^3)^{\mathbb{Z}_n} = S^1$. Now $\mathbb{Z}_p$ acts on this S^1, but p is odd, so the action has to be trivial. Thus $(S^3)^G = ((S^3)^{\mathbb{Z}_n})^{\mathbb{Z}_p} = S^1$, and G is cyclic by Corollary 1. □

Corollary 2 . *If $|G|$ is odd, then G is cyclic.*

Proof . This is because all odd order groups are solvable. (cf. [14]). □

The assertion A.2 is more delicate, and we have to discuss several cases, divided by the number of 2-factors in the order of the cyclic group.

Proposition 3 . *Suppose there is a short exact sequence* $0 \to \mathbb{Z}_n \to G \to \mathbb{Z}_2 \to 0$, *$n$ odd. Then G is cyclic or dihedral.*

Proof . Let $p_1^{n_1} \dots p_k^{n_k}$ be the prime decomposition of n. Since $(n, 2) = 1$, the sequence splits (cf. [6] p.93). Thus

$$G \cong \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2, \varphi : \mathbb{Z}_2 \to \operatorname{Aut} \mathbb{Z}_n = \operatorname{Aut} \prod_i \mathbb{Z}_{p_i^{n_i}} = \prod_i \operatorname{Aut} \mathbb{Z}_{p_i^{n_i}}$$

The component of $\varphi(1)$ on each $\operatorname{Aut} \mathbb{Z}_{p_i^{n_i}}$ is a multiplication by ± 1 (because $\varphi(1)$ has order ≤ 2, and $\operatorname{Aut} \mathbb{Z}_{p_i^{n_i}}$ is cyclic thus have a unique element of order 2).

If all components are $+1$, then $G \cong \mathbb{Z}_n \times \mathbb{Z}_2 \cong \mathbb{Z}_{2n}$ is a cyclic group.

If all components are -1, $\varphi(1)$ is multiplication of -1 on $\mathbb{Z}_n$, then $G \cong D_{2n}$ is the dihedral group.

If both ± 1 exist, take p, q prime factor of n where $\varphi(1)$ is multiplication of $+1$, -1 respectively on the corresponding components. Now there is a canonical subgroup $G_0 \subset G$, i.e. $G_0 := (\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes_\varphi \mathbb{Z}_2 \subset G$. But G_0 is O.K. of Type 0; a contradiction.

In conclusion, the only possibilities for G are: G-cyclic or G-dihedral. $\square$

Proposition 4 . *Suppose there is a short exact sequence* $0 \to \mathbb{Z}_{2n} \to G \to \mathbb{Z}_2 \to 0$, *$n$ odd. Then G is cyclic or dihedral.*

Proof . Note that in the above extension, there is an induced action φ of $\mathbb{Z}_2$ on $\mathbb{Z}_{2n}$. As in the previous proof, $\varphi(1)$ is a multiplication of ± 1 on each factor of the prime decomposition $\mathbb{Z}_{2n}$. For convenience, we name those prime with $\varphi(1)$ being $+1$ as $p_i, 1 \leqslant i \leqslant k$, and those corresponding to -1 as $q_j, 1 \leqslant j \leqslant l$ (the action on the $\mathbb{Z}_2$ component is always trivial). Let $P = \prod_i p_i^{n_i}, Q = \prod_j p_j^{m_j}, n = 2PQ$. An easy computation shows that $H^2(\mathbb{Z}_2, \mathbb{Z}_{2n})$, the cohomology of $\mathbb{Z}_2$ with coefficient $\mathbb{Z}_{2n}($

being a $\mathbb{Z}_2$-module via φ), is $\mathbb{Z}_2$ for any φ. Thus up to equivalence there are two extensions for each fixed φ.

A representative from each equivalence class can be given as:

Split Extension: $0 \longrightarrow \mathbb{Z}_{2n} \longrightarrow \mathbb{Z}_{2n} \rtimes_\varphi \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$, the semi-direct product.

Non-split Extension: $0 \longrightarrow \mathbb{Z}_{2n} \xrightarrow{\alpha} \mathbb{Z}_Q \rtimes_\phi \mathbb{Z}_{4P} \xrightarrow{\beta} \mathbb{Z}_2 \longrightarrow 0$, here $\phi(1)$ is multiplication by -1 on $\mathbb{Z}_Q$, α is induced by $\mathbb{Z}_{2n} \longleftrightarrow \mathbb{Z}_Q \times \mathbb{Z}_{2P} = \mathbb{Z}_Q \rtimes_\phi \mathbb{Z}_{2P} \subset \mathbb{Z}_Q \rtimes_\phi \mathbb{Z}_{4P}$, $\alpha(1) = (1, 2) \in \mathbb{Z}_Q \rtimes_\phi \mathbb{Z}_{4P}$, β is the projection on the quotient.

Now we investigate each case.

<u>Split Extension</u>:

1) If $Q = 1$, then φ is trivial and $G = \mathbb{Z}_{2n} \times \mathbb{Z}_2$. For $n \geqslant 2$ this is O.K. of Type 1, which is impossible. If $n = 1$ then G is dihedral.

2) If $P = 1$, then the action of φ on $\mathbb{Z}_{2n}$ is multiplication by -1, so G is the dihedral group D_{4n}.

3) If neither P or Q is 1, then taking primes p, q from their respective family we once again obtain $(\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes_\varphi \mathbb{Z}_2 \subset G$, which is O.K. of Type 0. Hence this case cannot occur.

<u>Non-split Extension</u>:

1) If $Q = 1$, $G \cong \mathbb{Z}_{4P}$ is cyclic.

2) If $Q > 1$, then take one prime factor q from its family. There is a canonical subgroup $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_4 \subset \mathbb{Z}_Q \rtimes_\phi \mathbb{Z}_{4P} = G$. Since P is odd, $\phi(1)$ acts on $\mathbb{Z}_q$ by -1. This is O.K. of Type 2, so cannot occur.

In conclusion, G is cyclic or dihedral. □

Proposition 5 . *Suppose there is a short exact sequence* $0 \to \mathbb{Z}_{2^k n} \to G \to \mathbb{Z}_2 \to 0$, *$n$ odd, $k \geqslant 2$. Then G is cyclic or dihedral.*

Proof . Consider the induced action of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^k n}$,

$$\varphi : \mathbb{Z}_2 \longrightarrow \operatorname{Aut} \mathbb{Z}_{2^k n} = \prod_i \operatorname{Aut} \mathbb{Z}_{p_i^{n_i}} \times \operatorname{Aut} \mathbb{Z}_{2^k}$$

where $n = 2^k p_1^{n_1} \dots p_l^{n_l}$ is the prime decomposition of n. As before we rename the p_i's as p_i and q_j according to the sign of $\varphi(1)$ on the corresponding components, and define $P = \prod_i p_i, Q = \prod_j q_j$.

Now for fixed n, P, Q, k, the possible φ's are classified by their actions on $\mathbb{Z}_{2^k}$. Since $\operatorname{Aut} \mathbb{Z}_{2^k} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{k-2}}$, there is no more than four elements of order ≤ 2 (when $k = 2$ there are two). It is not hard to see the possibilities for $\varphi(1)$ are multiplications by the following numbers on $\mathbb{Z}_{2^k}$:

1)$+1$, in which case $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k n}) = \mathbb{Z}_2$.

2)-1, in which case $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k n}) = \mathbb{Z}_2$.

3)$2^{k-1} + 1$, $(k > 2)$, in which case $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k n}) = 0$.

4)$2^{k-1} - 1$, $(k > 2)$, in which case $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k n}) = 0$.

Therefore the equivalence classes of extensions in each case are:

Case 1)There are two equivalence classes:

<u>Split</u>: The semi-direct product $0 \longrightarrow \mathbb{Z}_{2^k n} \longrightarrow \mathbb{Z}_{2^k n} \rtimes_\varphi \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$. Note that G contains $\mathbb{Z}_{2^k} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_{2^k} \times \mathbb{Z}_2$ since φ is trivial on $\mathbb{Z}_{2^k}$. But this is O.K. of Type 1; impossible.

<u>Non-split</u>: $0 \longrightarrow \mathbb{Z}_{2^k n} \longrightarrow (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes_\phi \mathbb{Z}_{2^{k+1}} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$, where $\phi(1)$ is multiplication by $+1, -1$ on $\mathbb{Z}_P$, $\mathbb{Z}_Q$ respectively, and $\mathbb{Z}_{2^k n} \cong \mathbb{Z}_P \times \mathbb{Z}_Q \times \mathbb{Z}_{2^k} \hookrightarrow \mathbb{Z}_P \times \mathbb{Z}_Q \rtimes_\phi \mathbb{Z}_{2^{k+1}}$ canonically. This extension induces φ and does not split. The group G contains an O.K. of Type 2 $\mathbb{Z}_q \rtimes \mathbb{Z}_{2^{k+1}}$ unless $Q = 1$. So the only possible G is $\mathbb{Z}_P \rtimes_\phi \mathbb{Z}_{2^{k+1}} \cong \mathbb{Z}_P \times \mathbb{Z}_{2^{k+1}}$, which is cyclic.

Case 2)Again there are two equivalence classes:

<u>Split</u>: $0 \longrightarrow \mathbb{Z}_{2^k n} \longrightarrow \mathbb{Z}_{2^k n} \rtimes_\varphi \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$. The group $G = (\mathbb{Z}_P \oplus \mathbb{Z}_Q \oplus \mathbb{Z}_{2^k}) \rtimes_\varphi \mathbb{Z}_2$ contains $(\mathbb{Z}_P \oplus \mathbb{Z}_2) \rtimes_\varphi \mathbb{Z}_2$ ($\mathbb{Z}_P \oplus \mathbb{Z}_2$ is the fixed point set of the action φ) unless $P = 1$. But $(\mathbb{Z}_P \oplus \mathbb{Z}_2) \rtimes_\varphi \mathbb{Z}_2 \cong \mathbb{Z}_{2P} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_{2P} \times \mathbb{Z}_2$, i.e. O.K. of Type 1. So $P = 1$, and $G = (\mathbb{Z}_Q \oplus \mathbb{Z}_{2^k}) \rtimes_\varphi \mathbb{Z}_2$, $\varphi(1)$ acts as multiplication by -1, so G is dihedral.

<u>Non-split</u>: $0 \longrightarrow \mathbb{Z}_{2^k n} \xrightarrow{\alpha} (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes_\phi Q_{4m} \xrightarrow{\beta} \mathbb{Z}_2 \longrightarrow 0$, where $m = 2^{k-1} \geqslant 2$. Here ϕ is defined as follows. Let $x = e^{\frac{i\pi}{m}}, y = j$ in Q_{4m}, define

$\varphi(x) = \text{id}, \varphi(y)$ be multiplication by $+1, -1$ on $\mathbb{Z}_P, \mathbb{Z}_Q$ respectively (Checking relations of Q_{4m} we see this is well-defined). The embedding α comes from $\mathbb{Z}_{2^k n} \cong \mathbb{Z}_P \oplus \mathbb{Z}_Q \oplus \mathbb{Z}_{2^k} \longleftrightarrow \mathbb{Z}_P \oplus \mathbb{Z}_Q \rtimes_\phi \langle x \rangle \subset (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes_\phi Q_{4m}$, and β is projection on the quotient. It can be readily verified that this extension is non-split and realizes φ. Now G contains Q_{4m}, which is O.K. of Type 3; impossible.

Case 3) Since $H^2(\mathbb{Z}_2, \mathbb{Z}_{2^k n}) = 0$, there is up to equivalence only one extension, the semi-direct product $(\mathbb{Z}_P \oplus \mathbb{Z}_Q \oplus \mathbb{Z}_{2^k}) \rtimes_\varphi \mathbb{Z}_2$. This group contains O.K. of Type 4.

Case 4)Again there is nothing other than the semi-direct product. Now $G = (\mathbb{Z}_P \oplus \mathbb{Z}_Q \oplus \mathbb{Z}_{2^k}) \rtimes_\varphi \mathbb{Z}_2$ contains $\mathbb{Z}_{2^k} \rtimes_\varphi \mathbb{Z}_2$. Checking the definition of φ, we see that this subgroup is precisely O.K. of Type 4; a contradiction.

In conclusion: G has to be cyclic or dihedral. □

Combining Propositions 2,3,4 and 5, we obtain the desired result:

Theorem 2 . *Suppose there is an extension* $0 \longrightarrow \mathbb{Z}_n \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0$, $n \geqslant 1$, p *prime. Then* G *is cyclic or dihedral.*

2.3.2 Extension of Dihedral Groups

We will prove B.1 and B.2 of the outline. We start with an extension by odd primes.

Proposition 6 . *Suppose there is an extension* $0 \longrightarrow D_4 \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0$, p *an odd prime. Then* $p = 3$ *and* $G \cong A_4$.

Proof . Since $(4, p) = 1$, the sequence splits and $G = D_4 \rtimes_\varphi \mathbb{Z}_p$ for some $\varphi : \mathbb{Z}_p \longrightarrow \text{Aut}\, D_4 \cong S_3$ (here S_3 is the permutation group of the nontrivial elements of D_4).

If $p > 3$, then φ is trivial and $G = D_4 \times \mathbb{Z}_p \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$, which is an O.K. of Type 1.

Thus $p = 3$. We observe that φ cannot be trivial, for in that case we would have again $G = D_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \times \mathbb{Z}_2$; impossible. Thus $\varphi(1)$ is a 3-cycle in Aut $D_4 = S_3$. This $G = D_4 \rtimes_\varphi \mathbb{Z}_p$ is precisely A_4. □

Proposition 7 . *There is no G that act faithfully and orientation preservingly on $\mathbb{R}^3$ and is represented by an extension $0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0$, $n > 2$, p an odd prime.*

Proof . Assume the existence of such G. If $p \nmid n$, then the sequence splits and $G = D_{2n} \rtimes_\varphi \mathbb{Z}_p$. There is a subgroup $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_p \subset G$ (because any automorphism of D_{2n} preserves the $\mathbb{Z}_n$ in D_{2n}). But $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_p \cong \mathbb{Z}_{np}$ since it is an extension of $\mathbb{Z}_n$ by $\mathbb{Z}_p$ and we have proven this in Proposition 2. Consequently G is an extension of this $\mathbb{Z}_{np}$ by $\mathbb{Z}_2$ and thus has to be dihedral by Theorem 2. However, a dihedral group cannot contain another dihedral group as a normal subgroup of odd index. Thus $p|n$.

Let $n = p^k l, (p, l) = 1$. There are subgroups $\mathbb{Z}_{p^k} \subset \mathbb{Z}_n \subset D_{2n} \subset G$. Let the standard generators of D_{2n} be a, b as mentioned before. Let P be a p-Sylow subgroup containing $\mathbb{Z}_{p^k}$. Take any $x \in P - D_{2n}$. Suppose $x^{-1}ax \notin \mathbb{Z}_n$, then $x^{-1}ax = a^r b$ and $(a^r b)^l = x^{-1}a^l x = a^l$, a contradiction. Thus $x^{-1}ax \in \mathbb{Z}_n$, which implies $N_G(\mathbb{Z}_n) = G$, which implies $\mathbb{Z}_n \lhd G$. Let $Q = \langle \mathbb{Z}_n \cup P \rangle$. Now $P \nsubseteq D_{2n}$, and $[G : \mathbb{Z}_n] = 2p$, and hence $[G : Q] \leqslant 2$. But $G = Q$ implies $D_{2n} \subset P$; impossible. Thus $[G : Q] = 2$. Now $\mathbb{Z}_n \lhd Q, [Q : \mathbb{Z}_n] = p$, i.e. Q is $\mathbb{Z}_n$ extended by a $\mathbb{Z}_p$. By Proposition 2 it has to be cyclic. As a result, G is diheral by Theorem 2 (it is an extension of Q by $\mathbb{Z}_2$). Again this is impossible, so there is no G that assume an action and an extension as described in the statement of the Proposition. □

Next we turn to extension by $\mathbb{Z}_2$. The discussion will be divided into two cases depending on the parity of n.

Proposition 8 . *Suppose there is an extension $0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$, $n \geqslant 2$ odd. Then G is dihedral.*

Proof . Let $n = p_1^{n_1} \dots p_k^{n_k}$ be its prime decomposition.

Since $\operatorname{Aut} D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$, and the inner automorphism $\operatorname{Inn} D_{2n} \cong \mathbb{Z}_n \rtimes \{\pm 1\}$, as a result $\operatorname{Out} D_{2n} = \operatorname{Aut} D_{2n} / \operatorname{Inn} D_{2n} \cong \mathbb{Z}_n^*/\{\pm 1\}$.

Since center of D_{2n}, denoted by C, is trivial, then $H^3(\mathbb{Z}_2, C) = 0$, $H^2(\mathbb{Z}_2, C) = 0$. In other words, each abstract kernel $\psi : \mathbb{Z}_2 \longrightarrow \operatorname{Out} D_{2n}$ has a unique extension up to equivalence. We will construct explicitly an extension for each given kernel.

Let ψ be given, then

$$\mathbb{Z}_n^*/\{\pm 1\} = \mathbb{Z}_{p_1^{n_1}}^* \times \dots \times \mathbb{Z}_{p_k^{n_k}}^* / \{\pm(1, \dots, 1)\}$$

$$\cong \ \mathbb{Z}_{(p_1-1)p_1^{n_1-1}} \times \dots \times \mathbb{Z}_{(p_k-1)p_k^{n_k-1}} / \langle \left(\frac{p_1 - 1}{2} p_1^{n_1-1}, \dots, \frac{p_k - 1}{2} p_k^{n_k-1} \right) \rangle.$$

Denote by $[(b_1, \dots, b_k)]$ the element corresponding to $\psi(1)$ in the middle group, and by $[(a_1, \dots, a_k)]$ the element corresponding to $\psi(1)$ in the bottom group, where $b_i \in \mathbb{Z}_{p_i^{n_i}}, a_i \in \mathbb{Z}_{(p_i-1)p_i^{n_i-1}}$.

Since $\psi(1)$ has order 2, either $2a_i = 0, 1 \leqslant i \leqslant k$, or $2a_i = \frac{p_i-1}{2} p_i^{n_i-1}, 1 \leqslant i \leqslant k$.

<u>Case 1</u>: If $(2a_1, \dots, 2a_k) = 0$, then $b_i = \pm 1$. Define

$$\varphi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_{p_1^{n_1}}^* \times \dots \times \mathbb{Z}_{p_k^{n_k}}^* = \mathbb{Z}_n^* \subset \mathbb{Z}_n \rtimes \mathbb{Z}_n^* \cong \operatorname{Aut} D_{2n}, \varphi(1) = (b_1, \dots, b_k)$$

This is a well-defined homomorphism, and $0 \longrightarrow D_{2n} \longrightarrow D_{2n} \rtimes_\varphi \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ induces the abstract kernel ψ. Hence $G = D_{2n} \rtimes_\varphi \mathbb{Z}_2$.

Recall that a, b denote standard generators of D_{2n}. The action φ is trivial on $\langle b \rangle$, whence $\langle b \rangle \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Now $\mathbb{Z}_n \lhd G$, and G is easily seen to be the inner semi-direct product of this copy of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with $\mathbb{Z}_n$. To be precise (checking induced inner automorphisms), $G \cong \mathbb{Z}_n \rtimes_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ where $\varphi(1, 0)$ is multiplication by (-1), $\varphi(0, 1)$ is determined by $(b_1, \dots, b_k) \in \mathbb{Z}_{p_1^{n_1}}^* \times \dots \times \mathbb{Z}_{p_k^{n_k}}^* \cong \mathbb{Z}_n$.

Now $b_i = \pm 1$, and there are three subcases:

i)If both ± 1 exist among them, take the semi-direct product of $\mathbb{Z}_n$ with the second $\mathbb{Z}_2$ component, we see there is an O.K. of Type 0 within as before. This case is thus excluded.

ii)If $b_i = +1$ for all i, take again the semi-direct product with the second $\mathbb{Z}_2$, $\mathbb{Z}_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2 \subset \mathbb{Z}_n \rtimes_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ because the action restricts to trivial action on $\mathbb{Z}_n$ and n is odd. Since G contains a cyclic subgroup of index 2, G has to be dihedral by Theorem 2.

iii)If $b_i = -1$ for all i, then $(\mathbb{Z}_n \rtimes \langle(1,1)\rangle) \subset \mathbb{Z}_n \rtimes_\phi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. Now that $\mathbb{Z}_n \rtimes \langle(1,1)\rangle \cong \mathbb{Z}_{2n}$ since the action is trivial. And conjugation of either copy of $\mathbb{Z}_2$ on $\mathbb{Z}_2 n$ is multiplication by -1. Thus G is dihedral.

This concludes Case 1.

Case 2: If $(2a_1, \ldots, 2a_k) = \left(\frac{p_1-1}{2}p_1^{n_1-1}, \ldots, \frac{p_k-1}{2}p_k^{n_k-1}\right)$, all $p_i \equiv 1 (\mathrm{mod}\, 4)$ since $2|\frac{p_i-1}{2}$. $b_i = m_i$ where $m_i^2 = -1$ in $\mathbb{Z}^*_{p_i^{n_i}}$

Let $G = \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_4$, where $\varphi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_n^* = \prod_i \mathbb{Z}^*_{p_i^{n_i}}$ sends 1 to $\prod_i m_i$

Restricting φ to $\mathbb{Z}_2 \subset \mathbb{Z}_4$, the resulted semi-direct product is precisely D_{2n}. The extension

$$0 \longrightarrow D_{2n} = \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2 \longrightarrow \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces the abstract kernel ψ, and up to equivalence it is the only one. This extension however can not exist, since G contains $\mathbb{Z}_{p_1} \rtimes_\varphi \mathbb{Z}_4$ where $\varphi(1) = m_1 \in \mathbb{Z}_p^*, m_1^2 = -1$, an O.K. of Type 5.

This concludes Case 2.

In conclusion, G has to be dihedral.

□

Proposition 9 . *Suppose there is an extension* $0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$, *$n$ even. Then G is dihedral.*

Proof . If $n = 2$, then G is a 2-group and hence must be dihedral by Theorem 1.

Let $n > 2$ and let $n = 2^l p_1^{n_1} \ldots p_k^{n_k}$ be the prime decomposition. It turns out that the sequence splits. To see this, take a Sylow-2 subgroup of D_{2n}. It has to be of the form $D_{2^{l+1}}$. Let P be a Sylow-2 subgroup of G containing $D_{2^{l+1}}$,

$[P : D_{2^{l+1}}] = 2$. By Theorem 1, P has to be dihedral, in particular, there exist $c \in P - D_{2^{l+1}}$ of order 2. Note that $c \notin D_{2n}$, since otherwise $P \subset D_{2n}$ which is impossible. Now the map $\mathbb{Z}_2 \to G$ sending 1 to c is a splitting.

As a consequence $G \cong D_{2n} \rtimes_\varphi \mathbb{Z}_2$ for some $\varphi : \mathbb{Z}_2 \longrightarrow \operatorname{Aut} D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$. Let $\varphi(1) = (t, s)$.

There is a canonical subgroup $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2 \subset D_{2n} \rtimes_\varphi \mathbb{Z}_2$. Since $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ is an extension of $\mathbb{Z}_n$ by $\mathbb{Z}_2$ then it is cyclic or dihedral by previous results (Theorem 2). We discuss these two cases separately:

i)Suppose $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ is cyclic. Then G is an extension of a cyclic group by $\mathbb{Z}_2$, so G is dihedral.

ii)Suppose $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ is dihedral. Then $\varphi(1) \in \operatorname{Aut} \mathbb{Z}_n$ is multiplication by -1, and whence $s = -1$ (see remark previous to O.K. of Type 6). Now we divide the discussion by the parity of t:

1)If t is odd, take $i = \frac{t+1}{2}$. Then $2i - t = 1$ which implies $(a^i b, 1)^2 = (a^{2i-t}, 0) = (a, 0)$; thus $(a^i b, 1)$ is of order $2n$. Thus G contains a cyclic subgroup of order $2n$. It is not hard to see that G is the semidirect product of this cyclic group with $\langle (b, 0) \rangle$. An easy computation shows that $(b, 0)(a^i b, 1)(b, 0) = (a^i b, 1)^{-1}$, whence G is dihedral.

2)If t is even, then G is O.K. of Type 6; impossible.

In conclusion, G is dihedral. □

Combining the results in this subsection, we have:

Theorem 3 . *Suppose there is an extension* $0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0, n \geqslant 2, p$ *prime. Then* G *is dihedral or* A_4*.*

2.3.3 Extension of A_4

There is an isomorphism $\operatorname{Aut} A_4 \cong S_4$, induced by the conjugation of elements of S_4 on the normal subgroup A_4. Let D_4 denote the unique order 4 subgroup of A_4. Then $D_4 \lhd A_4$, and any automorphism of A_4 preserves D_4. Now we are ready the state and prove:

Theorem 4 . *Suppose there is an extension* $0 \longrightarrow A_4 \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0$*,* p *prime. Then* $p = 2$ *and* $G \cong S_4$

Proof . The conjugation of any element of G restricts to an automorphism of A_4. By the discussion above, such an automorphism has to preserve D_4, so $D_4 \lhd G$.

i)If $p > 3$, $(12, p) = 1$ then the sequence splits, so $G = A_4 \rtimes_\varphi \mathbb{Z}_p$ for some φ. In fact φ has to be trivial since $|\operatorname{Aut} A_4| = |S_4| = 24$ and $p > 3$. Now G contains $D_4 \rtimes_\varphi \mathbb{Z}_p = D_4 \times \mathbb{Z}_p \cong \mathbb{Z}_{2p} \oplus \mathbb{Z}_2$. This is O.K. of Type 1; contradiction.

ii)If $p = 3$, let P be the Sylow-3 subgroup. By Corollary 2, $P \cong \mathbb{Z}_{3^n}$ for some n. Since $|D_4| = 4$, $|P| = 9$, $D_4 \lhd G$, we have $G \cong \mathbb{D}_4 \rtimes_\varphi \mathbb{Z}_9$ for some φ. $\varphi(1)^9 = \varphi(9) = \mathrm{id} \in \operatorname{Aut} D_4 \cong S_3$. $|S_3| = 6$ implies $\varphi(1)^3 = \mathrm{id}$. Consequently there is a subgroup $D_4 \times \mathbb{Z}_3 \subset \mathbb{D}_4 \rtimes_\varphi \mathbb{Z}_9$. Now $D_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \times \mathbb{Z}_2$, an O.K. of Type 1.

iii)If $p = 2$, define P as the Sylow-2 subgroup of G containing D_4. The group P is diheral by Theorem 1, thus there is $c \in P - D_4$ with $c^2 = \mathrm{id}$. Obviously $c \notin A_4$, so we obtain a splitting. Therefore $G \cong A_4 \rtimes_\varphi \mathbb{Z}_2$ for some φ, where $\varphi(1) \in \operatorname{Aut} A_4 \cong S_4$ is of order 2. We consider two cases:

<u>Case 1</u>: If $\varphi(1) \in A_4 \subset S_4$, then $\varphi(1) \in D_4$ implies $\varphi(1)_{|D_4} = \mathrm{id}$ ($\varphi(1)$ is conjugation and D_4 is Abelian). This however implies that the subgroup $D_4 \rtimes_\varphi \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This is impossible by Theorem 1, so this case is excluded.

<u>Case 2</u>: If $\varphi(1) \notin A_4$. Consider the canonical $0 \longrightarrow A_4 \longrightarrow S_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$. $\varphi : \mathbb{Z}_2 \to S_4$ is a splitting. Thus $S_4 \cong A_4 \rtimes_\varphi \mathbb{Z}_2 \cong G$.

This proves the theorem. □

2.3.4 Extension of S_4

Theorem 5 . *There is no extension of* S_4 *of the form* $0 \longrightarrow S_4 \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0$*,* p *prime, with* G *acting faithfully on* $\mathbb{R}^3$*.*

Proof . Suppose there is an extension $0 \longrightarrow S_4 \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 0$, p prime. Let C be the center of S_4, $C = \{0\}$. Now $\operatorname{Aut} S_4 = \operatorname{Inn} S_4 \cong S_4$, the second isomorphism being the obvious one. Thus Out S_4 is trivial and there exist only one abstract kernel: the trivial homomorphism $\mathbb{Z}_p \to \text{Out}\, S_4$. The product extension $0 \longrightarrow S_4 \longrightarrow S_4 \times \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \longrightarrow 0$, and the fact $H^2(\mathbb{Z}_p, C) = 0$ imply that this extension is the only one up to equivalence. Thus $G = S_4 \times \mathbb{Z}_p$.

If $p = 2$, then G contains $D_4 \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$; impossible by Theorem 1.

If $p > 2$, then G contains $D_4 \times \mathbb{Z}_p \cong \mathbb{Z}_{2p} \oplus \mathbb{Z}_2$, which is O.K. of Type 1; a contradiction. □

2.3.5 Conclusion

Combining the results of previous subsections and using an inductive argument, we obtain:

Corollary 3 . *If G is solvable, then G is isomorphic to a subgroup of $SO(3)$.*

2.4 Arbitrary Finite Group

Now we can prove the Main Theorem for the orientation preserving actions:

Proof . Our first observation is that by previous discussions (Theorem 1 and Corollary 2), the Sylow-p subgroups of G are cyclic for odd p, and cyclic or dihedral for $p = 2$. If the Sylow-2 subgroups are cyclic, then G is solvable (cf. [18] p.143), and Corollary 3 takes care of this situation. Thus it suffices to assume that Sylow-2 subgroups of G are dihedral.

By the result of Suzuki (cf. [27]), there exist a subgroup G_1 of G having the following properties: $G_1 \lhd G$, $[G : G_1] \leqslant 2$, $G_1 \cong Z \times L$ where Z is a solvable group and $L = PSL(2, p)$ the projective linear group for prime p.

i)If $p = 2$ or 3, then $L \cong S_3$ or $L \cong A_4$, in either case G_1 is solvable, thus so is G. Corollary 3 gives the desired result.

ii)If $p = 5$, then $L = A_5$. Z has to be trivial. This is because otherwise it contains a copy of $\mathbb{Z}_q$, q prime. This $\mathbb{Z}_q$ together with $D_4 \subset A_5$ produce

$\mathbb{Z}_q \times D_4 \subset Z \times L = G_1$. But $\mathbb{Z}_q \times D_4 \cong \mathbb{Z}_{2q} \oplus \mathbb{Z}_2$, an O.K. of Type 1. So $Z = 1$, $G_1 \cong A_5$.

Suppose $[G : G_1] = 2$. We have an extension $0 \longrightarrow A_5 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$. This implies either $G \cong S_5$ or $G \cong A_5 \times \mathbb{Z}_2$. The fact is purely algebraic, but we include a proof for completion. We have $\operatorname{Aut} A_5 = S_5$, and $\operatorname{Inn} A_5 = A_5$. Hence there are only two possible abstract kernels $\mathbb{Z}_2 \to \operatorname{Out} A_5 \cong \mathbb{Z}_2$. Since A_5 is centerless ($C = \{0\}$), then $H^2(\mathbb{Z}_2, C) = 0$ in either case. In other words, there are only two extensions up to isomorphism. Now $G \cong S_5$ and $G \cong A_5 \times \mathbb{Z}_2$ (together with obvious short exact sequences) are nonequivalent extensions exhausting all possibilities.

Case 1: $G \cong S_5$. In this case G contains a general affine group $GA(1,5)$ (e.g. the subgroup $\langle (12345), (2354) \rangle$ of S_5). Note that $GA(1,5) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_5^*$., where the semidirect product is the canonical one. This subgroup is solvable. By Corollary 3, $\mathbb{Z}_5 \rtimes \mathbb{Z}_5^*$ is isomorphic to either a cyclic/dihedral group, A_4 or S_4. This is however impossible (for the dihedral case it is convenient to use the fact $\mathbb{Z}_5^* \cong \mathbb{Z}_4$).

Case 2: $G \cong A_5 \times \mathbb{Z}_2$. Then G contains $D_4 \times \mathbb{Z}_2$. As we have seen before, this is impossible.

Thus the assumption $[G : G_1] = 2$ lead to contradiction, $G = G_1 \cong A_5$ is isomorphic to a subgroup of $SO(3)$.

This finishes the case $p = 5$.

iii)If $p > 5$, then L contains (as the normalizer of a Sylow-p subgroup) a group $\mathbb{Z}_p \rtimes_\varphi (\mathbb{Z}_p^*/\{\pm 1\})$ where $\varphi([a])$ is the multiplication by a^2 for $a \in \mathbb{Z}_p^*$. Let b be a generator of the cyclic group $\mathbb{Z}_p^*$. Then $\mathbb{Z}_p \rtimes_\varphi (\mathbb{Z}_p^*/\{\pm 1\}) \cong \mathbb{Z}_p \rtimes_\phi \mathbb{Z}_{\frac{p-1}{2}}$ where $\phi(1)$ is multiplication by b^2. This group is solvable and contains $\mathbb{Z}_p, p > 5$, so it is cyclic or dihedral (Corollary 3). The element $(0,1) \in \mathbb{Z}_p \rtimes_\phi \mathbb{Z}_{\frac{p-1}{2}}$ has order $\frac{p-1}{2} > 2$, thus the conjugation of $(0,1)$ in $\mathbb{Z}_p$ has to be trivial in either case (in the dihedral case $\mathbb{Z}_p$ is in the unique cyclic subgroup of index 2, so does $(0,1)$). Thus $b^2 = 1 \in \mathbb{Z}_p$ which implies $|\mathbb{Z}_p^*| = 1$ or 2, consequently $p = 2$ or 3; a contradiction.

In conclusion G is isomorphic to a subgroup of $SO(3)$. $\square$

3 Proof of the orientation reversing case

3.1 Preliminaries

Lemma 2 . *If f is an orientation reversing involution of S^3 with fixed point set S^2, then f permutes the two components of $S^3 - S^2$.*

Proof . Suppose the contrary holds. Let $S^3 - S^2$ be $A \cup B$ where A, B are the two components. f restricts to a homeomorphism on A, i.e., $\mathbb{Z}_2$ acts on A. Since A is acyclic, $A^{\mathbb{Z}_2}$ is nonempty according to Smith Theory (cf. [4] p.145 Theorem 7.11). This is impossible since $(S^3)^{\mathbb{Z}_2} = S^2$. □

Theorem 6 . *If $D_{2n}(n > 2)$ acts on $\mathbb{R}^3$ such that $\mathbb{Z}_n \subset D_{2n}$ is the collection of orientation preserving homeomorphisms, then $(S^3)^{D_{2n}} \cong S^1$.*

Proof . Assume this is not the case. Then $(S^3)^{D_{2n}} = [(S^3)^{\mathbb{Z}_n}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2} = S^0$. Let a, b be standard generators of D_{2n}. For $0 \leqslant i \leqslant n-1$, $a^i b$ is an orientation reversing homeomorphism, so $(S^3)^{\langle a^i b\rangle} \cong S^0$ or S^2.

Case 1: If for all i, $(S^3)^{\langle a^u b\rangle} = S^0$, then the fixed point set of any nontrivial subgroup of D_{2n} is in $S^1 = (S^3)^{\mathbb{Z}_n}$. Thus D_{2n} acts on $S^3 - S^1$ freely. But this implies (since $S^3 - S^1$ is a homological 1-sphere) that D_{2n} is cyclic, which is impossible.

Case 2: If there exist $0 \leqslant i \leqslant n-1$ such that $(S^3)^{\langle a^i b\rangle} \cong S^2$. Fix i and denote $(S^3)^{\langle a^i b\rangle}$ as S^2_0. Now $a(S^3)^{\langle a^i b\rangle} = (S^3)^{a\langle a^i b\rangle a^{-1}} = (S^3)^{\langle a^{i+2} b\rangle}$, denote this set as S^2_1(as the name suggests, it is homeomorphic to S^2). $S^2_1 \cap S^2_0 = (S^3)^{\langle a^i b, a^{i+2} b\rangle} = (S^3)^{\langle a^2, a^i b\rangle} = [(S^3)^{\langle a^2\rangle}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2} = S^0$. Note that this S^0 is the fixed point set of D_{2n} and $S^0 \subset S^2_j$, $j = 0, 1$.

Since $\langle a\rangle$ is a cyclic group acting orientation preservingly, $(S^3)^{\langle a\rangle} \cong S^1$. $S^0 \subset S^1$, and $S^1 - S^0$ has two components. Denote them as $\mathbb{R}_+$ and $\mathbb{R}_-$.

Claim: For $j = 0, 1$, $\mathbb{R}_+$ and $\mathbb{R}_-$ lie in different components of $S^3 - S^2_j$.

Proof: Suppose otherwise. The homeomorphism $a^{i+2j}b$ fixes S_j^2, and by the preceding lemma it permutes the two components. On the other hand, $D_{2n}/\langle a\rangle$ acts on $(S^3)^{\langle a\rangle} = S^1$. In particular $a^{i+2j}b(\mathbb{R}_+ \cup \mathbb{R}_-) = \mathbb{R}_+ \cup \mathbb{R}_-$, but $\mathbb{R}_+ \cup \mathbb{R}_-$ is mapped to the other component of $S^3 - S_j^2$ which contains none of $\mathbb{R}_+$ or $\mathbb{R}_-$. This is a contradiction.

For $j = 0, 1$, let A_j, B_j be the components of $S^3 - S_j^2$ where $\mathbb{R}_- \subset A_j, \mathbb{R}_+ \subset B_j$. The intersection $S^1 \cap S_j^2 = (S^3)^{\langle a, a^{i+2j}b\rangle} = (S^3)^{D_{2n}} = S^0$, whence $S_1^2 - S^0 \subset S^3 - S_0^2$. Now we must have either $S_1^2 - S^0 \subset A_0$ or $S_1^2 - S^0 \subset B_0$.

If $S_1^2 - S^0 \subset B_0$, then $A_0 \cup S_0^2 \subset A_1 \cup B_1 \cup S^0$ by taking complement. So $A_0 \subset A_1 \cup B_1$. Since A_0 is connected, either $A_0 \subset A_1$ or $A_0 \subset B_1$. The latter is not possible because $\mathbb{R}_- \not\subset B_1$. Thus $A_0 \subset A_1$. Obviously $A_0 \neq A_1$ and $A_0 \subset A_1$. By definition, $aA_0 = A_1$ and $a^n A_0 = A_0$. Then $a^n A_0 = a^{n-1}A_1 \supset a^{n-1}A_0 \supset a^{n-2}A_0 \supset \ldots \supset A_0$, a contradiction.

This forces $S_1^2 - S^0 \subset A_0$ to be the case. But this is impossible by an analogous argument as the one above.

Therefore all possibilities lead to contradictions and the initial assumption fails.

□

3.2 Obstruction Kernels

Proposition 10 . *(Obstruction Kernel of Type A) Let p, q be distinct primes and φ be an action of $\mathbb{Z}_2$ on $\mathbb{Z}_p \oplus \mathbb{Z}_q$ such that $\varphi(1)$ is multiplication by 1(resp. -1) on $\mathbb{Z}_p$ (resp. $\mathbb{Z}_q$). Then $G = (\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes_\varphi \mathbb{Z}_2$ cannot act faithfully on $\mathbb{R}^3$.*

Proof . Assume such action exists. Note that the subgroup $\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_2$ is a dihedral group D_{2q}. If G acts orientation preservingly, this is Obstruction Kernel of Type 0, thus impossible. So we may assume the action to be not orientation preserving.

The group $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is the only subgroup of index 2, whence it is the collection of orientation preserving homeomorphisms. In particular $\mathbb{Z}_q$ acts orientation preservingly.

Now $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2 = \mathbb{Z}_p \oplus \mathbb{Z}_2$. Therefore $(S^3)^{\mathbb{Z}_p\oplus\mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_2}]^{\mathbb{Z}_p}$. Since the generator of $\mathbb{Z}_2$ reverses orientation, $(S^3)^{\mathbb{Z}_2} = S^2$ or S^0. In either case $[(S^3)^{\mathbb{Z}_2}]^{\mathbb{Z}_p} = S^0$. Now $(S^3)^{\mathbb{Z}_p\oplus\mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_p}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2}$. Note that $(S^3)^{\mathbb{Z}_p} = (S^3)^{\mathbb{Z}_p\oplus\mathbb{Z}_q} = (S^3)^{\mathbb{Z}_q} \cong S^1$, whence $(S^1)^{\mathbb{Z}_2} = S^0 = [(S^3)^{\mathbb{Z}_q}]^{\mathbb{Z}_2} = (S^3)^{D_{2q}}$. But by Theorem 6, $(S^3)^{D_{2q}} = S^1$, a contradiction. □

Proposition 11 . *(Obstruction Kernel of Type B) Let q be an odd prime, φ be an action of $\mathbb{Z}_2$ on $\mathbb{Z}_4 \oplus \mathbb{Z}_q$ such that $\varphi(1)$ is multiplication by 1 (resp. -1) on $\mathbb{Z}_4$ (resp. $\mathbb{Z}_q$). Let $G = (\mathbb{Z}_4 \oplus \mathbb{Z}_q) \rtimes_\varphi \mathbb{Z}_2$, Then there is no faithful action of G on $\mathbb{R}^3$ such that $\mathbb{Z}_4 \oplus \mathbb{Z}_q$ is the collection of orientation preserving homeomorphisms.*

Proof . Assume such action exists. By Theorem 6, to produce a contradiction it suffices to prove that $(S^3)^{\mathbb{Z}_q\rtimes_\varphi\mathbb{Z}_2} = S^0$ ($\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_2$ is dihedral and $\mathbb{Z}_q$ is the orientation preserving subgroup). We start with computations of the fixed point sets of various subgroups of G.

Since $\mathbb{Z}_4 \oplus \mathbb{Z}_q$ is cyclic, we have $(S^3)^{\mathbb{Z}_4\oplus\mathbb{Z}_q} = (S^3)^{\mathbb{Z}_q} \cong S^1$.

The fixed point set $(S^3)^{\mathbb{Z}_q\rtimes_\varphi\mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_q}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2}$ is either S^1 or S^0.

The standard copy of $\mathbb{Z}_2$ acts orientation reversingly, whence $(S^3)^{\mathbb{Z}_2} = S^2$ or S^0.

Case 1: If $(S^3)^{\mathbb{Z}_2} = S^0$, then this forces $(S^3)^{\mathbb{Z}_q\rtimes_\varphi\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2}$ to be S^0 and we obtain the contradiction we are looking for.

Case 2: If $(S^3)^{\mathbb{Z}_2} = S^2$, then $(S^3)^{\mathbb{Z}_4\rtimes_\varphi\mathbb{Z}_2} = (S^3)^{\mathbb{Z}_4\oplus\mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_2}]^{\mathbb{Z}_4} = (S^2)^{\mathbb{Z}_4}$. According to [13], any action on S^2 is conjugate to an orthogonal one. Thus $(S^2)^{\mathbb{Z}_4}$ is either S^0 or empty. On the other hand, we have computed that $(S^3)^{\mathbb{Z}_q\rtimes_\varphi\mathbb{Z}_2}$ is either S^1 or S^0, Combining the two results, $(S^3)^{\mathbb{Z}_q\rtimes_\varphi\mathbb{Z}_2} = S^0$. □

Proposition 12 . *(Obstruction Kernel of Type C) Let p be an odd prime , k be an integer, $k \geqslant 1$. Let φ be an action of $\mathbb{Z}_{2^{k+1}}$ on $\mathbb{Z}_p$ such that $\varphi(1)$ is multiplication by -1. Then $G = \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{2^{k+1}}$ cannot act faithfully on $\mathbb{R}^3$.*

Proof . Assume such action exists. If the action is orientation preserving, then

G is Obstruction Kernel of Type 2, which is impossible. So assume the action is not orientation preserving.

It is not hard to see that $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{2^k} \cong \mathbb{Z}_{2^k p}$ is the only subgroup of G with index 2. Thus this subgroup has to be the collection of homeomorphisms preserving orientation.

Let $b = (0,1) \in \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{2^{k+1}}$. Then b reverses orientation of $\mathbb{R}^3$, whence $(S^3)^{\langle b \rangle} \cong S^0$ or $(S^3)^{\langle b \rangle} \cong S^2$.

<u>Case 1</u>: $(S^3)^{\langle b \rangle} \cong S^2$. In this case $(S^3)^{\langle b^2 \rangle} \supseteq (S^3)^{\langle b \rangle} = S^2$. But b^2 is orientation preserving and thus $(S^3)^{\langle b^2 \rangle} \cong S^1$, a contradiction. So this case is not possible.

<u>Case 2</u>: $(S^3)^{\langle b \rangle} \cong S^0$. In this case $(S^3)^G = S^0$. Since $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{2^k} \cong \mathbb{Z}_{2^k p}$ acts orientation preservingly, $(S^3)^{\mathbb{Z}_{2^k p}} \cong S^1$. The quotient $G/\mathbb{Z}_{2^k p} \cong \mathbb{Z}_2$ acts on this copy of S^1, so $G = \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{2^{k+1}}$ acts on the complement $S^3 - S^1$. The fixed point set of any nontrivial subgroup H of G is either S^1 (when $H \subset \mathbb{Z}_{2^k p}$) or S^0 (otherwise). Therefore the restriction to $S^3 - S^1$ is free. As before, this implies that G is cyclic, which is obviously not the case.

Thus either case leads to a contradiction and the proposition is proven. □

Proposition 13 . *(Obstruction Kernel of Type D) Let p be an odd prime with $p \equiv 1 (\mathrm{mod}\, 4)$. Let φ be an action of $\mathbb{Z}_2$ on $\mathbb{Z}_p$ such that $\varphi(1) = n \in \mathbb{Z}_p^*$ where $n^2 = -1 (\mathrm{mod}\, p)$. Then $G = \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2$ cannot act faithfully on $\mathbb{R}^3$.*

Proof . The proof of Obstruction Kernel of Type 5 carries verbatim. □

Proposition 14 . *(Obstruction Kernel of Type E) Let p be an odd prime and φ be an action of $\mathbb{Z}_2$ on $\mathbb{Z}_p \oplus \mathbb{Z}_4$ such that $\varphi(1)$ is multiplication by 1 (resp. -1) on $\mathbb{Z}_p$ (resp. $\mathbb{Z}_4$). Then $G = (\mathbb{Z}_p \oplus \mathbb{Z}_4) \rtimes_\varphi \mathbb{Z}_2$ cannot act faithfully on $\mathbb{R}^3$ such that the collection of orientation preserving homeomorphisms is $\mathbb{Z}_p \oplus \mathbb{Z}_4$.*

Proof . Assume such action exists. Note that $\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_2 \cong D_8$. To produce a contradiction, it suffices (by Theorem 6) to prove $(S^3)^{D_8} \cong S^0$.

By our assumption that the action restricted to $\mathbb{Z}_p \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{4p}$ is orientation preserving, we have $(S^3)^{\mathbb{Z}_p} = (S^3)^{\mathbb{Z}_{4p}} = (S^3)^{\mathbb{Z}_4} \cong S^1$.

The generator of $\mathbb{Z}_2$ reverses orientation, whence $(S^3)^{\mathbb{Z}_2} \cong S^2$ or S^0.

If $(S^3)^{\mathbb{Z}_2} = S^0$, then $(S^3)^{D_8} = [(S^3)^{\mathbb{Z}_4}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2} = S^0$, and we are done.

If $(S^3)^{\mathbb{Z}_2} = S^2$, consider the subgroup $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2 \subset G$. By definition it is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_2$. And $(S^3)^{\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2} = (S^3)^{\mathbb{Z}_p \oplus \mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_2}]^{\mathbb{Z}_p} = (S^2)^{\mathbb{Z}_p} = S^0$ since any action on S^2 is conjugate to an orthogonal one. On the other hand, $(S^3)^{\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_p}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2}$. Thus $(S^1)^{\mathbb{Z}_2} = S^0$. Since $(S^1)^{\mathbb{Z}_2}$ is also $[(S^3)^{\mathbb{Z}_4}]^{\mathbb{Z}_2} = (S^3)^{D_8}$, we obtain $(S^3)^{D_8} = S^0$. □

Proposition 15 . *(Obstruction Kernel of Type F) $G = Q_{4m}$ (the generalized quaternion group) cannot act faithfully on $\mathbb{R}^3$.*

Proof . The proof follows verbatim from the proof of Obstruction Kernel of Type 3 , using the remark following Theorem 1 and Obstruction Kernel of Type C. □

3.3 The Cyclic Case

In the orientation preserving case, we considered extensions of (finite) subgroups of SO(3) by $\mathbb{Z}_p$, p prime. In the following sections, the algebraic aspects of the situations are almost the same as their counterparts in the orientation preserving case. The discussion of the orientation preserving case has given an algebraic description to the possible results of extensions. Thus we will not repeat the algebraic consideration of those proofs, but to filter them with the new obstruction kernels.

Theorem 7 . *If G acts on $\mathbb{R}^3$ such that orientation preserving subgroup is cyclic, then G is isomorphic to a subgroup of $O(3)$.*

Proof . It suffice to consider the case where the orientation preserving subgroup is of index 2. There is an short exact sequence

$$0 \longrightarrow \mathbb{Z}_n \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

where $\mathbb{Z}_n$ is the subgroup of orientation preserving homeomorphisms in G.

The algebraic possibilities are known (c.f. Proposition 3,4,5). We will investigate the possibilities analogously as in the orientation preserving case. Let $n = 2^k m$, m odd. Let φ be the induced action of $\mathbb{Z}_2$ on $\mathbb{Z}_n$. Let $m = P \cdot Q$ where $\varphi(1)$ restrict to a multiplication by $+1$(resp. -1) on $\mathbb{Z}_P$(resp. $\mathbb{Z}_Q$)

Case 1: $k = 0$(n is odd).

As in Proposition 3, G is either cyclic, dihedral or contains an Obstruction Kernel of Type A. In the last case G cannot act. So G is isomorphic to a subgroup of $O(3)$.

Case 2: $k = 1$($n = 2m$, m odd)

We proceed as in Proposition 4.

i)Split Case:

If $Q = 1$, $G = \mathbb{Z}_n \oplus \mathbb{Z}_2$, which is a subgroup of $O(3)$.

If $P = 1$, G is dihedral, thus G is isomorphic to a subgroup of $O(3)$

If neither P nor Q is 1, G contains an Obstruction Kernel of Type A, which is impossible.

ii)Non-split Case:

If $Q = 1$, then G is cyclic and G is isomorphic to a subgroup of $O(3)$.

If $Q > 1$, then G contains an Obstruction Kernel of Type C, a contradiction.

In sum, for $n = 2m$, m odd, G is isomorphic to a subgroup of $O(3)$.

Case 3: $k \geqslant 2$

We argue as in Proposition 5. There are four possibilities for the restriction of φ on the standard copy of $\mathbb{Z}_{2^k}$ in $\mathbb{Z}_n$.

i)$\varphi(1)$ is multiplication by 1.

Split Case: $G = \mathbb{Z}_{2^k m} \rtimes_\varphi \mathbb{Z}_2$. Either $P = 1$ or $Q = 1$ since otherwise there will be an Obstruction Kernel of Type A in G.

If $Q = 1$, then $G = \mathbb{Z}_n \oplus \mathbb{Z}_2$, whence G is isomorphic to a subgroup of $O(3)$.

If $P = 1$, then $G = (\mathbb{Z}_{2^k} \oplus \mathbb{Z}_Q) \rtimes_\varphi \mathbb{Z}_2$, thus contains Obstruction Kernel of Type B. This is impossible.

Non-split Case: In this case $G \cong (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes_\phi \mathbb{Z}_{2^{k+1}}$ where $\phi(1)$ is multiplication by 1(resp. -1) on $\mathbb{Z}_P$(resp. $\mathbb{Z}_Q$)

If $Q = 1$, G is cyclic thus isomorphic to a subgroup of $O(3)$.

If $Q > 1$, G contains an Obstruction Kernel of Type C.

ii)$\varphi(1)$ is multiplication by -1

Split Case: $G \cong \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$. Again either $P = 1$ or $Q = 1$ since an Obstruction Kernel of Type A will show up otherwise.

If $Q = 1$, then G contains an Obstruction Kernel of Type E, therefore this case is excluded.

If $P = 1$, G is dihedral.

Non-Split Case: In such case $G \cong (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes_\phi Q_{4m}$ where for some ϕ, but Q_{4m} is Obstruction Kernel of Type F, which implies this case cannot occur.

iii)$\varphi(1)$ is multiplication by $2^{k-1}+1$, then G contains $\mathbb{Z}_{2^k} \rtimes_\varphi \mathbb{Z}_2$, a 2-group. This however contradicts the remark following Theorem 1. Thus this case is impossible.

iv)$\varphi(1)$ is multiplication by $2^{k-1} - 1$. A same argument as above produces a contradiction.

In all the possible cases above, G has to be isomorphic to a subgroup of $O(3)$.

□

Remark 2 . The proof actually shows that G is either $\mathbb{Z}_{2n}$, $\mathbb{Z}_n \oplus \mathbb{Z}_2$ or D_{2n}.

3.4 The Dihedral Case

The proof of the dihedral case is very much in the spirit of Proposition 8 and 9.

Theorem 8 . *If G acts on $\mathbb{R}^3$ such that the subgroup of orientation preserving homeomorphisms is D_{2n}, n odd, $n \geqslant 3$, then G is isomorphic to a subgroup of $O(3)$.*

Proof . It suffice to consider the case where the action is not orientation preserving. The first half of the proof of Proposition 8 carries verbatim. We have two cases(notations are borrowed from that proof):

Case 1: If $(2a_1, 2a_2, \ldots, 2a_n) = (0, 0, \ldots, 0)$

In this case the short exact sequence

$$0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

splits and thus $G \cong D_{2n} \rtimes_\varphi \mathbb{Z}_2$, φ as defined in Proposition 8.

We have $b_i = \pm 1$ for all i. There are three subcases:

i) Both ± 1 appears. In this case G contains an Obstruction Kernel of Type A, which is impossible.

ii) $b_i = 1$ for all i. then the action φ is trivial on the cyclic subgroup $\mathbb{Z}_n$. φ is always trivial on the period 2 generator b of D_{2n} by definition. Thus φ is trivial and $G \cong D_{2n} \times \mathbb{Z}_2$, a subgroup of $O(3)$.

iii) $b_i = -1$ for all i, in this case G has been computed to be dihedral.

Case 2: if $(2a_1, 2a_2, \ldots, 2a_n) = \left(\frac{p_1-1}{2} p_1^{n_1-1}, \ldots, \frac{p_k-1}{2} p_k^{n_k-1}\right)$

In this case G(as computed in the orientation preserving case) contains an Obstruction Kernel of Type D, which is a contradiction.

Combining the above results, we see in all possible cases G is isomorphic to a subgroup of $O(3)$. □

Proposition 16 . *Suppose G acts on $\mathbb{R}^3$ such that the subgroup of orientation* *preserving homeomorphisms is D_{2n}, n even. If $n > 2$, then the extension $0 \to D_{2n} \to G \to \mathbb{Z}_2 \to 0$ has to split.*

Proof . Assume this is not the case.

Take the Sylow 2-subgroup of D_{2n}. It must be a copy of $D_{2^{l+1}}$. Let P be the Sylow 2-subgroup of G containing this $D_{2^{l+1}}$. As a 2-group, $P \subseteq O(3)$, whence it is either $\mathbb{Z}_{2^{l+2}}$, $\mathbb{Z}_{2^{l+1}} \oplus \mathbb{Z}_2$, $D_{2^{l+2}}$ or $D_{2^{l+1}} \times \mathbb{Z}_2$. Containing $D_{2^{l+1}}$, P is not cyclic. It cannot be $D_{2^{l+2}}$ or $D_{2^{l+1}} \times \mathbb{Z}_2$ either since that would make the extension $0 \to D_{2n} \to G \to \mathbb{Z}_2 \to 0$ split ($P - D_{2^{l+1}}$ contains an element of order 2). Thus $P \cong \mathbb{Z}_{2^{l+1}} \oplus \mathbb{Z}_2$. This group is cyclic, and the same has to be true for $D_{2^{l+1}}$, whence $l = 1$. In other word, $n = 2m$, m odd.

We have seen that $\operatorname{Aut} D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$. It is not hard to compute that $\operatorname{Inn} D_{2n} = \mathbb{Z}_m \rtimes \{\pm 1\} \subset \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ where the embedding $\mathbb{Z}_m \subset \mathbb{Z}_n$ is canonical. Thus $\operatorname{Out} D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^* / \mathbb{Z}_m \rtimes \{\pm 1\}$.

The exact sequence

$$0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces an abstract kernel $\phi : \mathbb{Z}_2 \to \operatorname{Out} D_{2n}$. $\phi(1)$ is represented by any conjugation of an element of $G - D_{2n}$ on D_{2n}.

Consider the subgroup $\{e, a^m, b, a^m b\} \subset D_{2n}$ where e stands for identity. This is a copy of D_4. Let P' be a Sylow 2-subgroup of G containing D_4. $P' \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$. In particular, P' is abelian. Take $x \in P' - D_4$. the conjugation of x on D_4 is trivial. Now $x \in G - D_{2n}$. Let $(t, s) \in \mathbb{Z}_n \rtimes \mathbb{Z}_n^* = \operatorname{Aut} D_{2n}$ be the conjugation by x. This automorphism sent b to $a^t b$. Thus $t = 0 \in \mathbb{Z}_n$. So $\phi(1)$ can be represented by $(0, s)$. $\phi(1)^2 = 0$ implies $s^2 = \pm 1 \in \mathbb{Z}_n^*$.

Now assume $m = p_1^{n_1} \ldots p_k^{n_k}$ is the prime decomposition, then

$$\mathbb{Z}_n^* = \mathbb{Z}_2^* \times \mathbb{Z}_{p_1^{n_1}}^* \times \ldots \times \mathbb{Z}_{p_k^{n_k}}^* \cong \mathbb{Z}_{p_1^{n_1}}^* \times \ldots \times \mathbb{Z}_{p_k^{n_k}}^*$$

Let $(b_1, \ldots, b_k)$ be the element in the rightmost group above corresponding to s.

Since D_{2n} is centerless for $n > 2$, then (as computed in Proposition 8) each abstract kernel corresponds to one and only one extension.

<u>Case 1</u>: If $s^2 = 1$, then $b_i = \pm 1$. Consider the homomorphism

$$\varphi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_n \rtimes \mathbb{Z}_n^* = \mathrm{Aut}\, D_{2n}$$

where $\varphi(1) = (0, s)$. The extension

$$0 \longrightarrow D_{2n} \longrightarrow D_{2n} \rtimes_\varphi \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces the abstract kernel ϕ. By uniqueness this split extension is equivalent to $0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$, contradicting to the non-splitting assumption.

<u>Case 2</u>: If $s^2 = -1$, then $b_i = m_i$ where $m_i^2 \equiv -1 (\mathrm{mod}\, p_i^{n_i})$. Consider

$$f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_n^* \cong \prod_i \mathbb{Z}_{p_i^{n_i}}^*$$

where $f(1) = \prod_i m_i$. The canonical subgroup $\mathbb{Z}_n \rtimes_f \mathbb{Z}_2 \subset \mathbb{Z}_n \rtimes_f \mathbb{Z}_4$ is dihedral since $m^2 \equiv -1$, and

$$0 \longrightarrow D_{2n} \longrightarrow \mathbb{Z}_n \rtimes_f \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces the abstract kernel ϕ. By uniqueness $G \cong \mathbb{Z}_n \rtimes_f \mathbb{Z}_4$. This however contains an Obstruction Kernel of Type D, which is thus impossible.

In conclusion, either case leads to a contradiction and the assumption fails. □

Theorem 9 . *If G acts on $\mathbb{R}^3$ such that the subgroup of orientation preserving homeomorphisms is D_{2n}, n even, then G is isomorphic to a subgroup of $O(3)$.*

Proof . If $n = 2$, then G is a 2-group and the result is known. So it suffice to consider the case where $n > 2$.

By the preceding proposition, the extension $0 \to D_{2n} \to G \to \mathbb{Z}_2 \to 0$ splits. Thus $G \cong D_{2n} \rtimes_\varphi \mathbb{Z}_2$ for some $\varphi : \mathbb{Z}_2 \longrightarrow \mathrm{Aut}\, D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$. Let $\varphi(1) = (t, s)$.

Consider the subgroup $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2 \subset G$. By Remark 2, this subgroup is isomorphic to either $\mathbb{Z}_{2n}$, $\mathbb{Z}_n \oplus \mathbb{Z}_2$ or D_{2n}. Since n is even, $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ cannot be cyclic. Thus there are two possibilities.

Case 1: $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2 \cong \mathbb{Z}_n \oplus \mathbb{Z}_2$

In this case $s = 1$. $\varphi(1)^2 = (t,1)^2 = (2t,1) = (0,1)$. Thus $t = 0$ or $t = m$, $m = \frac{n}{2}$.

i)If $t = 0$, then $\varphi(1) = (0,1)$, which is the identity isomorphism of D_{2n}, thus $G = D_{2n} \times \mathbb{Z}_2$.

ii)If $t = m$, then $\varphi(1) = (m,1)$. Consider the subgroup $\{e, a^m, b, a^m b\}$ of D_{2n}. This is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $\varphi(1)$ restrict to an isomorphism of this group. It is not hard to see the resulted $(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes_\varphi \mathbb{Z}_2$ is isomorphic to Q_8. But this is Obstruction Kernel of Type F, a contradiction.

Case 2: $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2 \cong D_{2n}$

In this case $s = -1$. We divide the discussion by parity of t.

i)If t is odd. Then G is dihedral as computed in Proposition 9.

ii)If t is even. Let $r = \frac{t}{2}$. The element $(a^r b, 1)$ is of order 2, and $(a^r b, 1) \in G - D_{2n}$. Thus $G \cong D_{2n} \rtimes_\phi \mathbb{Z}_2$ where $\phi(1)$ is the conjugation by $(a^r b, 1)$. An easy computation shows that

$$(a^r b, 1) \cdot (a, 0) \cdot (a^r b, 1) = (a, 0)$$
$$(a^r b, 1) \cdot (0, 1) \cdot (a^r b, 1) = (0, 1)$$

Thus $\phi(1)$ is the identity and $G \cong D_{2n} \times \mathbb{Z}_2$.

We have seen that in all possible cases, G is isomorphic to a subgroup of $O(3)$.

□

Remark 3 . The last case in the proof actually gives an alternative proof to Obstruction Kernel of Type 6, since G then contains a copy of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, which is not allowed in the orientation preserving case.

Combining the above results, we have:

Corollary 4 . *If G acts on $\mathbb{R}^3$ such that the subgroup of orientation preserving homeomorphisms is dihedral, then G is isomorphic to a subgroup of $O(3)$.*

3.5 The A_4 Case

Theorem 10 . *If G acts on $\mathbb{R}^3$ such that the subgroup of orientation preserving homeomorphisms is A_4, then G is isomorphic to a subgroup of $O(3)$.*

Proof . There are up to isomorphism 15 groups of order 24, among which only $A_4 \times \mathbb{Z}_2$ and S_4 contains A_4. Both are subgroup of $O(3)$. □

3.6 The S_4 Case

Theorem 11 . *If G acts on $\mathbb{R}^3$ such that the subgroup of orientation preserving homeomorphisms is S_4, then G is isomorphic to a subgroup of $O(3)$.*

Proof . The only extension of S_4 by $\mathbb{Z}_2$ is $S_4 \times \mathbb{Z}_2$. □

3.7 The A_5 Case

Theorem 12 . *If G acts on $\mathbb{R}^3$ such that the subgroup of orientation preserving homeomorphisms is A_5, then G is isomorphic to a subgroup of $O(3)$.*

Proof . There are only two extensions of A_5 by $\mathbb{Z}_2$: $A_5 \times \mathbb{Z}_2$ and S_5 (cf the proof in section 2.4). It suffice to prove that S_5 cannot act on $\mathbb{R}^3$. Now suppose there is such an action.

There is a subgroup of S_5 isomorphic to $GA(1,5) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_5^*$. This group cannot be embedded in A_5, thus the action restrict to an orientation reversing one on it (alternatively one can observe that the group is not isomorphic to a subgroup of $SO(3)$ and use the result of the orientation preserving case). There is only one subgroup in $\mathbb{Z}_5 \rtimes \mathbb{Z}_5^*$ on index 2 and it is a copy of D_{10}. This D_{10} then has to be the subgroup of orientation preserving homeomorphisms.

By the dihedral case, $\mathbb{Z}_5 \rtimes \mathbb{Z}_5^*$ has to be a subgroup of $O(3)$. It is not abelian, thus has to be either D_{20} or $D_{10} \times \mathbb{Z}_2$. Neither can be the case(the former is discussed in Section 2.4, while the latter can be done by comparing Sylow 2-subgroups). This contradicts the assumption. □

4 Remarks

This short section contains the following two results (again all actions are assumed to be faithful):

Remark 4.1: *Let G be a finite group acting (orientation preservingly), locally linearly or smoothly on* $\mathbb{R}^4$*. Then G is isomorphic to a subgroup of* $O(4)$ *(*$SO(4)$*).*

Remark 4.2: *There are finite groups G which act topologically and orientation preservingly on* $\mathbb{R}^4$ *and G is not isomorphic to any subgroup of* $SO(4)$ *(in fact G is not isomorphic to any subgroup of* $O(7)$*).*

These two results are known but are included for the completeness reasons. A considerable effort was made to make proofs of these remarks short and self-contained.

Proof of 4.1: Our first observation is the following:

FACT: The only finite simple group which acts orientation preservingly on $\mathbb{R}^4$ is A_5.

The above fact follows from the direct inspection of all finite non-Abelian simple groups, in the atlas of Finite Groups (cf. [9]). The point here is that each such group except A_5 has a solvable subgroup too large to act faithfully on $\mathbb{R}^4$. (We recall that each solvable group acting on $\mathbb{R}^4$ always has a fixed point). For example, in A_6 one can take the normalizers of Sylow-3 subgroups.

Suppose now that G is NOT simple. Let $H \neq \{0\}$ be a maximal normal proper subgroup of G (i.e. G/H is simple).

Case 1: H is a non-Abelian simple group (hence $H \cong A_5$). Then we have an extension

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0 \qquad (1)$$

In order to classify such extensions (cf. [6] p.105), let $\psi : G/H \to \mathrm{Out}(A_5) \cong \mathbb{Z}_2$ be a homomorphism. Then $\psi : G/H \to \mathbb{Z}_2$ is trivial except when $G/H \cong \mathbb{Z}_2$. (note that G/H is a simple group).

The set of extensions (1) with fixed ψ is classified by $H^2(G/H; Z(H))$, where $Z(H)$ is the center of H (cf. [6] p.105). Consequently there is only one extension $G \cong H \times G/H$ for $G/H \neq \mathbb{Z}_2$ and two extensions $G \cong H \times \mathbb{Z}_2$, and $G \cong S_5$ for $G/H \cong \mathbb{Z}_2$.

It follows that both $G \cong A_5 \times \mathbb{Z}_2$ and S_5 are subgroups of $SO(4)$ and it is not difficult to see that $G \cong A_5 \times G/H$, $G/H \neq \mathbb{Z}_2$ cannot act on $\mathbb{R}^4$

Case 2: H is simple Abelian. In this case $(\mathbb{R}^4)^G = ((\mathbb{R}^4)^H)^{G/H} = \{\mathrm{pt}\}$. Consequently G is isomorphic to a subgroup of $SO(4)$

Case 3: H is not simple. Repeating the argument from Case 1 and Case 2 with H replacing G one easily concludes G is isomorphic to a subgroup of $SO(4)$.

Case 4: Suppose G has an orientation reversing element. Let $K \triangleleft G$ be the normal subgroup of orientation preserving elements, so that $G/K \cong \mathbb{Z}_2$. Then either $(\mathbb{R}^4)^K \neq \emptyset$ and hence $(\mathbb{R}^4)^G \neq \emptyset$ and consequently G is isomorphic to a subgroup of $O(4)$ or $K \cong A_5$ and hence we have an extension

$$0 \longrightarrow K \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \qquad (2)$$

This however was handled in Case 1 and hence the proof of Remark 4.1 is concluded.

Proof of 4.2: Let $Q(8p, q)$ be the generalized quaternionic group (cf. [10]), given by the extension

$$0 \longrightarrow \mathbb{Z}_p \times \mathbb{Z}_q \longrightarrow Q(8p, q) \longrightarrow Q(8) \longrightarrow 0$$

where $Q(8)$ is the standard quaternionic group of order eight.

It turns out that $\pi = Q(8p, q)$ i a 4-periodic group and hence acts freely on a simply connected CW complex $\widetilde{X}$ with $\widetilde{X} \simeq S^3$. (cf. [10]).

There are conditions on p, q (cf. [10]) (for example $(p, q) = (3, 313), (3, 433), (7, 113), (5, 461), \ldots$) which imply that π acts freely on a closed 3-manifold $\mathcal{M}^3$ which is a homology 3-sphere. Moreover there is a $\mathbb{Z}[\pi]$-homology equivalence

$$k : \mathcal{M}^3 \longrightarrow X = \widetilde{X}/\pi$$

Consider the map $h = k \times \mathrm{id} : \mathcal{M}^3 \times I \longrightarrow X \times I$ and let $\lambda(h) \in L_0^h(\pi)$ be the surgery obstruction for changing h to a homotopy equivalence without modifying anything on the boundaries.

Now let $\mathcal{F} : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$ be the identity homomorphism and $\Gamma_0(\mathcal{F})$ be the Cappell-Shaneson homological surgery obstruction group as in [7].

The natural homomorphism $j_* : L_0^h(\pi) \to \Gamma_0(\mathcal{F})$ is an isomorphism (see CappellShaneson p.288) and clearly $j_*(\lambda(h)) = 0$ so that $\lambda(h) = 0$ in $L_0^h(\pi)$.

Let $\bar{h} : (\mathcal{W}^4; \mathcal{M}^3, \mathcal{M}^3) \longrightarrow (X \times I; X, X)$ be a homotopy equivalence. Form a two ended open manifold

$$\mathcal{W}_0^4 := \ldots \cup \mathcal{W}^4 \underset{\mathcal{M}^3}{\cup} \mathcal{W}^4 \underset{\mathcal{M}^3}{\cup} \mathcal{W}^4 \cup \ldots$$

by stacking together copies of $\mathcal{W}^4$.

Observe that $\pi_1(\mathcal{W}^4) \cong \pi$ and the universal cover $\widetilde{\mathcal{W}_0^4}$ of $\mathcal{W}_0^4$ is a manifold properly homotopy equivalent to $S^3 \times \mathbb{R}$ and hence homeomorphic to $S^3 \times \mathbb{R}$ by [15].

One point compactification of one end of $\widetilde{\mathcal{W}_0^4}$ yields an action of π on $\mathbb{R}^4$ with one fixed point. Since π is not isomorphic to a subgroup of $O(7)$ (cf. [1]) the proof of 3.2 is complete.

References

[1] S. Bentzen and J. Madsen. On the Swan subgroups of certain periodic groups. *Math. Ann., Vol. 162, 447-474*, 1983.

[2] R.H. Bing. homeomorphism between the 3-sphere and sum of two solid horned spheres. *Ann. of Math., Vol.56, 354-362*, 1952.

[3] R.H. Bing. Inequivalent families of periodic homeomorphisms of E^3. *Ann. of Math., Vol.80, 78-93*, 1964.

[4] G. Bredon. *Introduction to Compact Transformation Groups*, volume 46 of *Pure and Applied Mathematics.* Academic Press, New York-London, 1972.

[5] L. E. J. Brouwer. Über die periodischen transformationen der kugel. *Math. Ann., Vol. 90, 39-41*, 1919.

[6] K. Brown. *Cohomology of Groups*, volume 87 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, Berlin, Heidelberg, 1982.

[7] S. Cappell and J. Shaneson. The codimension two placement problem and homology equivalent manifolds. *Ann. of Math., Vol. 99, 277-348*, 1974.

[8] W. Chen, S. Kwasik, and R. Schultz. Finite symmetries of S^4. *Forum Math., Vol. 28, 295-310*, 2016.

[9] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of Finite Groups.* Oxford Univ. Press, 1985.

[10] J. Davis and R. Milgram. A survey of the spherical space form problem. *Math. Reports, Vol. 2, Harvard Academic Publishers. London, pp. 223-283.*, 1985.

[11] R. Dotzel and G. Hamrick. *p*-group actions on homology spheres. *Invent. Math., Vol. 62, 437-442*, 1981.

[12] A. Edmonds. A topological proof of the equivariant Dehn lemma. *Trans. Amer. Math. Soc. 297, 605-615*, 1986.

[13] S. Eilenberg. Sur les transformations périodiques de la surface de sphère. *Fund. Math, Vol. 22, 28-41*, 1934.

[14] W. Feit and J. Thompson. Solvability of groups of odd order. *Pacific Journal of Mathematics, Vol. 13, 775-102*, 1963.

[15] M. Freedman. The topology of four-dimensional manifolds. *J. Diff. Geom. 17, 357-453*, 1982.

[16] C. H. Giffen. The generalized Smith conjecture. *Amer. J. Math., 88, 187-198*, 1966.

[17] McA Gordon. On the higher dimensional smith conjecture. *Proc. London Math. Soc.(3), Vol. 29, 98-110*, 1974.

[18] M. Hall. *The Theory of Groups.* The Macmillan Company, New York, 1962.

[19] W. Jaco and H. Rubinstein. Pl equivariant surgery and invariant decompositions of 3-manifolds. *Adv. Math. 73, 149-191*, 1989.

[20] B. Kerekjarto. Über die-periodischen transformationen der kreisscheibe und der kugelfläche. *Math. Ann. 80, 36-38*, 1919.

[21] S. Kwasik and R. Schultz. Desuspension of group actions and the Ribbon Theorem. *Topology Vol. 27, 443-457*, 1988.

[22] S. Kwasik and R. Schultz. Pseudofree group actions on S^4. *Amer. J. Math 112, 47-70*, 1990.

[23] S. Kwasik and R. Schultz. Icosahedral group actions on $\mathbb{R}^3$. *Invent. Math. 108, 385-402*, 1992.

[24] W. Meeks and S. T. Yau. The equivariant Dehn's lemma and Loop Theorem. *Comment. Math. Helv. 56, 225-239*, 1981.

[25] P. A. Smith. Transformations of finite period. *Ann. of Math. (2) 39 127-164*, 1938.

[26] E. Stein. Surgery on products with finite fundamental group. *Topology 16, 473-493*, 1977.

[27] M. Suzuki. On finite groups with cyclic Sylow subgroups for all odd primes. *Amer. J. Math., 77, 657-691*, 1955.

Biography

Fang Sun was born on January 11th, 1990 in Longkou, Shandong Province, P. R. China. He received a Bachelor's degree in Mathematics from Nankai University on 2012 and went to Tulane University in pursue of a Ph. D. in Mathematics. He received his degree on 2018.