YANG-MILLS THEORY FOR THE NUCLEAR GEOMETRICAL COLLECTIVE MODEL

AN ABSTRACT
SUBMITTED ON THE FIRST DAY OF DECEMBER, 2017
TO THE DEPARTMENT OF PHYSICS
OF THE GRADUATE SCHOOL OF
TULANE UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
BY

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Abstract

The Bohr-Mottelson collective model is central to the study of nuclear structure physics. The model treats nuclei as ellipsoids and focuses on their vibrational and rotational degrees of freedom. The rotational part of the model regards the moment of inertia of the nuclei as a falling between the extreme of a rigid body and an irrotational fluid. The true moment of inertia, as revealed by experiment, provides a parameter between these two extremes and acts as a way of interpolating the data. In this work, we show how the interpolating parameter between the two extreme moments of inertia can be treated theoretically using an algebraic and differential geometric framework. The essential idea is to couple the angular momentum of the nucleus with a “magnetic” term that involves the Kelvin circulation into a covariant derivative. This coupling term or connection can be found by solving a Yang-Mills equation. Measuring the nuclear Kelvin circulation then reveals a theoretical justification for the determining the correct moment of inertia.
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Acknowledgments

I would like to begin by thanking my advisor, George Rosensteel, for his tremendous patience and dedication in helping me complete my Ph.D. He has been an amazing friend and wonderful mentor and I am grateful to him for teaching me how to think deeply about simple things. I would also like to thank Lev Kaplan and Fred Wietfeldt for being excellent teachers and for their support during the writing of this thesis. Both have been an inspiration for the way that I teach my own students.

I wouldn’t be where I am without the support of many people. My mom encouraged me from a young age to seek truth and beauty. It’s not a surprise she encouraged me to pursue physics and I am thankful for her wisdom and love. I’d also like to thank my dad for his humor and rhythm. I’d like to thank Connor Whitaker for the great conversations about physics and life. I would like to thank Noah Rahman for helpful discussions and recommending some great sources for this work. I am grateful to the New Orleans Catholic Worker and its patrons for giving me the opportunity to help the people of New Orleans. I would like to thank Willie Hill and Cori Jenkins for helping me acclimate to life in New Orleans and for their continued friendship. I would also like to thank Rebecca Stilling for her wisdom and for helping me become a better swimmer.

Lastly, I would like to thank the faculty and staff at Tulane University and the Department of Science and Engineering for their support in my academic endeavors.
Foreword

Having the opportunity to perform research at Tulane and the writing of this thesis has been a great honor and a lot of fun. Having the opportunity to study nuclear collective motion has greatly expanded my knowledge concerning emergent phenomena. The subject of this dissertation has been perfect in guiding my pursuits towards this field of research. After all, if all elementary particles are indistinguishable, what better place to begin? The techniques used in this work such as the covariant derivative, differential geometry, Lie groups and algebras, and the Yang-Mills equation have been everything I could possibly want out of a graduate degree. I really love this subject and where some questions have been answered, many more have been created!

It is my hope that this dissertation is as much fun to read as it was to write. My goal in writing it was to be as readable as possible in the hopes that it might help further knowledge in the field of nuclear structure, but also to be a useful resource for those starting out in the field. Hopefully this goal has been achieved.
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Chapter 1

Introduction

How do deformed nuclei rotate? This is a central question in the science of nuclear structure physics. The aim of this work is to shed some light on answering this fundamental question. The philosophy of this thesis is broken down into providing an overview of the types of phenomena and associated models that the field of nuclear physics studies. Because nuclei are collections of fermions, the two dominant models discussed will include the Shell and Collective models. The first addresses the individual particle nature of nuclei, while the latter addresses the bulk collective properties and the phenomena that emerges. By seeking to address the concept of a nucleus rotating, we are implicitly dealing with the collective model and so our focus will be heavily skewed in this direction.

The primary problem that this work addresses is providing a theoretical justification regarding the moment of inertia of a rotating nucleus. The moment of inertia of the nucleus is a model dependent phenomenon. One can model the nucleus as a rotating fluid, as was done in the work of Bohr and Mottelson. Alternatively, one can model the nucleus as a rigid object. The true moment of inertia as revealed by experiment falls somewhere in between these two extremes. One can then naturally wonder if there is some interpolating parameter that correctly combines these two extremes: $I_r = r^2I_{\text{rigid}} + (1 - r^2)I_{\text{fluid}}$. This work seeks to provide a theoretical framework for
finding such a parameter.

Nuclei have angular momentum. What is going on inside of the nucleus that produces such angular momentum? Clearly, the motion of the individual nucleons determine how the whole nucleus rotates. It is this observation that leads one to speak of angular momentum of the whole nucleus and circulation of the internal nucleons. These two types of motion must be related to one another. We might imagine if rigid motion is at work that the angular momentum and circulation are essentially the same thing. If we imagine the nucleus as a rotating liquid drop, the angular momentum must be related to the circulation in some non-trivial way. The relationship between the angular momentum and the circulation must be related to the same interpolation parameter mentioned above.

The central idea of this work is that a Yang-Mills equation correctly determines this interpolation parameter. Solutions to the Yang-Mills equation provide mathematical objects called connections, which physicists might think of as “magnetic terms” highly analogous to the vector potential in electrodynamics. By writing down and solving the Yang-Mills equation for the nuclear geometrical collective model, the coupling between the angular momentum and the circulation can be fully determined. This work is important because it addresses a central problem in nuclear structure physics which can be experimentally realized and is hence falsifiable. Answering it through experiment sheds tremendous light on the nature of how nuclei rotate.
Chapter 2

Background

2.1 Nuclear Structure Preliminaries

The heart of scientific endeavor is based upon creating falsifiable models of the natural world, using them to predict phenomena, relating various phenomena to one another, and ultimately building upon (or rejecting!) the model as new data becomes available. Nuclear models are created from general patterns that have emerged from experimental data. Here, we will introduce several key concepts and an overview of the dominant models found in nuclear structure physics. Nuclear collective models will be the primary focus of the discussion.

2.1.1 General Structure of the Nucleus as a Bulk Object

Nuclear Radius

Since the focus of this discussion is on the collective (bulk) structure of the nucleus, there are natural questions that one can ask. A starting point is to address how “large” the nucleus is. Data suggests that the radius of the nucleus increases as the number of nucleons is added, but in a seemingly peculiar way: $R_0 = 1.2 A^{1/3}$ fm [1], where $A$ is the number of nucleons. The general form of this result can be deduced as follows: If the volume of the nucleus is proportional to the number of nucleons
that comprise it (i.e. \( V \propto A \)) and the volume takes on a roughly spherical shape
\( (V \approx \frac{4\pi}{3}R^3) \), then the result for the radius follows. As is the case with dimensional
analysis, specific numbers such as the one present, must be empirically deduced from
experiment. This formula for the radius is ubiquitous in nuclear collective structure.

**Nucleus as a Collection of Charges**

One of the most central ideas to nuclear collective structure is how the nu-
cleons are arranged within the nucleus. While addressing this question completely is
fundamentally difficult, we can look for trends in the data that can be viewed from
a classical electrodynamics perspective. The nucleus is composed of discrete charges,
but as an approximation, it can be regarded as a continuum of nuclear matter. With
this said, a general approach to treating this idea is to look at the general multipole
expansion of the potential due to these charges. Following Jackson [2]:

\[
\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')}{\|\vec{x} - \vec{x}'\|} \, d^3x' \quad (2.1)
\]

\[
= \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm}(\theta, \phi) \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \quad (2.2)
\]

\[
= \frac{1}{\varepsilon_0} \sum_{l,m} \left[ \int Y_{lm}^*(\theta', \phi') r'^l \rho(\vec{x}') \, d^3x' \right] \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \quad (2.3)
\]

As usual, primes indicate the region over which the charge distribution is
being summed over. Here we have essentially set up a definition for the “multipole
moments” by expanding the potential in terms of spherical harmonics \( Y_{lm} \). Our
reasons for choosing this expansion are due to symmetry considerations and also for
the reason that the spherical harmonics are eigenstates of the angular momentum
operators. The multipole moments are given by:

\[
q_{lm} = \int Y_{lm}^*(\theta', \phi') r'^l \rho(\vec{x}') \, d^3x' \quad (2.4)
\]
For the purposes of structure theory for low energy (< 5 MeV), we will focus primarily on the first three moments called the monopole moment (l = 0), dipole moment (l = 1), and quadrupole moment (l = 2). For the purposes of studying structure, the monopole moment tells us the size of the nucleus, the dipole tells us about the center of charge, and the quadrupole tells us the overall shape of the nucleus. Of these three multipole moments, the quadrupole will be the focus of this discussion.

As we will see later, the quadrupole moments will serve as mathematical coordinates when we discuss the geometrical collective model, and an alternative expression for the potential can be written with this in mind in the following Cartesian way:

\[
\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{r} + \frac{\vec{p} \cdot \vec{x}}{r^3} + \frac{1}{2} \sum Q_{ij} x_i x_j r^3 + \ldots \right] \tag{2.5}
\]

\(Q_{ij}\) is called the traceless moment of inertia tensor and is given by:

\[
Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij}) \rho(\vec{x}') \, d^3x' \tag{2.6}
\]

For the purposes of convention, “the” intrinsic (body fixed) quadrupole moment when measured experimentally is chosen to be the moment about the z-axis, \(Q_{33} = \int (3z'^2 - r'^2) \rho(\vec{x}') \, d^3x'\).

**Nucleus as a Rigid Object**

Now that some of the basic language inherited from classical electrodynamics has been laid out, we can discuss some of these ideas further. As mentioned above, we measure the quadrupole moment about the z-axis in the nucleus’s body fixed frame of reference. With that in mind, we can immediately see that if \(Q_{33} > 0\) the distribution is cigar-shaped or “prolate”, and for \(Q_{33} < 0\) the distribution is pancake-shaped or
"oblate". This is of course all with respect to the z-axis in the body-fixed frame of the nucleus where the other two axes are of equal size to one another. In general, we might measure the quadrupole moments about the x or y axes and deduce similar ideas of shape, but up to a rotation. The shape of a nucleus is often approximated as an ellipsoid:

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \] (2.7)

The values of a, b, and c all correspond to the axes lengths of the ellipsoid. For the sake of this work, the axes lengths of an ellipsoid follow the definition given by Hill and Wheeler [3], namely:

\[ a = R_0 \exp \left( \sqrt{\frac{5}{4\pi}} \beta \cos(\gamma - \frac{2\pi}{3}) \right) \] (2.8)

\[ b = R_0 \exp \left( \sqrt{\frac{5}{4\pi}} \beta \cos(\gamma - \frac{4\pi}{3}) \right) \] (2.9)

\[ c = R_0 \exp \left( \sqrt{\frac{5}{4\pi}} \beta \cos(\gamma) \right) \] (2.10)

Here, \( \beta \) and \( \gamma \) are called "deformation" parameters and are related to the quadrupole moments defined above in the following way:

\[ Q_0 = \frac{3}{\sqrt{5\pi}} R_0^2 Z \beta \] (2.11)

The deformation parameter \( \gamma \) is slightly trickier to treat presently, however, it will be addressed in an appendix.

The Nuclear Data Sheets provide a great deal of information concerning some of the ideas expressed above. The deformation parameters produced by experiment
can be found here and as we’ll see, these allow us to calculate the interpolation parameter (coupling constant) \( r \). The appendices discuss how one can start from the data provided by experiment and deduce much of the material discussed above.

2.2 Background: Nuclear Models

The atomic nucleus is surprisingly complicated to understand. This fact is best seen by the large number of models and their variations that have gone into attempting to understanding the nucleus’s various properties. Since the nucleus can split apart via fission into daughter nuclei, early attempts of modeling the nucleus as a fluid were created. Due to the nucleus being composed of both protons and neutrons, both fermions, a shell model was devised to model similar characteristics to that of the electronic shell model. Finally, because bulk properties such as rotational and vibrational bands were observed in heavy nuclei, the collective model was developed.

We will now look at each of these pictures of the atomic nucleus, but with primary focus on the collective model.

2.2.1 Liquid Drop Model

Early experiments in radioactivity by Becquerel, the Curies, and others showed that some elements could split into new elements via a process called fission. To model this phenomena, it was suggested that the nucleus should be modeled as a high density, incompressible, electrically charged fluid composed of nuclear matter. It was thought if the attractive forces holding the nucleus together were overcome by repulsive forces trying to make it fly apart, that the nucleus would split apart. In 1935, Weizächer produced the following formula for the binding energy of the nucleus [4]:

\[
E_B = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A} + \delta(A,Z)
\] (2.12)
Here, $A$ represents the number of total nucleons. $Z$ represents the number of protons, and each of the coefficients are best fit parameters that account for the volume, surface tension, Coulomb repulsion, asymmetry, and nuclear pairing term. It should be noted that empirically the nuclear radius is approximated as $R_0 = 1.2A^{1/3}$fm. It is clear by dimensional analysis why the first two terms of the binding energy equation take the form that they do. The third term called the Coulomb term is of particular interest. For large nuclei, $Z(Z - 1) \approx Z^2$ and the "self energy" of assembling a collection of charges is $E = \frac{3}{5} \frac{Q^2}{R}$. The total charge, $Q$, is simply given by adding up $Z$ fundamental charges for a total charge of $Ze$. The term $Z(Z - 1)$ originates because there is no potential for a single proton. Relying on the formula for the nuclear radius, we obtain the Coulomb term in terms of $Z$ and $A$. The last two terms are quantum mechanical in origin. The asymmetry term and pairing terms both require knowledge of the shell model to derive. For this reason, they will be suppressed.

To this day, the liquid drop model has still garnered research interest. The original formulation of the model only included the terms discussed above, however, some recent models have as many as 10 terms some of which are quite elaborate! [4]

The liquid drop model essentially breaks down both empirically and theoretically. Empirically, the model doesn’t account for collections of nuclei with special numbers of stability called 'magic numbers'. It also fails to predict the correct binding energy for light nuclei. This of course makes sense when one takes into account that the model was originally used to describe heavy nuclei that fission. Theoretically, the model isn’t purely quantum mechanical. For these reasons, subsequent models followed and one the most successful of these was the shell model.
2.2.2 Shell Model

The Nuclear Shell Model has its origins in recognition that some nuclear isotopes are more stable than others and that nucleons are fermions like the electron. As in the case of the atomic model of the atom, fermions occupy certain fixed energy regions called shells. Shells can be filled or unfilled based on the particular number of nucleons available to the system. Filled shells correspond to regions of high stability. Unlike the atomic model which treats the nucleus as a being a positively charged particle with the electrons "orbiting" it, the nuclear model can't quite assume such a picture because the nucleons are all very close to one another on the order of a femtometer (1 fm = 10^{-15} m). Modeling the potential of how this interaction might work presents a challenge. A good starting point to describe how the nucleons realistically interact with one another (without appealing to quantum chromodynamics) is via an admixture of the square well potential and the harmonic oscillator potential called the Woods-Saxon potential. It is given by:

\[ V(r) = -\frac{V_0}{1 + e^{(r-R_0)/a}} \]  \hfill (2.13)

Here, \( a \) is the surface thickness which is roughly 0.5 fm, \( R_0 \) is the average nuclear radius given above, and \( V_0 \) is the depth of the potential typically around a value of 50 MeV [4]. Although realistic, using this potential involves non-analytic approaches to finding the energy of the system. Instead, to describe nuclear shell structure analytically, we model the potential as being a harmonic oscillator. To get the experimentally revealed shell structure, a spin-orbit interaction becomes the necessary ingredient. It is said, possibly apocryphally, that Maria Goeppert-Mayer (Nobel Prize 1963) took advice from Enrico Fermi to include the spin-orbit term and within several minutes of work was able to recover the regions of stability known
as the "magic numbers". Magic numbers are regions of shell closure and hence are large stability for both protons and neutrons separately. Nuclei can be doubly magic, meaning that both shells are filled and are extremely stable. The magic numbers are given by the sequence: 2, 8, 20, 28, 52, 126, and possibly 184 for neutrons and 114 for protons [1]. Such a large region of nuclei is sometimes regarded as the "island of stability" and is a subject of modern research.

The shell model while successful at predicting regions of stability also predicts that the ground state of even-even nuclei (even number of protons and even number of neutrons) must have an angular momentum of zero and have positive parity. Unfortunately, the shell model suffers from practical problems. As is the case with atomic physics, for large numbers of particles that can occupy various energy levels, computation of the shell states becomes combinatorially intractable. Even the most advanced super computers must truncate the model space of states to perform even the most basic of calculations. For this reason and for experimentally realizing that nuclei exhibit collective "bulk" properties, the Collective Model was developed.

2.3 The Nuclear Collective Model

The collective model is the main focus of this work. As was eluded to above, it was developed as an answer to explain phenomena that the shell model couldn't account for both practically and theoretically. This will be discussed in greater detail. Since most isotopes aren't near closed shells, the collective model is crucial to understanding much of nuclear structure. Philosophically, the collective model is the ideal "playground" for understanding emergent phenomena. This is because all protons and neutrons are identical to every other proton and neutron. This suggests that the simplest model in physics with truly identical objects that are capable of displaying emergent phenomena in a collective manner can be found within the field of nuclear
physics. What we mean by collective is best described by the following plot:

![Figure 2.1: B(E2) experiment to single particle ratio](image)

We see from this plot that the electric quadrupole transition rates for heavy nuclei far away from closed shells are many times larger than for a single particle estimate! This means that the large rates must be due to many nucleons acting together, collectively. It is appropriate to use single particle estimates when one is studying nuclei with one or two nucleons close to a closed shell.

How then does one construct a theory to describe collective nuclei? Simply put, one must guess some model Hamiltonian and determine if it makes the right predictions. Fortunately, these guesses can be educated. By virtue of looking at patterns within nuclear data, one can deduce that for nuclei far away from closed shells that vibrational and rotational bands become apparent. This suggests that modeling the nucleus as a vibrating and rotating ellipsoid may be most appropriate for the lowest order non-trivial degrees of freedom. Again, the validity of any guess at a model must be backed up by experimental data. Fortunately, the quadrupole
moments provide much information for describing collective phenomena.

The form of the Hamiltonian for a collective nucleus must have both a kinetic and a potential piece, however, due to the large number of degrees of freedom present (each nucleon gets its own set!), there is also an additional term which we refer to as the ‘intrinsic Hamiltonian’. This is an important idea and is fortunately ubiquitous across science. A claim made in most physical models is that only some degrees of freedom are dominant in describing the phenomena at hand. This means that any additional degrees of freedom play a passive role and can be neglected. Amazingly, such a treatment of disregarding most of the extra degrees of freedom works well in describing nuclear collective motion. The form of our Hamiltonian then looks like:

\[
\hat{H} = \hat{H}_{\text{rot}} + \hat{H}_{\text{vib}} + \hat{H}_{\text{intr}}
\]

\[
= \sum_i \frac{\hat{I}_i^2}{2I_i} + \frac{1}{2} B (\beta^2 + \beta^2 \gamma^2) + V(\beta, \gamma) 
\]

\[
= \sum_i \frac{\hbar^2 J(J+1)}{2I_i} + \frac{1}{2} B (\beta^2 + \beta^2 \gamma^2) + V(\beta, \gamma) 
\]

Here, the rotational term imagines that the nucleus is rotating about some axis not unlike a rotating football whose surface is vibrating. Any dependence on the intrinsic degrees of freedom are absorbed into the potential term since all of the dominant observations fit into the kinetic terms. This is an oversimplification, but in practice it works very well. As with any model, one may build off of the success of prior models. This current thesis attempts just that. In the Bohr-Mottelson collective model, which is structurally similar to the Hamiltonian written above, the idea of the Kelvin circulation is omitted. That is, some of the degrees of freedom to describe such an effect aren’t taken into account. To truly answer the question of how nuclei rotate must involve the motion of the nucleons in a way that relates to the overall motion that is observed. That is, the angular momentum and the circulation must
be coupled in some way. The idea of how to incorporate this is the subject of this work and will be discussed in the following chapters.

Due to the Hamiltonian having rotational and vibrational components, the collective model quantum states should involve parts that contain a radial and a vibrational term. A product of two functions for each type of respective type of variable is not unlike the standard separation of variables solution to a partial differential equation. With that said, the collective states take on the form [1]:

\[
\Psi_{J,M} = \sqrt{\frac{2J + 1}{8\pi^2}} \sum_{K=-J}^{J} \Phi_{J,K}(r) \mathcal{D}^{J}_{M,K}(\Omega)
\]  
(2.17)

![Diagram](image)

Figure 2.2: Deformed Nucleus

[7]

Here, \( J \) and \( I \) (seen above in the figure) represent the same total angular momentum. The collective wavefunction is a radial wavefunction expanded in terms of Wigner D-functions, \( \mathcal{D}^{J}_{M,K}(\Omega) \). This is not unlike a Fourier expansion, as the Wigner D-functions form a basis for an irreducible representation of the rotation group \( SO(3) \) in the same manner that the Fourier functions form a basis for the group \( U(1) \), or equivalently the rotation group \( SO(2) \). A particularly nice feature about the form of the collective wavefunction is that it allows us to easily transform from the laboratory frame to the intrinsic frame. This is simply a matter of applying
an appropriate rotation from one frame to the other via multiplication by another Wigner D-function [1].

Now that the foundation for the nuclear model has been laid out, we ask 'how do nuclei rotate?'. To answer this question as simply as possible, we must start by acknowledging that some of the extra degrees of freedom that are left out of the original Bohr-Mottelson model must be incorporated into the nuclear geometrical collective model. It seems reasonable that the rotational part of the Bohr-Mottelson model might include an additional term that relates the angular momentum to the nuclear Kelvin circulation. The reasons for proposing such an assertion comes from the need to reconcile the experimental moment of inertia with the extreme cases of rigid body and irrotational flow moments of inertia. The following plot makes the problem explicit:

![Image](image.png)

Figure 2.3: Moments of Inertia vs. A

[6], [7]

The observed discrepancy is numerically is given roughly by \( I_r = r^2 I_{\text{rigid}} + (1 - r^2) I_{\text{irrotational}} \) where \( r \approx 2/3 \). This work's goal is provide a theoretical justification for this value.
2.4 Gauge Theory and All that Jazz

Gauge theory is of central importance to modern theoretical physics. The essential idea is that the derivatives of geometrical objects must transform in the same way that the geometrical objects themselves transform under the action of some symmetry, that is, they “co vary”. There are two major reasons why gauge theory is so important. The first reason is that Noether’s theorem asserts that symmetries correspond to conserved quantities. The second reason is that building symmetries into a theory by ensuring that the action is invariant under some symmetry is the modern recipe for creating physical theories. To see how this works, consider the following examples that illustrate the geometrical origins of where covariance arises.

2.4.1 A Simple Example: Local Rotation of a Vector

Suppose that we have some two dimensional vector whose components each depend on time. Moreover, consider what happens if we apply some rotation to this vector where the angle also depends on time, \( \theta = \theta(t) \).

\[
\begin{pmatrix}
  x'(t) \\
  y'(t)
\end{pmatrix}
= \begin{pmatrix}
  \cos(\theta) & \sin(\theta) \\
  -\sin(\theta) & \cos(\theta)
\end{pmatrix}
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix}
= \begin{pmatrix}
  \cos(\theta) \, x(t) + \sin(\theta) \, y(t) \\
  -\sin(\theta) \, x(t) + \cos(\theta) \, y(t)
\end{pmatrix}
\]
Differentiating with respect to time and factoring out the original matrix gives:

\[
\begin{pmatrix}
\dot{x}'(t) \\
\dot{y}'(t)
\end{pmatrix} = \begin{pmatrix}
-\sin(\theta) \dot{\theta} x(t) + \cos(\theta) \dot{\theta} \dot{x}(t) + \cos(\theta) \dot{\theta} y(t) + \sin(\theta) \dot{\theta} \dot{y}(t) \\
-\cos(\theta) \dot{\theta} x(t) - \sin(\theta) \dot{\theta} \dot{x}(t) - \sin(\theta) \dot{\theta} y(t) + \cos(\theta) \dot{\theta} \dot{y}(t)
\end{pmatrix}
= \begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{pmatrix} \begin{pmatrix}
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{pmatrix} + \dot{\theta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}
\end{pmatrix}
= \begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{pmatrix} \frac{dx}{dt} + \dot{\theta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}
= \begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{pmatrix} D \begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}
\]

A summary of what we have done can be written in the following way:

\[
\begin{align*}
\dot{x}'(t) &= g \bar{x}'(t) \\
\dot{y}'(t) &= g D \bar{x}'(t)
\end{align*}
\]

Here \( g \in SO(2) \), the special orthogonal group, and \( D \) is called the covariant derivative. We say that the derivative varies "with" the original vector, hence the name covariant derivative. It will usually be the case that we are interested in continuous symmetries such as the one above. With that said, all covariant derivatives take the form of this expression.

2.4.2 Another Simple Example: Local Rotation of a Function

At first glance, it seems unnatural to speak of rotating a function, however it seems natural to rotate the plane. The idea is simply understanding that passive and
active rotations are equivalent. Mathematically, we should write:

\[(D(R)\psi)(\vec{x}) = \psi(R^{-1}\vec{x})\]  \hspace{1cm} (2.18)

To appreciate the form of this expression, it is helpful to look at an example. Consider a counterclockwise rotation about the z-axis by angle \(\theta\). The matrix that accomplishes such a rotation is given by:

\[R_z(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (2.19)

Its inverse is simply given by replacing each value of \(\theta\) with \(-\theta\). To see explicitly why the definition above makes sense, we write:

\[\psi(R^{-1}\vec{x}) = \psi(\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y, z)\]  \hspace{1cm} (2.20)

\[\approx \psi(x - \theta y, \theta x + y, z) \text{ (for } \theta \approx 0\text{)}\]  \hspace{1cm} (2.21)

\[\approx \psi(\vec{x}) + \theta \frac{\partial \psi}{\partial \theta}(\vec{x}) + O(\theta^2)\]  \hspace{1cm} (2.22)

\[= \psi(\vec{x}) + \theta \left( \frac{\partial \psi}{\partial x}(-y) + \frac{\partial \psi}{\partial y}x \right) + O(\theta^2)\]  \hspace{1cm} (2.23)

\[= \psi(\vec{x}) + \theta \left( \frac{i}{\hbar} [\hat{L}_z, \psi]\right) + O(\theta^2)\]  \hspace{1cm} (2.24)

\[\approx (1 + \frac{i}{\hbar} \theta \hat{L}_z)\psi(\vec{x})\]  \hspace{1cm} (2.25)

\[= (e^{i\theta \hat{L}_z}\psi)(\vec{x})\]  \hspace{1cm} (2.26)

\[= (D(R)\psi)(\vec{x})\]  \hspace{1cm} (2.27)

Of course in the limit of small \(\theta\), both sides are equal and not just approximate. The
first thing to appreciate by this description is that equivalent rotations can occur in possibly different spaces. The left hand side of the equation fixes the underlying spatial point and rotates the function to a new value. Equivalently, the underlying space can be rotated to a new point where the function takes on a new value at that point. Another idea is that group multiplication holds in both of the spaces. To see this, let $R_1$ and $R_2$ be two rotations we write:

$$
(D(R_1 R_2)\psi)(\vec{x}) = \psi((R_1 R_2)^{-1}\vec{x})
$$

(2.29)

$$
= \psi((R_2)^{-1}(R_1)^{-1}\vec{x})
$$

(2.30)

$$
= \psi((R_2)^{-1}(R_1)^{-1}\vec{x})
$$

(2.31)

$$
= (D(R_2)\psi)((R_1)^{-1}\vec{x})
$$

(2.32)

$$
= (D(R_1)D(R_2)\psi)(\vec{x})
$$

(2.33)

We are now in a position to discuss the covariant derivative of a function. If both the position and the angle of rotation become local variables (i.e. depend on some parameter like time), we can write:

$$
\psi'(\vec{x}(t)) = (e^{\frac{i}{\hbar}(\vec{L}\cdot \psi)}(\vec{x}(t))
$$

(2.34)
Taking a derivative with respect to this parameter, we obtain:

\[
\frac{d\psi'(\bar{x}(t))}{dt} = \frac{d}{dt}(e^{i\theta(t)\hat{L}_z}\psi'(\bar{x}(t)))
\]

\[= e^{i\theta(t)\hat{L}_z}\frac{i}{\hbar}\frac{d\theta}{dt}\hat{L}_z\psi(\bar{x}(t)) + e^{i\theta(t)\hat{L}_z}\frac{d\psi(\bar{x}(t))}{dt}
\]

\[= e^{i\theta(t)\hat{L}_z}\left(\frac{d\psi(\bar{x}(t))}{dt} + \frac{i}{\hbar}\frac{d\theta}{dt}\hat{L}_z\psi(\bar{x}(t))\right)
\]

\[= e^{i\theta(t)\hat{L}_z}\left(\frac{d}{dt} + \frac{i}{\hbar}\frac{d\theta}{dt}\hat{L}_z\right)\psi(\bar{x}(t))
\]

\[= e^{i\theta(t)\hat{L}_z}D\psi(\bar{x}(t))
\]

(2.39)

(2.40)

\[= g\ D\psi(\bar{x}(t))
\]

(2.40)

2.4.3 Why do we need a Covariant Derivative?

We have now seen a couple of examples of how to construct covariant derivatives, namely when a rotation is involved. However, any continuous symmetry will take this form. One of the deepest ideas in physics is that the way that we choose describe a physical system cannot depend on our choice of coordinates. However, when we work in a curved space, whether in general relativity or over some abstract space with a group structure, it is important that geometrical information be correctly dealt with in a covariant manner. The underlying mathematical reason for this feature is that points in different tangent spaces cannot be directly compared. To remedy this issue, the notion of parallel transport and a connection must be specified. We now investigate this idea in greater detail.

2.4.4 Parallel Transport

Following the excellent description in Nakahara [5], if we want to build up an idea of comparing different tangent spaces, we need a way of “connecting” them together. What kind of properties should such a way of connecting two different tangent spaces have? Let’s begin by looking at the structure of what a derivative has
and build off of that. Given a vector \( \vec{V} = V^\mu e_\mu \), its derivative is given by:

\[
\frac{\partial \vec{V}}{\partial x^\nu} = \frac{\partial (V^\mu e_\mu)}{\partial x^\nu} = \lim_{\Delta x^\nu \to 0} \frac{V^\mu (x^\nu + \Delta x^\nu) e_\mu (x^\nu + \Delta x^\nu) - V^\mu (x^\nu) e_\mu (x^\nu)}{\Delta x^\nu}
\] (2.42)

Notice that this standard way of treating a derivative assumes that the components of the vector are already in the same tangent space (i.e. \( e_\mu (x^\nu + \Delta x^\nu) = e_\mu (x) \)). That is, they can simply be directly compared and subtracted. What happens if the vector we wish to compare is in another tangent space that has been “rotated” away from the original? Now the derivative takes a form that might look like:

\[
\frac{\partial \vec{V}}{\partial x^\nu} = \lim_{\Delta x^\nu \to 0} \frac{\vec{V}^\mu (x^\nu + \Delta x^\nu) e_\mu (x^\nu + \Delta x^\nu) - \vec{V}^\mu (x^\nu) e_\mu (x^\nu)}{\Delta x^\nu}
\] (2.43)

Two properties that this derivative must satisfy at least in the trivial case are:

\[
\vec{V}^\mu (x^\nu + \Delta x^\nu) = V^\mu (x^\nu) \propto \Delta x^\nu
\] (2.44)

\[
(\vec{V}^\mu + \vec{W}^\mu) (x^\nu + \Delta x^\nu) = \vec{V}^\mu (x^\nu + \Delta x^\nu) + \vec{W}^\mu (x^\nu + \Delta x^\nu)
\] (2.45)

To see why the first of these properties are important, consider what happens if we expand the numerator to order \( \Delta x^\nu \) of the derivative:

\[
\vec{V}^\mu (x^\nu + \Delta x^\nu) e_\mu (x^\nu + \Delta x^\nu) \approx \left( \vec{V}^\mu (x^\nu) + \frac{\partial \vec{V}^\mu}{\partial x^\nu} (x^\nu) \Delta x^\nu \right) \left( e_\mu (x^\nu) + \frac{\partial e_\mu}{\partial x^\nu} (x^\nu) \Delta x^\nu \right) \\
\approx \left( \vec{V}^\mu e_\mu (x^\nu) + \frac{\partial \vec{V}^\mu}{\partial x^\nu} e_\mu (x^\nu) + \vec{V}^\mu \frac{\partial e_\mu}{\partial x^\nu} (x^\nu) \right) \Delta x^\nu
\] (2.46)

If we apply the first property, we can compute the derivative:

\[
\frac{\partial \vec{V}}{\partial x^\nu} = \frac{\partial \vec{V}^\mu}{\partial x^\nu} e_\mu + \vec{V}^\mu \frac{\partial e_\mu}{\partial x^\nu}
\] (2.47)
We say that if $\frac{\partial v^i}{\partial x^r} = 0$ that the vector has been parallel transported. However, this raises a fundamental question. How do we evaluate $\frac{\partial v^i}{\partial x^r}$? This is question is of central importance to differential geometry because it is asking 'what is the connection?'

2.4.5 Connections

A connection is essentially a recipe for how to connect two different tangent spaces so that covariant derivatives can be calculated. The connection is a central object of study of modern differential geometry. The standard connection often used in Riemannian geometry is called the Levi-Civita connection. This connection is sometimes written as $\nabla_{e_j} e_k = \Gamma^i_{jk} e_i$, where $\Gamma^i_{jk}$ is called the Christoffel symbol. The Christoffel symbol is related to the metric via:

$$
\Gamma^i_{jk} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right)
$$

The Levi-Civita connection isn’t the only connection that makes an appearance in physics. One of the more striking examples of a connection can be found when discussing the generalized momentum for the Hamiltonian of a charged particle in an electromagnetic field. The usually momentum get ‘upgraded’ to include an addition term that involves the vector potential via:

$$
\hat{p}_\mu \rightarrow \hat{p}_\mu - e \hat{A}_\mu
$$

This additional piece is called the connection and by adding it, the momentum becomes covariant. It is this idea that this thesis uses as a launching point. The notion of a connection is an object of central importance to modern differential geometry and plays a vital role in the nuclear geometrical collective model. To understand a
connection in a more elegant way, the notion of a bundle naturally arises.
Chapter 3

Connexions for the Nuclear Geometric Collective Model

G Rosensteel and N Sparks

3.1 Introduction

This chapter investigates the differential geometry of the Bohr-Mottelson-Frankfurt collective model [7, 1, 8]. The model's motion group is \( P = \text{GL}_+(3, \mathbb{R}) \) because \( P \) acts on a nucleus by rotating and deforming it. Wave functions in the nuclear geometrical collective model are sections of an associated bundle. The bundle is \( E = P \times_\rho V \) for \( \rho \) an irreducible representation on the vector space \( V \) of the gauge group \( \text{SO}(3) \subseteq P \). In the legacy Bohr-Mottelson model, the representation \( \rho \) is trivial. However, nature allows any irreducible representation of the gauge group.

The Riemannian connexion on \( P \) determines the Bohr-Mottelson irrotational flow kinetic energy. This energy is proportional to the Laplace-Beltrami operator, which depends on \( P \)’s Riemannian geometry. When the Riemannian metric and the differential connexion become independent structures, the Laplace-de Rham operator creates a new collective model kinetic energy.
The noncompact 15-dimensional Lie group GCM(3), also called the General Collective Model, provides the group structure. This group contains the subgroup $P = GL_+(3, \mathbb{R})$, which is the motion group of the Bohr-Mottelson model. The 6-dimensional abelian Lie group generated by the inertia tensor is a normal subgroup of GCM(3). GCM(3) is also the dynamical group for the classical Riemann ellipsoid model of rotating stars and galaxies [9, 10].

The importance of the group structure for the nuclear collective model is twofold: First, a single irreducible representation of GCM(3) is the Bohr-Mottelson model space. The predictions from an irreducible GCM(3) representation for E2 transition rates, deformations, and energy levels are identical to results from the Bohr-Mottelson model [11, 12]. Second, the Fock space of many-fermions is a reducible GCM(3) representation. The decomposition of this reducible representation into a direct sum of irreducible representations achieves the fully microscopic realization of the collective model [13, 14, 15].

The Mackey inducing construction determines all the irreducible representations of GCM(3), which is a semi-direct product Lie group with an abelian normal subgroup. The mathematical set-up is the same as the Wigner construction of the irreducible representations of the Poincaré semi-direct product group [16, 17]. The inequivalent GCM(3) irreducible representations are indexed by a non-negative integer $C$ which labels the $(2C+1)$-dimensional irreducible representations of the subgroup $G = SO(3) \subset P \subset GCM(3)$. For such an irreducible representation $\rho$ of $G$ on the vector space $V$, the Hilbert space $\mathcal{H}_\rho$ of the corresponding irreducible GCM(3) representation $\pi_\rho$ consists of functions $\Psi : P \rightarrow V$ of type-$\rho$.

$$\Psi(\xi g) = \rho(g^{-1})\Psi(\xi). \text{ for all } \xi \in P, g \in G. \quad (3.1)$$

which are square-integrable $\int ||\Psi(\xi)||^2 d\mu(\xi) < \infty$. where $d\mu(\xi)$ denotes the Haar
measure on \( P \). The representation of \( x \in P \) is

\[
(\pi_\rho(x)\Psi)(\xi) = \Psi(x^{-1}\xi), \text{ for all } \xi \in P.
\] (3.2)

There are two relevant actions of \( SO(3) \). When the subgroup \( SO(3) \) of \( P \) acts on the left, Eq.(3.2), the group \( SO(3) \) is interpreted physically as the usual rotation group for Euclidean space. The group \( G \cong SO(3) \) also acts on the right for type-\( \rho \) wave functions, Eq.(3.1). The right action is interpreted physically as vortex rotation. For clarity, \( G \) is reserved for the right action and \( SO(3) \) designates the rotation group.

When \( \rho \) is the trivial identity representation, \( \mathcal{H}_\rho \) is the Bohr-Mottelson model space. However, there exists an invariant subspace of many-fermion Fock space that is isomorphic to each and every irreducible GCM(3) representation space \( \mathcal{H}_\rho \). Hence, every Hilbert spaces \( \mathcal{H}_\rho \) has a microscopic foundation and is a possible physical model of geometrical collective motion. The details of the decomposition follow from a reciprocity theorem of Rowe and Repka [18]

The differential-geometric structure becomes evident when each type-\( \rho \) wave function is viewed as a section of the vector bundle \( E = P \times_\rho V \). The Laplacian \( \Delta \), which is proportional to the collective kinetic energy, depends on \( P \)'s Riemannian geometry and \( E \)'s bundle connexion. In the Bohr-Mottelson model, the Riemannian connexion is adopted and the rotational kinetic energy has the irrotational flow moment of inertia. This moment of inertia is about five times smaller than the measured moment of inertia of yrast nuclear rotational bands.

In this chapter, the bundle connexion on \( P \) is not required to be Riemannian. For a general Ehresmann connexion [19, 20], the moment of inertia takes the irrotational flow value for the one special case of a Riemannian connexion. An Ehresmann connexion can be adopted for which the moment of inertia is an interpolation between the rigid and irrotational moments. Eq.(3.32). In addition, the general Laplacian in-
cludes a covariant derivative term that introduces a kind of "magnetic" force. The word "magnetic" is used here solely in analogy with the covariant derivative of abelian E&M gauge theory.

The chapter's plan is to formulate the Bohr-Mottelson irrotational flow model in differential geometric terms in §3.2-3.4. Then these methods for the Riemannian connexion are applied to a general connexion in §3.5-3.6.

Section 3.2 introduces for P a coordinate chart that separates rotational from vibrational degrees of freedom. Each $\xi \in P$ may be factorized as the product of matrices $R, S \in SO(3)$ and a diagonal positive-definite matrix $A$. $\xi = RAS$. The orientation of the nucleus relative to an inertial frame determines $R$, the semi-axis lengths of the inertia ellipsoid are $A$'s entries, and $S$ refers to vortex rotation/Kelvin circulation.

The de Rham Laplacian is derived for three different, but related, problems. Section 3.3 evaluates the Laplacian $\Delta = \ast \ast d$ for the space of complex-valued functions on the manifold $P$, where $d$ is the exterior derivative and $\ast$ denotes the Hodge star. This Laplacian coincides with the Laplace-Beltrami operator [21, 22], and it is expressed directly in terms of an orthogonal basis of physically-relevant vector fields on $P$, which includes the angular and vortex momenta and the vibrational momentum.

The right action of $G$ on $P$ commutes with $P$'s Laplace-Beltrami operator. This defines a gauge symmetry and $G$ is the gauge group. The Noether symmetry corresponding to the gauge group is conservation of Kelvin circulation. Because of this gauge symmetry, the Laplacian is well-defined on the space of wave functions of type-$\rho$. Section 3.4 computes this Laplacian explicitly; it has a crucial covariant derivative $\nabla$ term which depends on the Riemannian bundle connexion for $E$.

For a general connexion on the vector bundle $E$, Section 3.5 attains the major
goal of the chapter: an explicit formula for the bundle Laplacian \( \Delta = \star d_\nabla \star d_\nabla \), where \( d_\nabla \) denotes the exterior covariant derivative [20]. The covariant derivative in the Laplacian operator depends on the bundle connexion and is reminiscent of the covariant derivative and Laplacian for U(1) E&M and for the U(2) electroweak interaction.

The chapter’s Conclusion discusses the implications of the new differential geometry for the Bohr-Mottelson model. The covariant derivative or, colloquially, the “magnetic-like” term in the kinetic energy, enables the inclusion into the model of special collective interactions. These interactions are not limited to couplings associated with the Riemannian irrotational flow connexion of the legacy Bohr-Mottelson model. The generalized result spans the continuum from rigid rotation to irrotational flow.

3.2 Coordinate Chart for P

Many nuclei are deformed and their primary anisotropy is a quadrupole mode. The deformation often persists among a sequence of low energy quantum states, called a band, that differ in their angular momentum. Large electric quadrupole transitions between neighboring states in the band provide direct experimental evidence for a fixed deformation and the shape fluctuations are small for band members [7]. The Bohr-Mottelson model aims to describe and explain such sequences of quadrupole-deformed low energy bands.

The inertia tensor \( q_{ij} \) characterizes both the size and quadrupole deformation of a nucleus. Let \( \mathcal{Q} \) denote the cone of all positive-definite symmetric real \( 3 \times 3 \) matrices \( q \). The space of all ellipsoids centered at the origin in 3-dimensional Euclidean space is in one-to-one correspondence with \( \mathcal{Q} \), because a point \( q \in \mathcal{Q} \) determines an ellipsoid’s surface as the solutions \((x_1,x_2,x_3)\) to the quadratic equation
\[ \sum_{ij} q_{ij}^{-1} x_i x_j = 1. \] If \( q \) is a diagonal matrix, \( q = A^2 \) for \( A = \text{diag}(a_1, a_2, a_3) \), then the principal axes of the corresponding inertia ellipsoid are aligned with the Euclidean space axes, \( (x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 = 1 \). The lengths of this ellipsoid's principal semi-axes are \( a_1, a_2, a_3 \).

The group \( \mathcal{P} \) acts on the space \( \mathcal{Q} \) by \( q \mapsto \xi q \xi^t \) for any \( \xi \in \mathcal{P} \). This action is transitive and the one orbit contains the identity matrix \( I \). Hence, any \( q \in \mathcal{Q} \) may be expressed as \( q = \xi \xi^t \). The isotropy subgroup at the identity is \( \mathcal{G} = \text{SO}(3) \) because \( \xi \xi^t = I \) iff \( \xi \in \mathcal{G} \). Hence, \( \mathcal{Q} \) is a homogeneous space \( \mathcal{P}/\mathcal{G} \). This isotropy subgroup \( \mathcal{G} \) is called the vortex or circulation group.

Any matrix \( q \in \mathcal{Q} \) can be diagonalized by a special orthogonal matrix \( R \) and its real eigenvalues are positive, \( q = RA^2 R^t \). However, this decomposition is not unique because, although the eigenvalues are unique, their order in the diagonal matrix \( A \) is not. When the eigenvalues are distinct, choose \( A \) to be an element of \( \mathcal{D} \), the space of diagonal matrices with 3 distinct real, positive entries in descending order. Let \( U \) denote the open neighborhood of \( \mathcal{Q} \) consisting of those \( q \) with distinct eigenvalues. The decomposition of \( q \in U \) into the product \( RA^2 R^t \) for \( R \in \text{SO}(3) \) and \( A \in \mathcal{D} \) is still not unique because, although each eigenspace is unique, its three eigenvectors, which are the columns of \( R \) are not. Each column of \( R \) may be multiplied by \( \pm 1 \) without changing \( q \).

Let \( M \) denote the finite subgroup of \( \text{SO}(3) \) consisting of diagonal matrices with entries \( \pm 1 \). \( M \) is isomorphic to the dihedral group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). The mapping

\[
\text{SO}(3)/M \times \mathcal{D} \to U
\]

\[
(RM, A) \mapsto RA^2 R^t.
\]

is a diffeomorphism onto the open neighborhood \( U \) of \( \mathcal{Q} \).

This diffeomorphism determines a smooth manifold of coset representatives for
an open neighborhood of \( P/G \). Every element \( \xi \in P \) may be expressed as a product \( \xi = RAS \), where \( R \in \text{SO}(3) \), \( A \in \mathcal{D} \), \( \xi \xi' = RA^2R' \), and \( S = A^{-1}R' \xi \in G \). If \( R \) is replaced by any \( Rm \) for \( m \in M \), then \( S \) must be replaced by \( mS \). A function \( f \) on the Cartesian product space \( \text{SO}(3) \times \mathcal{D} \times G \) is well-defined on \( P \) iff \( f(R, A, S) = f(Rm, A, mS) \). Such \( M \)-invariant functions on \( P \) are more convenient to work with than picking a space of coset representatives for \( \text{SO}(3)/M \).

A matrix \( R \) is called an element of the rotation group, a matrix \( S \) is said to be in the circulation or vortex group. \( A \) is a member of the vibrational group. When \( q = RA^2R' \), the element \( R \in \text{SO}(3) \) rotates the ellipsoid’s principal axes (body-fixed frame) to \( q \), the ellipsoid in the laboratory or inertial frame.

### 3.2.1 Vector Fields on \( P \)

The decomposition \( \xi = RAS \) suggests three sets of three vector fields on \( P \). Let \( e_{ij} \) denote the elementary \( 3 \times 3 \) matrix with only one nonzero entry, which is equal to one at the intersection of row \( i \) and column \( j \). Set \( e_i = -(e_{jk} - e_{kj}) \) for \( i, j, k \) cyclic, which is a basis for the antisymmetric matrices, the Lie algebra of \( \text{SO}(3) \) and \( G \).

1. The curve \( t \mapsto \xi(t) = \exp(t e_i) \xi = (\exp(t e_i)R)AS \) through \( \xi \) is well-defined on \( P \).

The derivative of the curve at \( t = 0 \) determines the right invariant vector field

\[
l_i = \mathcal{R}_e = \sum_{kl} (e_i, \xi)_{kl} \frac{\partial}{\partial \xi_{kl}}. \tag{3.4}
\]

These three vector fields are the components of the angular momentum vector in the laboratory frame. In terms of the \( RAS \) coordinates, \( l_i \) is a right invariant vector field on \( \text{SO}(3) \).

2. The curve \( t \mapsto \xi(t) = \xi \exp(t e_i) = RA(S \exp(t e_i)) \) through \( \xi \) is well-defined on
P. The derivative of the curve at \( t = 0 \) defines the left invariant vector field

\[
c_i = \mathcal{L}_{\xi} = \sum_{kl} (\xi e_i)_{kl} \frac{\partial}{\partial \xi_{kl}}. \tag{3.5}
\]

These three are the components of the Kelvin circulation vector in the laboratory frame. For the RAS decomposition, \( c_i \) is a left invariant vector field on \( G \).

3. The curve \( t \mapsto \xi(t) = R \exp(t \varepsilon_{aa})AS \) through \( \xi \) is well-defined on \( P \) because \( \xi(t) = (Rm) \exp(t \varepsilon_{aa})A(mS) \) for all \( m \in M \). The derivative of the curve at \( t = 0 \) gives the vector field

\[
t_\alpha = \sum_{kl} (Re_{\alpha \alpha} R^k \xi)_{kl} \frac{\partial}{\partial \xi_{kl}}. \tag{3.6}
\]

In the RAS coordinates, this vector field is

\[
t_\alpha = a_\alpha \frac{\partial}{\partial a_\alpha} = \frac{\partial}{\partial x_\alpha}. \tag{3.7}
\]

where \( x_\alpha = \ln a_\alpha \). The domain is \( x_1 > x_2 > x_3 \), an open region of \( \mathbb{R}^3 \).

The set of nine vector fields, \( l_i, c_i, t_\alpha \), make a basis for the tangent bundle of \( P \). An alternative basis replaces \( l_i \), the right invariant vector fields on \( \text{SO}(3) \), with the left invariant vector fields \( l_\alpha = \sum l_i R_{i\alpha} \), which are the angular momenta in the body-fixed frame. Similarly, the right invariant vector fields \( c_\alpha = \sum S_{ai} c_i \) is an alternative basis for the vector fields on \( G \).
3.2.2 Riemannian Metric on P

P is a Riemannian manifold and the vector fields $\partial / \partial \xi_{ij}$ are an orthonormal basis.

$$g(\frac{\partial}{\partial \xi_{ij}}, \frac{\partial}{\partial \xi_{kl}}) = \delta_{ik} \delta_{jl}. \quad (3.8)$$

For two general vector fields $U$ and $V$ on $P$.

$$U = \sum_{ij} U(\xi)_{ij} \frac{\partial}{\partial \xi_{ij}}, \quad V = \sum_{ij} V(\xi)_{ij} \frac{\partial}{\partial \xi_{ij}}.$$  

their inner product is

$$g(U, V) = \text{Tr} (UV^t). \quad (3.9)$$

The Riemannian metric is rather simple for the vector field basis. $l_\alpha, c_\alpha, t_\alpha$:

$$g(l_\alpha, l_\beta) = g(c_\alpha, c_\beta) = I^{R}_{\alpha\beta} = \delta_{\alpha\beta} I^{R}_{\alpha}.$$  

$$g(t_\alpha, t_\beta) = A^2_{\alpha\beta} = \delta_{\alpha\beta} a^2_\alpha.$$  

$$g(l_\alpha, c_\beta) = J_{\alpha\beta} = \delta_{\alpha\beta} J_{\alpha} \quad (3.10)$$  

$$g(l_\alpha, t_\beta) = g(c_\alpha, t_\beta) = 0.$$  

where the matrices $I^{R}$, $A^2$, and $J$ are diagonal.

$$I^{R} = \text{diag}(a^2_2 + a^2_3, a^2_1 + a^2_2 + a^2_3).$$

$$J = 2 \text{diag}(a_2 a_3, a_1 a_3, a_3 a_2). \quad (3.11)$$

Note that $I^{R}$ is the rigid rotor moment of inertia in the body-fixed frame.
3.2.3 Orthogonal Basis of Vector Fields

The vector field basis \( \{l_\alpha, l_\alpha, c_\alpha\} \) is not an orthogonal basis because \( g(l_\alpha, c_\alpha) = J_\alpha \) is not zero. An orthogonal basis is formed by replacing \( l_\alpha \) by \( D_\alpha = l_\alpha - A^I_\alpha c_\alpha \) and choosing the coefficient \( A^I_\alpha(\xi) \) so that

\[
0 = g(D_\alpha, c_\beta) = g(l_\alpha - A^I_\alpha c_\alpha, c_\beta) = \delta_{\alpha\beta}(J_\alpha - A^I_\alpha T^R_\alpha). \tag{3.12}
\]

Hence the coefficients are

\[
A^I_\alpha = \frac{J_\alpha}{T^R_\alpha} = \frac{2a_{\beta a_\gamma}}{a_\beta^2 + a_\gamma^2}, \quad \text{for } \alpha, \beta, \gamma \text{ cyclic.} \tag{3.13}
\]

The squared length of the vector field \( D_\alpha \) is the irrotational flow moment of inertia

\[
T^I_\alpha = g(D_\alpha, D_\alpha) = \frac{(a_\beta^2 - a_\gamma^2)^2}{a_\beta^2 + a_\gamma^2}. \tag{3.14}
\]

The vector fields \( l_\alpha, c_\alpha, \) and \( D_\alpha \) are not well-defined on the bundle \( P \) because they are not invariant with respect to \( M \). When \( R \in SO(3) \) is replaced by \( Rm \) and \( S \in G \) is replaced by \( mS \) for \( m \in M \), the vector fields change, correspondingly, to \( l_\alpha \mapsto m_\alpha l_\alpha, c_\alpha \mapsto m_\alpha c_\alpha, \) and \( D_\alpha \mapsto m_\alpha D_\alpha \). Nonetheless, the orthogonal set of vector fields is preferred whenever possible. Care must be taken to only work with combinations that are \( M \)-invariant. E.g., the second order differential operator \( l^2_\alpha \) is well-defined on \( P \) because \( m^2_\alpha = 1 \). In the next section, the Laplacian, a second order differential operator on \( P \), is derived explicitly in the orthogonal basis. Because the Laplacian is, by definition, well-defined on \( P \), it consequently must be \( M \)-invariant in the orthogonal basis.
3.3 Laplacian on \( P \)

When acting on a smooth function \( \Psi \) on \( P \), the Laplace-de Rham operator \( \Delta \) satisfies \( d \star d\Psi = \Delta (\Psi) \omega \) where \( d \) denotes the exterior derivative, \( \star \) is the Hodge star operator, and \( \omega \) is the volume form. On the Riemannian space \( P \), the Laplace-Beltrami operator, which is constructed in [22], coincides with the Laplace-de Rham operator. Nonetheless, this chapter works out the de Rham \( \Delta \) because geometrical structures are revealed in the derivation.

The calculation of the Laplace-de Rham operator on \( P \) requires the explicit evaluation of the Hodge star and exterior derivative with respect to our chosen orthogonal basis.

3.3.1 Volume Form \( \omega \)

Given an orthogonal basis of vector fields \( \{ f_s \} \) with squared lengths \( g(f_s, f_s) = B_s \) (\( s = 1, 2, \ldots, n = \dim P = 9 \)) and its corresponding dual basis of 1-forms \( \{ f^s \} \) with squared lengths \( g(f^s, f^s) = 1/B_s \), the usual Riemannian volume element is

\[
\text{vol} = \sqrt{\prod B_s} f^1 \wedge f^2 \wedge \ldots \wedge \ldots. \tag{3.15}
\]

For the vector field basis \( \{ D_\alpha, t_\alpha, c_\alpha : \alpha = 1, 2, 3 \} \) for \( P \), the product is

\[
\sqrt{\prod B_s} = a_1 a_2 a_3 (a_2^2 - a_3^2)(a_3^2 - a_1^2)(a_1^2 - a_2^2). \tag{3.16}
\]

The volume form \( \text{vol} \) does not scale the same as the volume element for the Euclidean configuration space \( \mathbb{R}^{3N} \) of \( N \) particles. When every individual particle’s position vector is scaled by \( \lambda > 0 \), the Euclidean volume scales as \( \lambda^{3N} \). Or, after removing center-of-mass motion, it scales as \( \lambda^{3N-3} \). The equivalent scaling transformation for \( P \) is \( \xi \mapsto \lambda \xi \). Or, in our local trivialization, \( A \mapsto \lambda A \). However, \( \text{vol} \) is
the $GL_4(3, \mathbb{R})$ Haar measure and is invariant under scaling. The following volume element corrects this discrepancy:

$$\omega = (a_1 a_2 a_3)^{N-1} \text{vol} = h f^1 \wedge f^2 \wedge \ldots \wedge f^9, \quad (3.17)$$

where the positive measure function is

$$h(a_1, a_2, a_3) = (a_1 a_2 a_3)^N (a_2^2 - a_3^2)(a_3^2 - a_1^2)(a_1^2 - a_2^2). \quad (3.18)$$

### 3.3.2 Hodge Star

The Hodge star is a vector space isomorphism between $p$ forms and $n-p$ forms. This isomorphism depends on the volume $n$-form $\omega$ and the Riemannian geometry on the space of forms. For a one-form $\theta$, $*\theta$ is the $n-1$ form satisfying $\theta \wedge *\theta = g(\theta, \theta) \omega$.

For the chosen orthogonal basis of vector fields on $P$,

$$*f^s = (-1)^{s-1} B_s^{-1} h f^1 \wedge f^2 \wedge \ldots \wedge f^{s-1} \wedge f^{s+1} \wedge \ldots f^n$$

$$= B_s^{-1} h \hat{f}^s. \quad (3.19)$$

### 3.3.3 Exterior Derivative

If $\Psi$ is a smooth function on $P$, its differential is

$$d\Psi = f_s(\Psi) f^s \quad (\text{Einstein summation convention}). \quad (3.20)$$

Applying the Hodge star to this 1-form produces the $(n-1)$-form

$$*d\Psi = f_s(\Psi) * f^s = B_s^{-1} h f_s(\Psi) \hat{f}^s. \quad (3.21)$$
The exterior derivative of this form is an \( n \)-form

\[
d \ast d \Psi = d \left( B_s^{-1} h f_s(\Psi) \right) \wedge \hat{f}^s + B_s^{-1} h f_s(\Psi) \, d \hat{f}^s. \tag{3.22}
\]

The second term in this expression is zero because of the identity:

\[
d \hat{f}^s = 0 \text{ for all } s = 1, 2, \ldots, n. \tag{3.23}
\]

To prove this, note that the exterior derivative of \( \hat{f}^s \) is a sum of terms like

\[
f^1 \wedge \ldots \wedge df^{s'} \wedge \ldots \wedge f^{s-1} \wedge f^{s+1} \wedge \ldots \wedge f^n. \tag{3.24}
\]

For this \( n \)-form to be non-zero, the 2-form \( df^{s'} \) must include a non-zero \( f^{s'} \wedge f^s \) contribution. We will show that no term of this kind appears in \( df^{s'} \) and, thereby, prove Eq.(3.23).

The exterior derivative of the one-form \( f^{s'} \) is the two-form

\[
d f^{s'} = -\frac{1}{2} \sum_{s''} f^{s'} ([f_{s''}, f_s]) \, f^{s''} \wedge f^s. \tag{3.25}
\]

The exterior derivatives of the 1-forms \( \{D^\gamma, t^\gamma, c^\gamma; \gamma = 1, 2, 3\} \) are

\[
d D^\gamma = -\frac{1}{2} \epsilon_{\alpha \beta \gamma} D^\alpha \wedge D^\beta
\]

\[
d t^\gamma = 0 \tag{3.26}
\]

\[
d c^\gamma = -\frac{1}{2} \epsilon_{\alpha \beta \gamma} \left[(A_{\alpha}^1 A_{\beta}^1 + A_{\beta}^1) D^\alpha \wedge D^\beta - A_{\beta}^1 c^\alpha \wedge D^3 - c^\alpha \wedge c^\beta \right]
\]

\[+\frac{1}{2} t_\alpha (A_{\alpha}^1) t^\alpha \wedge D^\gamma.
\]

In no case does the exterior derivative \( df^{s'} \) contain a term \( f^{s'} \wedge f^s \) for any \( f^{s'} \) in the basis set \( \{D^\alpha, t^\alpha, c^\alpha\} \). Thus the identity (3.23) is established.
### 3.3.4 \triangle_1 on P

The differential of \( \ast d\Psi \) is an \( n \) form proportional to \( \omega \).

\[
d \ast d\Psi = d \left( B_s^{-1} h f_s(\Psi) \right) \wedge \hat{f}^s
\]
\[
= f_s \left( B_s^{-1} h f_s(\Psi) \right) f^s \wedge \hat{f}^s
\]
\[
= h^{-1} f_s \left( B_s^{-1} h f_s(\Psi) \right) \omega
\]
\[
= \left( f_s \left( B_s^{-1} f_s(\Psi) \right) + B_s^{-1} f_s (\ln h) f_s(\Psi) \right) \omega
\]

(3.27)

Therefore, the Laplacian operator is

\[
\Delta_1 = \sum_{\alpha=1}^{3} \left\{ \frac{1}{T^I_\alpha} D^2_\alpha + \frac{1}{T^R_\alpha} \epsilon^2_\alpha + t_\alpha \frac{1}{a^2_\alpha} t_\alpha + \frac{1}{a^2_\alpha} t_\alpha (\ln(h)) t_\alpha \right\}
\]

(3.28)

This expression agrees with the Laplace-Beltrami operator reported in [22]. Note that \( \Delta \) is M-invariant and, consequently, well-defined on \( P \). It is a hermitian operator on the space \( \mathcal{L}^{(2)}(P, \omega) \).

### 3.4 Laplacian for Functions of Type-\( \rho \)

The Laplacian (3.28) commutes with the angular momentum components \( l_i \) in the inertial lab frame for two reasons. First the right invariant vector field \( l_i \) commutes with the left invariant vector field \( l_\alpha \). Second, angular momentum vector fields commute with the vibrational \( t_\alpha \) and circulation \( c_\alpha \) vector fields because both \( l_i \) and \( l_\alpha \) act on \( R \) coordinates whereas \( t_\alpha \) acts on \( A \) and \( c_\alpha \) act on \( S \) in the coordinate chart \( RAS \). The property \( [l_i, \Delta] = 0 \) implies conservation of angular momentum.

In addition, the Laplacian commutes with the three circulation components \( c_i \) because these left-invariant vector fields commute with the right invariant \( c_\alpha \), \( D_\alpha \) and also \( t_\alpha \). This commutative property \( [c_i, \Delta] = 0 \) determines a gauge symmetry and
the gauge group is $G$.  

The group $P = \text{GL}_+(3,\mathbb{R})$ is a principal bundle with structure group $G = \text{SO}(3)$ and base manifold $P/G$. The base manifold is identified with the cone $\mathcal{Q}$ of positive-definite $3 \times 3$ real symmetric matrices. This identification arises because the $P$-action on $\mathcal{Q}$, $q \in \mathcal{Q} \mapsto \xi_q \xi^r$ for $\xi \in P$, has one orbit containing the identity matrix at which the isotropy subgroup is $G$. The base manifold may be identified with the space of ellipsoids in $\mathbb{R}^3$. The bundle projection is 

$$
\pi : P \to P/G \\
\xi \mapsto \xi \xi^r. \tag{3.29}
$$

If $\rho$ is an irreducible unitary representation of the structure group $G$ on the complex vector space $V$, then it determines an associated $G$-bundle $E = P \times_\rho V$. The sections of this vector bundle are smooth vector-valued functions $\Psi : P \to V$ that transform equivariantly with respect to the gauge group $G$. 

$$
\Psi(\xi g) = \rho(g^{-1})\Psi(\xi). \text{ for all } \xi \in P, g \in G. \tag{3.30}
$$

Such functions are said to be of type-$\rho$. The Hilbert space of all square-integrable functions of type-$\rho$ carries an irreducible representation of the general collective motion group $GCM(3)$ [11].

The inequivalent irreducible representations of $G$ are indexed by a non-negative integer $C$. The dimension $V$ of the representation space corresponding to $C$ is an odd number, $2C + 1$. The physical interpretation of $C$ is that the squared length of the Kelvin circulation, $\sum c_i^2 = \sum c_i^2$, equals a constant $C(C + 1)$.

In terms of the coordinate chart, a type-$\rho$ function $\Psi$ determines a vector-
valued function \( \Psi \) on the Cartesian product space \( \text{SO}(3) \times \mathcal{D} \times G \) that satisfies

\[
\Psi(R, A, S) = \Psi(RAS) = \rho(S^{-1})\Psi(RA).
\]

(3.31)

For a type-\( \rho \) function, the value of it on \( \xi = RA \) is sufficient to extend it to all of \( P \). Let \( \psi \) denote the vector-valued function from \( \text{SO}(3) \times \mathcal{D} \) to \( V \) given by

\[
\psi(R, A) = \Psi(RA).
\]

(3.32)

For any \( m \in M \), the function \( \psi \) satisfies

\[
\psi(Rm, A) = \Psi(RmA) = \Psi(RAm) = \rho(m^{-1})\Psi(RA) = \rho(m)\psi(R, A)
\]

(3.33)

because \( m^{-1} = m \).

Conversely, a smooth vector-valued function \( \psi \) that satisfies Eq.(3.33) determines a function \( \tilde{\Psi}(R, A, S) = \rho(S^{-1})\psi(R, A) \) and \( \tilde{\Psi} \) determines a unique type-\( \rho \) function \( \Psi \) because \( \tilde{\Psi}(Rm, A, mS) = \tilde{\Psi}(R, A, S) \). Let \( L \) denote the vector space isomorphism from the space of smooth functions \( \psi \) satisfying Eq.(3.33) to the space of smooth sections \( \Psi \) of \( E \).

The right-invariant vector field \( c_\alpha \) on \( P \) extends to a right-invariant vector field on vector-valued functions on \( P \). The derived matrix representation of the Lie algebra \( \mathfrak{g} \) of \( G \) on \( V \) is

\[
\hat{\rho}(c_\alpha) = \frac{d}{dt}\rho(\exp(tc_\alpha))|_{t=0}.
\]

(3.34)

defines a natural representation of this Lie algebra on the space of vector-valued functions \( \psi \).

The isomorphism \( L \) intertwines the action of the vector field \( c_\alpha \) with the matrix
action $\hat{\rho}(c_\alpha)$ because, when $\Psi = L \psi$,

$$
(c_\alpha \Psi)(\xi) = \frac{d}{dt} \tilde{\psi}(R.A.e^{-t c_\alpha}S)|_{t=0} \\
= \frac{d}{dt} \rho(S^{-1}e^{t c_\alpha})\psi(R.A)|_{t=0} \\
= \rho(S^{-1})(\dot{\rho}(c_\alpha)\psi)(R.A). \\
= (L(\dot{\rho}(c_\alpha)\psi))(\xi)
$$

(3.35)

Hence, when acting on functions of type-$\rho$, the action of the vector field $c_\alpha$ is the same as the simple matrix action $\rho(c_\alpha)$. The isomorphism $L$ intertwines the two actions. $L \circ \dot{\rho}(c_\alpha) = c_\alpha \circ L$.

The differential operators for the angular momentum, either $l_\iota$ or $l_\alpha$, and the vibrational momentum $t_\alpha$ act solely on the base manifold coordinates $R$ and $A$. Thus, these operators are well-defined on the space of type-$\rho$ smooth functions and commute with the isomorphism $L$.

The vector field $D_\alpha$ on $P$ does not commute with $L$. Define the covariant derivative $\nabla_\alpha = l_\alpha - A^1_\alpha \dot{\rho}(c_\alpha)$ which is a differential operator on type-$\rho$ functions. The isomorphism intertwines the vector field $D_\alpha$ with the covariant derivative. $L \circ \nabla_\alpha = D_\alpha \circ L$.

In the language of differential geometry, $D_\alpha$ is the horizontal lift to $P$ of the vector field $l_\alpha$ on $P/G$. The horizontal subspace of the tangent space to $P$ at $\xi$ is orthogonal to the vertical subspace spanned by the vertical vectors $c_\alpha$. The $A^1_\alpha$ are the Riemannian connexion coefficients.

The Laplacian $\Delta_2$ on sections of $E$ is defined by replacing the vector field $D_\alpha$ with the covariant derivative $\nabla_\alpha$ and the circulation $c_\alpha$ with the corresponding $\dot{\rho}(c_\alpha)$.

$$
\Delta_2 = \sum_{\alpha=1}^3 \left\{ \frac{1}{T^2_\alpha} \nabla_\alpha^2 + \frac{1}{T^R_\alpha} \dot{\rho}(c_\alpha)^2 + t_\alpha \frac{1}{a^2_\alpha} t_\alpha + \frac{1}{a^2_\alpha} t_\alpha (\ln(h)) t_\alpha \right\}.
$$

(3.36)
The Laplacians of this and the prior section intertwine. $L \circ \Delta_2 = \Delta_1 \circ L$.

### 3.5 Laplacian on the Associated G-Bundle

On the space $P$, the Laplacian $\Delta_1$ is a hermitian operator on square-integrable complex-valued functions on $P$. This Laplacian extends naturally to act on vector-valued functions on $P$, and it simplifies to $\Delta_2$ when acting on vector-valued functions of type-$\rho$. Such functions may be viewed as sections of the associated bundle $E = P \times_\rho V$, where $V$ is the representation space of $\rho$.

The exterior covariant derivative $d_\nabla$ determines a third Laplacian $\Delta_3$ that acts on vector-valued functions $\psi$ on just the base manifold $Q \simeq P/G$:

$$d_\nabla \star d_\nabla \psi = \Delta_3(\psi)\omega. \quad (3.37)$$

In this equation defining $\Delta_3$, the volume element $\omega$ and Hodge star $\star$ refer to the base manifold and its Riemannian geometry. A connexion one-form $A$ on the base manifold determines the exterior covariant derivative. Hence the calculation of $\Delta_3$ requires choices for several geometrical ingredients. In this section and the next, the Ehresmann connexion is not restricted to Riemannian, and $\Delta_3$ is not generally the Laplace-Beltrami operator.

#### 3.5.1 Riemannian Geometry on $Q$

A basis of six vector fields for a chart $U \subset Q$ is $\{f_\alpha; s = 1, 2, \ldots, 6\}$, where $f_\alpha = l_\alpha$, the angular momentum projected on the body-fixed frame, and $f_{3+\alpha} = t_\alpha$, the vibrational momentum, $(\alpha = 1, 2, 3)$.

The Riemannian geometry on $Q$ is inherited from the Riemannian geometry on $P$ in a natural way. Using the connexion on $P$, two tangent vectors on $Q$ may be lifted to horizontal vectors on $P$. The Riemannian geometry on $P$ determines the
inner product of the two lifted vectors. This inner product on $\mathcal{P}$ is set equal to the new inner product on $\mathcal{Q}$. In the case of the Riemannian connexion, the horizontal lift of $l_\alpha$ is $D_\alpha$, and $t_\alpha$ is already horizontal. Hence the Riemannian metric for the orthogonal basis of vector fields $\{f_s\}$ on $\mathcal{Q}$ is

$$g(f_s, f_s) = B_s. \tag{3.38}$$

where, for $\alpha = 1, 2, 3$. $B_\alpha = g(D_\alpha, D_\alpha) = I_\alpha^1$ are the irrotational flow moments of inertia and $B_{3+\alpha} = g(t_\alpha, t_\alpha) = a_\alpha^2$. The corresponding dual basis of one-forms $\{f^s\}$ is also orthogonal.

$$g(f^s, f^s) = B_s^{-1}. \tag{3.39}$$

### 3.5.2 Volume Element and Hodge Star on $\mathcal{P}/\mathcal{G}$

The Haar measure vol on $\mathcal{P}$ determines the Haar measure on the coset space $\mathcal{P}/\mathcal{G}$. The scaling property is resolved in a way similar to that for $\mathcal{P}$ because only the space $\mathcal{D}$ is involved. Thus the volume element on $\mathcal{Q}$ is

$$\omega = hf^1 \wedge f^2 \wedge \ldots \wedge f^6. \tag{3.40}$$

The Riemannian geometry and volume element on $\mathcal{Q}$ determines the Hodge star on this space.

$$\ast f^s = \frac{h}{B_s} \tilde{f}^s. \tag{3.41}$$

where $\tilde{f}^s = (-1)^{s-1}f^1 \wedge \ldots f^{s-1} \wedge f^{s+1} \wedge \ldots \wedge f^6$. Note that $f^{q'} \wedge \tilde{f}^s = \delta_{q'q}h^{-1}\omega.$
3.5.3 Connexion One-Form on $Q$

A connexion one-form $A$ defined on the base manifold is a Lie algebra-valued form,

$$A = - \sum_{s\beta} A_{s}^{\beta}(q) \dot{\rho}(\epsilon_{\beta}) f^{s}. \quad (3.42)$$

where $s$ ranges over the basis of vector fields on the base manifold. $\beta$ runs over a basis $\epsilon_{\beta}$ of the Lie algebra of the structure group, and the connexion coefficients $A_{s}^{\beta}(q)$ are smooth functions on the base manifold $q \in P/G$. The connexion one-form consists of $(\dim V)^2$ one-forms,

$$A_{u}^{u'} = - \sum_{s\beta} A_{s}^{\beta}(q) \dot{\rho}(\epsilon_{\beta})_{u}^{u'} f^{s}. \quad (3.43)$$

where $u, u'$ range over a basis for the representation space $V$ of $\dot{\rho}$.

For the Riemannian or irrotational flow connexion, the coefficients $A_{s}^{\beta}(m)$ are zero when either $q = 4, 5, 6$ or $\beta \neq q$.

$$A^{l} = - \sum_{\alpha} A_{\alpha}^{l}(\epsilon_{\alpha}) l^{\alpha}. \quad (3.44)$$

and the irrotational flow connexion coefficients are functions of $a_{1}, a_{2}, a_{3}$.

The covariant derivative in the direction of $f_{s}$ of a section $\psi$ of $E$ is another section of $E$.

$$\nabla_{s} \psi = f_{s} \psi + A(f_{s}) \psi. \quad (3.45)$$

In the first term of this expression, the vector field $f_{s}$ is applied to each of the components of the vector-valued function $\psi$, and, in the second term, the matrix $A(f_{s})$ is applied to the column vector $\psi$.

The exterior covariant derivative $d_{\nabla}$ maps vector-valued $p$-forms to vector-valued $(p + 1)$-forms. When $\psi$ is a 0-form, the covariant derivative determines its
exterior covariant derivative

\[ d\nabla \psi = (\nabla_s \psi) f^\alpha. \tag{3.46} \]

For the Riemannian connexion.

\[ d\nabla \psi = \sum_{\alpha=1}^3 \left\{ \left( (l_\alpha - A^1_\alpha \psi) + t_\alpha \right) l^\alpha + t_\alpha (\psi) t^\alpha \right\}. \tag{3.47} \]

Any E-valued p-form is a sum of monomials \( \eta \wedge \mu \), where \( \eta \) is a bundle section or E-valued 0-form and \( \mu \) is an ordinary p-form on the base manifold. The exterior covariant derivative of a monomial is

\[ d\nabla (\eta \wedge \mu) = d\nabla \eta \wedge \mu + \eta \wedge d\mu. \tag{3.48} \]

### 3.5.4 Laplacian \( \Delta_3 \)

If \( \psi \) is a bundle section, then Eq. (3.46) gives its exterior covariant derivative.

The Hodge star of the resulting 1-form is a 5-form.

\[ *d\nabla \psi = (\nabla_s \psi) \frac{h}{B_s} \hat{f}^s. \tag{3.49} \]

Because \( d\hat{f}^s = 0 \), the exterior covariant derivative of this 5-form is the 6-form.

\[ d\nabla * d\nabla \psi = \nabla_s' \left( (\nabla_s \psi) \frac{h}{B_s} \right) f^{s'} \wedge \hat{f}^s \]

\[ = h^{-1} \nabla_s \left( (\nabla_s \psi) \frac{h}{B_s} \right) \omega. \tag{3.50} \]

Hence the de Rham Laplacian is

\[ \Delta_3 = \sum_{\alpha=1}^3 \left\{ \frac{\nabla^2 \alpha}{T^\alpha} + t_\alpha \frac{1}{a_\alpha^2} t_\alpha + t_\alpha (\ln h) t_\alpha \right\}. \tag{3.51} \]
The Laplacians $\Delta_2$ and $\Delta_3$ differ by one term.

$$\Delta_2 - \Delta_3 = \frac{1}{I^a_a} \rho(c_a)^2.$$ \quad (3.52)

which acts exclusively on the vector space of the associated bundle.

Although the Riemannian connexion determines $\Delta_3$, the derivation applies to any Ehresmann connexion. A summary of the construction of the de Rham Laplacian for the associated bundle $E = P \times_p G$ is as follows:


2. Define the Riemannian metric on the base manifold by horizontally lifting vector fields on the base manifold to the Riemannian manifold $P$.

3. Choose the volume element $\omega$ on the coset space $P/G$ and find the corresponding Hodge star on the base manifold.

4. Determine the exterior covariant derivative $d_{\nabla}$.

5. Evaluate the de Rham Laplacian on bundle sections $\psi$. $d_{\nabla} \star d_{\nabla} \psi = (\Delta \psi) \omega$.

### 3.6 Conclusion

For the Riemannian connexion, §3.3 - 3.5 show the relationships among the Laplacians evaluated by different, but closely related, differential geometric methods. These derivations introduce the relevant physical quantities, viz., the angular momentum, vibrational momentum, and Kelvin circulation, at the outset and conclude with expressions for the Laplacian in terms of these observables. Because explicit coordinates like Euler angles are unnecessary, the derivations illuminate the roles of the metric and connexion.
The most significant conclusion is that the method of §3.5 for the irrotational flow collective kinetic energy generalizes to any bundle connexion. For example, for rigid rotation, the connexion 1-form \( A \equiv 0 \). The base manifold vector fields \( l_\alpha, t_\alpha \) are then horizontal. Hence the metric on the base manifold differs from Eqs. (3.38, 3.39) by \( B_\alpha = T_\alpha^R \) instead of the irrotational flow inertia. §3.5.2 gives the volume element and Hodge star. The result is the rigid rotor Laplacian.

\[
\Delta_R = \sum_{\alpha=1}^{3} \left\{ \frac{l_\alpha^2}{T_\alpha^R} + t_\alpha \frac{1}{q_\alpha^2} t_\alpha + t_\alpha (\ln h) t_\alpha \right\}. \tag{3.53}
\]

More generally, suppose \( r \) is a parameter, \( 0 \leq r \leq 1 \), and the connexion one-form is \( A^r = (1 - r) A^1 \). For \( r = 0 \), the Laplacian corresponds to irrotational flow and, for \( r = 1 \), rigid rotation. The horizontal lift of \( l_\alpha \) is \( D_\alpha = l_\alpha - (1 - r) A^1_\alpha c_\alpha \), and the metric in this case is an interpolation between the rigid and irrotational inertias.

\[
T_\alpha^r = B_\alpha = g(D_\alpha, D_\alpha) = r^2 T_\alpha^R + (1 - r^2) T_\alpha^1. \tag{3.54}
\]

The Laplacian for this connexion differs from the irrotational flow and rigid rotor by the interpolated inertia and by the covariant derivative \( \nabla_\alpha = l_\alpha - A^r_\alpha \dot{\rho}(t_\alpha) \).

\[
\Delta_r = \sum_{\alpha=1}^{3} \left\{ \frac{\nabla_\alpha^2}{T_\alpha^r} + t_\alpha \frac{1}{q_\alpha^2} t_\alpha + t_\alpha (\ln h) t_\alpha \right\}. \tag{3.55}
\]

For deformed nuclei, the experimental moment of inertia is about five times the irrotational flow inertia and half the rigid rotor inertia. Thus the interpolating parameter is about \( r = 2/3 \).

Gauge transformations provide a potential simplification of the connexion coefficients. For a principal bundle, a gauge transformation arises from a different choice for the coset representatives of \( P/G \). This chapter works with a set of coset
representatives that is compatible with the angular momentum group $\text{SO}(3)$. This compatibility is achieved by making a double coset decomposition $\text{SO}(3) \backslash P / G$ and choosing the double coset representatives to be the diagonal matrices. The latter have a physical interpretation as the inertia ellipsoid’s axes lengths which determine an isotope’s deformation. This choice of gauge is optimal for nuclear physics applications.

The connexion 1-form $A$ is not the most general. This chapter simplifies the general connexion, Eq. (3.42), to Eq. (3.44). This simplification is necessary and sufficient for the Riemannian connexion. The connexion’s physical interpretation is that it determines the coupling between base manifold degrees of freedom (orientation and deformation) and structure group degrees of freedom (vorticity). This article’s ansatz is that the body-fixed angular momentum components couple to circulation components about the same principal axis of the rotating nucleus.

The covariant derivative $\nabla$ is a generalization of the electromagnetic covariant derivative for which the connexion is the vector potential. Thus the Laplacian of this paper introduces a so-called “magnetic” interaction into the Bohr-Mottelson collective model. This interaction is velocity-dependent, i.e., it depends on the angular momentum. Because the connexion coefficients depend on the axes lengths, the interaction also depends on the deformation.

The Hamiltonian for this chapter’s generalization of the collective model is a hermitian operator on the space of square-integrable sections of the bundle $E$. In addition to the kinetic energy, the Hamiltonian must include a deformation-dependent potential $v$ and a Hamiltonian $H_\rho$ that acts in the representation space.

$$
H = -\frac{\hbar^2}{2m} \Delta_r + v(a_1, a_2, a_3) + H_\rho.
$$

(3.56)

The potential may be expressed in terms of the Bohr-Mottelson $\beta, \gamma$ deformation
coordinates. The difference $\Delta_2 - \Delta_3$ in Eq. (3.52) shows the necessity of $H_\rho$.

The next chapter determines an equation for the bundle connexion $A$, viz., the Yang-Mills equation.
Chapter 4

Yang-Mills Equation for the Nuclear Geometrical Collective Model Connexion

N Sparks and G Rosensteel

4.1 Introduction

The legacy Bohr-Mottelson collective model [7, 1, 8] describes the collective rotations and vibrations of deformed atomic nuclei. Although it achieves a quantitative explanation of nuclear deformations, rotational bands and shape changes, the model predicts the liquid irrotational flow moment of inertia which is a factor of five too small compared to measurement. It's a disconcerting outcome because the irrotational flow moment is derived in the legacy model from the microscopic kinetic energy.

A standard approach to address this problem is to make the moment of inertia an additional model parameter. This repair attains agreement between theory and experiment, but at the cost of abandoning an approach that aims at answering a
fundamental physics question in nuclear dynamics: How does a deformed nucleus rotate? The nucleus is neither a rigid body nor an irrotational liquid.

To address the issue, the legacy model is generalized to a differential-geometric theory based on the principal bundle \( P = \text{GL}_+(3, \mathbb{R}) \). This bundle's Ehresmann connexion is related directly to the moment of inertia. From an investigation of the connexion, the physics of nuclear rotation may be understood quantitatively.

The structure or gauge group of the bundle is the subgroup \( G = \text{SO}(3) \), and the base manifold \( P/G \) is diffeomorphic to the cone of positive-definite symmetric \( 3 \times 3 \) matrices, denoted by \( \mathcal{Q} \). The physical interpretation of the gauge group is as the kinematical vorticity group in the rotating body-fixed frame, and the base manifold is viewed as the space of all ellipsoidal shapes and orientations. The classical image of a rotating and vibrating ellipsoid with internal vortex motion is the Riemann ellipsoid model of stars and galaxies [9, 10]. Its natural quantization is the quantum collective model [11, 12, 14].

The quantum wave functions of the differential-geometric collective model are sections of an associated bundle. In terms of the Hodge star and the exterior covariant derivative \( d_{\mathcal{V}} \), the de Rham Laplacian, \( \Delta = \star d_{\mathcal{V}} \star d_{\mathcal{V}} \), is a second order differential operator acting on the associated bundle's sections. The generalized collective model kinetic energy is proportional to the de Rham Laplacian, which depends on the connexion.

This chapter builds on the theoretical framework of the previous chapter about the kinetic energy operator [23]. The legacy Bohr-Mottelson model adopts the Riemannian bundle connexion: the resulting kinetic energy yields the irrotational flow moment of inertia. Experiment evidently demands a non-Riemannian bundle connexion.

The main goal of the current chapter is to derive the Yang-Mills equation for
the collective model Ehresmann connexion.

Electroweak theory and the standard model use the mathematics of connexions, covariant derivatives, and the Yang-Mills equation. In these relativistic physical applications, the base manifold is Minkowski space and the fibres are irreducible representations of an elementary particle’s internal symmetry group. The connexion is a solution to the Yang-Mills equation that relates the bundle’s curvature to the field sources. These particle physics applications may be regarded as paradigms for the modern formulation of quantum mechanics and field theory [20] in terms of differential geometry [19].

This chapter casts the non-relativistic collective model in the modern framework. In this case, a point $q$ in the base manifold determines the deformation and orientation of a rotating and vibrating atomic nucleus. A point in the base manifold may be visualized by identifying the positive-definite matrix $q$ with the inertia ellipsoid whose boundary consists of points $(u^1, u^2, u^3)$ in Euclidean space $\mathbb{R}^3$ that satisfy the equation $\sum_{ij} q^{-1}_{ij} u^i u^j = 1$.

The action of $GL_+(3, \mathbb{R})$ on the space of inertia tensors is inherited from Euclidean space, $q \mapsto \xi \cdot q = \xi q \xi^T$ for $\xi$ in the general linear group. In the space of positive-definite inertia tensors, the general linear group has but one orbit, and it contains the $3 \times 3$ identity matrix. The isotropy subgroup at the identity matrix is $G = SO(3)$ and $Q$ is diffeomorphic to the right coset space $P/G$. The explicit diffeomorphism is $\xi G \mapsto q = \xi \xi^T$. The bundle projection is $\pi: P \to Q$. $\pi(\xi) = \xi \xi^T$: the bundle projection is right invariant. $\pi(\xi g) = \pi(\xi)$ for every $g$ in $G$. The subgroup $G$ is the structure group of the principal bundle $P$.

The projection of the left action of $SO(3)$ on the bundle. $\pi(R \xi) = Rq R^T$ for $R \in SO(3)$, corresponds to a rotation of the inertia ellipsoid. Each orbit of $SO(3)$ in the space of inertia tensors contains a positive diagonal matrix, $A^2$, where
\[ A = \text{diag}(a_1, a_2, a_3) \text{ with each } a_i > 0. \] The ellipsoid corresponding to the diagonal orbit representative is oriented in its principal axis frame, and its half-axes lengths are \( a_1, a_2, a_3 \).

The double coset decomposition \( \text{SO}(3) \backslash \text{P}/\text{G} \) achieves a local coordinate chart for \( \text{P} \) tied to the physical interpretation of the bundle \( \text{P} \). Every \( \xi \in \text{P} \) can be written as the product \( \xi = RaS \) with \( R \in \text{SO}(3), S \in \text{G}, \) and \( a = \text{diag}(a_1, a_2, a_3) \). The projection \( q = \pi(\xi) = Ra^2R^T \) determines the inertia ellipsoid and the model’s interpretation of \( R \) and \( a \) as nuclear orientation and deformation, respectively. The gauge group element \( S = a^{-1}R^T \xi \) in the structure group is independent of the ellipsoid \( q \); \( S \) lies in the fibre \( \pi^{-1}(q) \) and it determines collective vortex motion of particles internal to the ellipsoid’s boundary.

The double coset decomposition provides almost unique coordinates for \( \text{P} \). In the generic case, the axes lengths are distinct and unique, except for their order. When an order is chosen, the discrete dihedral group \( M \), which consists of the four diagonal \( \text{SO}(3) \) matrices, characterizes the only remaining ambiguity. \( \xi = RaS = (Rm)a(mS) \) for all \( m \in M \). c.f. [23] for more details. The principal bundle \( \pi : \text{P} \to Q \) has coordinates \((R,a)\) for the base manifold and \( S \) for the fibres. This choice of coset representatives for \( \text{P}/\text{G} \) exhibits clearly the physical interpretation of the generalized collective model.

The associated bundle is \( E = \text{P} \times_{\rho} V \), where \( \rho \) is an irreducible representation of the structure group \( G \) on the vector space \( V \). The Hilbert space \( \mathcal{H}_{\rho} \) of sections consists of vector-valued functions \( \Psi : \text{P} \to V \) of type-\( \rho \).

\[
\Psi(\xi g) = \rho(g^{-1})\Psi(\xi), \text{ for all } \xi \in \text{P}, g \in G. \tag{4.1}
\]

which are square-integrable \( \int \|\Psi(\xi)\|^2d\mu(\xi) < \infty \), where \( d\mu(\xi) \) denotes the Haar
measure on \( P \). The unitary representation of \( x \in P \) on \( \mathcal{H}_\rho \) is

\[
(\pi_\rho(x)\Psi)(\xi) = \Psi(x^{-1}\xi), \quad \text{for all } \xi \in \mathcal{P}.
\]

(4.2)

The vector space \( V \) is odd-dimensional, \( \dim V = 2C + 1 \), where \( C \) is an integer. The physical interpretation is that \( C\hbar \) is the value of the quantized Kelvin circulation. The legacy collective model is the trivial bundle. \( C = 0 \), corresponding to irrotational flow in which a Bohr-Mottelson wave function is complex-valued on the base manifold \( \mathcal{Q} \).

The first step in the derivation of the Yang-Mills equation is to evaluate the bundle curvature for a general connexion with respect to a basis of forms adapted to the physics of collective motion. §4.2 evaluates the field Lagrangian density which is a sum of source-free and interaction Lagrangians: the action’s critical points define the Yang-Mills equation for the connexion \( A \). The interaction Lagrangian depends on a source current \( J \), which is a Lie algebra-valued one-form. §4.3 determines the source \( J \) for irrotational flow. By adopting a current that is an interpolation between rigid and irrotational values, the solution to the Yang-Mills equation is \( A = gA \), where \( A \) denotes the irrotational flow connexion and the real number \( g = 1 - r \) indicates the strength of the interaction between the angular and vortex rotational modes, \( 0 \leq g \leq 1 \).

### 4.2 Bundle Curvature

Consider the six-dimensional base manifold \( \mathcal{Q} \) with coordinates \((R,a)\), \( R \in \text{SO}(3) \) and \( a_i \) three positive real numbers: notation is adopted from [23]. The angular and vibrational momenta are an apt orthogonal basis of vector fields, and its dual elements are an orthogonal basis of one-forms. For the vibrational momenta, choose the basis of three coordinate vector fields \( \partial/\partial x_\alpha \), where \( x_\alpha = \log(a_\alpha) \). Denote the
dual basis elements by \( dx^\alpha \).

For the angular momenta, the three right-invariant vector fields \( l_i \) on SO(3) are the components of the quantum angular momentum in the laboratory frame. This basis set of vector fields on the rotation group is not orthogonal. An orthogonal basis consists of the left-invariant SO(3) vector fields \( l_\alpha = \sum l_i R_{i\alpha} \), which are the three components of the angular momentum in the principal axis frame. Neither of these two SO(3) vector field bases are coordinate bases: the commutator of two angular momenta components is not zero. The corresponding dual one-form to \( l_\alpha \) is denoted by \( l^\alpha \).

The Riemannian metric \( g \) for the space of base manifold one-forms is:

\[
\begin{align*}
g(l^\alpha, l^\beta) &= \delta^{\alpha\beta}/I^R_\alpha, \\
g(dx^\alpha, dx^\beta) &= \delta^{\alpha\beta}/a_\alpha^2, \\
g(l^\alpha, dx^\beta) &= 0.
\end{align*}
\]  

(4.3)

where \( I^R \) denotes the rigid rotor moment of inertia in the body-fixed frame.

\[
I^R = \text{diag}(a_2^2 + a_3^2, a_1^2 + a_3^2, a_1^2 + a_2^2).
\]  

(4.4)

Hence, the desired orthogonal non-coordinate basis of six one-forms for the base manifold is \( \{ l^\alpha, dx^\alpha, \alpha = 1, 2, 3 \} \). The Riemannian volume element on \( Q \) is the six-form

\[
\omega = h l^1 \wedge l^2 \wedge l^3 \wedge dx^1 \wedge dx^2 \wedge dx^3.
\]  

(4.5)

where \( h = a_1 a_2 a_3 \sqrt{I^R_1 I^R_2 I^R_3 / 8\pi^2} \). By dividing by \( 8\pi^2 \), the volume of the compact 3-dimensional rotation group submanifold is unity.

Let \( \mathfrak{g} \) denote the three-dimensional Lie algebra of the structure group \( G \), and
let $\epsilon_a$ denote its basis of antisymmetric $3 \times 3$ real matrices. $(\epsilon_a)_{bc} = \epsilon_{abc}$, where $a, b, c = 1, 2, 3$. This is an orthogonal basis for the Lie algebra with respect to the trace form.

$$\text{Tr}(\epsilon_a \epsilon_b) = -2 \delta_{ab}. \quad (4.6)$$

### 4.2.1 Connexion and Curvature

The collective model connexion $A$ is a $\mathfrak{g}$-valued one-form on the base manifold $\mathcal{Q}$,

$$A = \epsilon_a A^a \rho_a. \quad (4.7)$$

Several physically reasonable constraints restrict the general mathematical form of the connexion. The model supposes that the dominant coupling is between angular and vortex degrees of freedom; hence, connexion terms involving $dx^a$ are assumed to vanish and are omitted from the above equation. The coefficients $A^a_\alpha(x)$ are smooth functions of the deformation $x = (x^1, x^2, x^3)$ and are assumed independent of the orientation. An orientation-dependent connexion fails to satisfy Noetherian rotational invariance. The strong coupling, as determined by the connexion, is restricted to angular momentum and Kelvin circulation components about the same axis. $A^a_\alpha = 0$ when $a \neq \alpha$. The connexion simplifies to

$$A = \sum_{\alpha=1}^{3} \epsilon_a A^a(x) \rho^\alpha. \quad (4.8)$$

where the connexion coefficients are abbreviated to $A^\alpha = A^a_\alpha$. 
The bundle curvature $F$ is the $\mathfrak{g}$-valued two-form

$$
F = dA + A \wedge A
$$

$$
= \epsilon_1 R^1 l^2 \wedge l^3 + \epsilon_2 R^2 l^3 \wedge l^1 + \epsilon_3 R^3 l^1 \wedge l^2
$$

$$
+ \epsilon_1 dA^1 \wedge l^1 + \epsilon_2 dA^2 \wedge l^2 + \epsilon_3 dA^3 \wedge l^3
$$

$$
= \frac{1}{2} \sum_{\alpha, \beta, \gamma} \epsilon_{\alpha \beta \gamma} \epsilon_\alpha R^\alpha l^\beta \wedge l^\gamma + \sum_{\alpha, \beta} \epsilon_\alpha \frac{\partial A^\alpha}{\partial x^\beta} dx^\beta \wedge l^\alpha.
$$


4.2.2 Yang-Mills Lagrangian

The Yang-Mills Lagrangian density $\mathcal{L}$ is a six-form defined on the base manifold; the density depends on the connexion coefficients $\mathcal{L}(A^1, A^2, A^3)$. The Yang-Mills action, $S = \int \mathcal{L}$, is a real-valued functional of the connexion coefficients. The Yang-Mills equations for the connexion are the Euler-Lagrange equations for this action.

The Lagrangian density is a sum of two densities, the source-free or free-field Yang-Mills Lagrangian,

$$
\mathcal{L}_{YM} = -\frac{1}{2} Tr(F \wedge *F).
$$

and the source or interaction Lagrangian,

$$
\mathcal{L}_{int} = Tr(A \wedge *J).
$$

where $J$ is a source "current," a $\mathfrak{g}$-valued one-form. Because of the connexion's special form, a non-zero source Lagrangian implies a current

$$
J = \sum_{\alpha} \epsilon_\alpha J^\alpha l^\alpha.
$$

Terms involving $dx^\alpha$ are omitted from the current one-form because they contribute
nothing to the interaction Lagrangian. The word "current" is adopted from conventional particle physics terminology.

The source-free Lagrangian depends on the curvature and the Hodge star of the curvature,

\[ \star F = \frac{1}{2} \sum_{\alpha, \beta, \gamma} e_{\alpha \beta \gamma} e_{\alpha} R^{\alpha} \star (I^{\beta} \wedge I^{\gamma}) + \sum_{\alpha, \beta} e_{\alpha} \frac{\partial A^{\alpha}}{\partial x^{\beta}} \star (dx^{\beta} \wedge I^{\alpha}). \tag{4.13} \]

which is a four-form. To evaluate the source-free Lagrangian, use orthogonality for the one-form basis, Eq.(4.3), to show that

\[ (I^{\beta} \wedge I^{\gamma}) \wedge \star (I^{\beta} \wedge I^{\gamma}) = \frac{1}{I_{\beta}^{R} I_{\gamma}^{R}} \omega \text{ when } \beta \neq \gamma. \]

\[ (dx^{\beta} \wedge I^{\alpha}) \wedge \star (dx^{\beta} \wedge I^{\alpha}) = \frac{1}{a_{\beta}^{2} I_{\alpha}^{R}} \omega. \tag{4.14} \]

and also use orthogonality for the $g$ basis, Eq.(4.6). The result is

\[ \mathcal{L}_{YM} = \frac{1}{I_{1}^{R} I_{2}^{R} I_{3}^{R}} \sum_{\beta} I_{\beta}^{R} (R^{\beta})^{2} \omega + \sum_{\alpha, \beta} \left( \frac{\partial A^{\alpha}}{\partial x^{\beta}} \right)^{2} \frac{1}{a_{\beta}^{2} I_{\alpha}^{R}} \omega. \tag{4.15} \]

Because this Lagrangian density is a rotation group scalar, the integral over SO(3) is trivial and the source-free action simplifies to an integration over the ellipsoid's deformation,

\[ S_{YM} = \int I_{\beta}^{R} \left( \frac{1}{I_{1}^{R} I_{2}^{R} I_{3}^{R}} (R^{\beta})^{2} + \sum_{\alpha} \left( \frac{\partial A^{\alpha}}{\partial x^{\beta}} \right)^{2} \frac{1}{a_{\beta}^{2} I_{\alpha}^{R}} \right) \cdot a_{1} a_{2} a_{3} dx^{1} dx^{2} dx^{3}. \tag{4.16} \]
The free variation of the source-free Yang-Mills action is

$$
\delta S_{YM} = 2 \sum_\alpha \int \delta A^\alpha \sum_\beta \left\{ \frac{1}{R_1 R_2 R_3} \left( R_3^R R_3 \frac{\partial R_3}{\partial A^\alpha} \right) \right. \\
\left. - \frac{1}{a_1 a_2 a_3} \frac{\partial}{\partial x^\beta} \left( \frac{a_1 a_2 a_3}{a_3^2 I_3^R} \frac{\partial A^\alpha}{\partial x^\beta} \right) \right\} a_1 a_2 a_3 \, dx^1 \, dx^2 \, dx^3.
$$

(4.17)

The variation may be expressed compactly in terms of the one-form \( \delta A = \sum_\alpha e_\alpha \delta A^\alpha \) and the exterior covariant derivative,

$$
\delta S_{YM} = - \int Tr(\delta A \wedge d_\nabla \star F).
$$

(4.18)

The variation of the interaction part of the action is simply

$$
\delta S_{int} = \int Tr(\delta A \wedge \star J).
$$

(4.19)

Therefore, the Yang-Mills equation \( \delta S = \delta S_{YM} + \delta S_{int} = 0 \) is satisfied if and only if

$$
\star J = d_\nabla \star F, \quad \text{or} \quad J^o = - \frac{1}{a_1 a_2 a_3} \sum_\beta \frac{\partial}{\partial x^\beta} \left( \frac{a_1 a_2 a_3}{a_3^2 I_3^R} \frac{\partial A^\alpha}{\partial x^\beta} \right) + \frac{1}{I_1 I_2 I_3} \sum_\beta \left( R_3^R R_3 \frac{\partial R_3}{\partial A^\alpha} \right).
$$

(4.20)

Note that the current components are functions of the axis lengths, \( J^o(x) \). Indeed, rotational symmetry requires the \( J \)-components to be independent of the angular momentum.

### 4.3 Irrotational Flow Source Current \( J \)

The source \( J \) determines the connexion from the Yang-Mills equation. Eq.(4.20). The current \( J \) models the strong coupling between angular momentum and vortex.
rotation in atomic nuclei. The model assumes and experimental moments of inertia confirm that the collective effect of strong nuclear forces among nucleons produces such a source current \( J \) and corresponding gauge field \( A \) [23]. \( J \) would be calculated from first principles if that ideal were not too hard at this time. Nonetheless, progress can be made from some general symmetry arguments and the special case of irrotational flow.

### 4.3.1 Symmetry Restrictions

Because the coupling between angular and vortex velocities is presumed independent of overall size and volume, the connexion coefficients are invariant under scaling. \( A^0(x_1 + \lambda, x_2 + \lambda, x_3 + \lambda) = A^0(x_1, x_2, x_3) \) for all real \( \lambda \). Note that the corresponding axis lengths scale as \( a_i \mapsto e^\lambda a_i \). The connexion coefficients, therefore, depend on differences \( x_i - x_j \) or, equivalently, on axis length ratios \( a_i/a_j \). Consequently, each connexion coefficient is a function of two independent variables, e.g., \( A^1(x) \) may be viewed as a function of \( x_2 - x_3 \) and \( 2x_1 - x_2 - x_3 = (x_1 - x_2) + (x_1 - x_3) \). A similar physical argument applies to the other two current components.

In the absence of an external field pointing in some special direction, the connexion \( A \) cannot depend on the numbering of the principal axes. For example, the connection is invariant under interchange of the 1- and 2-axes.

\[
A = c_1 A^1(x_1x_2x_3)l^1 + c_2 A^2(x_1x_2x_3)l^2 + c_3 A^3(x_1x_2x_3)l^3
\]

or

\[
= c_2 A^1(x_2x_1x_3)l^2 + c_1 A^2(x_2x_1x_3)l^1 + c_3 A^3(x_2x_1x_3)l^3. \tag{4.21}
\]

and, therefore, the connexion coefficients satisfy

\[
A^3(x_2x_1x_3) = A^3(x_1x_2x_3), \quad \text{and} \quad A^2(x_1x_2x_3) = A^1(x_2x_1x_3). \tag{4.22}
\]
Transposing (23) and (31) results in four more restrictions on the connexion coefficients. Their simultaneous solution requires that one function \( A(u, v) \) of two real variables determines all three connexion coefficients:

\[
\begin{align*}
A^1(x) &= A(u = x_2 - x_3, v = 2x_1 - x_2 - x_3). \\
A^2(x) &= A(u = x_3 - x_1, v = 2x_2 - x_3 - x_1). \\
A^3(x) &= A(u = x_1 - x_2, v = 2x_3 - x_1 - x_2).
\end{align*}
\]

(4.23)

The smooth function \( A(u, v) \) is symmetric in its first argument, \( A(-u, v) = A(u, v) \).

Henceforth, the connexion is assumed to be independent of \( v \): \( A^1 = A(u_1) \), \( A^2 = A(u_2) \), \( A^3 = A(u_3) \), where \( u_i = x_j - x_k \) for \( i, j, k \) cyclic. With this assumption, the Yang-Mills equation (4.20) simplifies to

\[
\frac{J^\alpha}{R^\alpha} = - \left( \frac{a_\alpha}{a_1 a_2 a_3} \right)^2 \left[ \tanh(u_\alpha) \frac{dA^\alpha}{du_\alpha} + \frac{d^2 A^\alpha}{du_\alpha^2} \right] + \frac{1}{R_{1} R_{2} R_{3}} \sum_\beta \left( R_{3} R_{3} \frac{\partial R_{3}}{\partial A^\alpha} \right). \tag{4.24}
\]

Because \( A(u) \) is a symmetric function, its first derivative \( dA/du \) is antisymmetric and its second derivative \( d^2 A/du^2 \) is symmetric. Using symmetry properties, permutations of one current component, say \( J^1 \), determines the other two current components.

\[
J^{p(1)}(x_{p(1)}, x_{p(2)}, x_{p(3)}) = J^1(x_1, x_2, x_3), \text{ for any permutation } p \in S_3. \tag{4.25}
\]
4.3.2 Irrotational Flow Connexion

For rigid rotation, the connexion is trivial, \( A = 0 \), and, hence, the current must vanish, \( J = 0 \). The irrotational flow connexion is \( \mathcal{A}(u) = 1 / \cosh(u) \), or

\[
\mathcal{A}^1 = \frac{1}{\cosh(u_1)} , \quad \mathcal{A}^2 = \frac{1}{\cosh(u_2)} , \quad \mathcal{A}^3 = \frac{1}{\cosh(u_3)} ,
\]

(4.26)

which is a consequence of the classical theory of Riemann ellipsoids [9], formulated geometrically as a principal bundle [10]. For this connexion, Eq. (4.24) determines the corresponding irrotational flow current,

\[
\mathcal{J}^\alpha = \left( \frac{a_\alpha}{a_1 a_2 a_3} \right)^2 \frac{1}{\cosh^3(u_\alpha)} + \frac{1}{I^R_1 I^R_2 I^R_3} \sum_\beta \left( I^R_\beta R^\beta \frac{\partial R^\beta}{\partial A^\alpha} \right),
\]

(4.27)

where \( R^\beta \) adopts the irrotational flow values for \( R^3 \), e.g., \( R^1 = A_2 A_3 - A_1 \).

Conversely, given the irrotational flow current \( \mathcal{J}^\alpha \), the Yang-Mills second order differential equation (4.24) has a unique solution with the initial conditions \( A(0) = 1 \) and \( dA/du(0) = 0 \), viz., \( A = \mathcal{A} \).

The irrotational flow \( R^\beta \) are small in typical nuclear physics applications in which deformations are small, \(|u_\alpha| << 1\). When the \( u \)'s are small, then, for \( \alpha, \beta, \gamma \) cyclic, \( R^\gamma \approx u_\alpha u_\beta \approx 1 \) can be ignored, and the irrotational flow current simplifies to

\[
\mathcal{J}^\alpha \approx \left( \frac{2a_\alpha}{a_1 a_2 a_3} \right) \exp(-u^2_\alpha).
\]

(4.28)

The connexion coefficient in the small deformation case is approximately \( A(u) \approx \exp(-u^2/2) \).

4.3.3 Coupling Constant \( g \)

An interpolation between the irrotational flow and rigid body connexions is \( A(u) = g A(u) \) for \( 0 \leq g \leq 1 \). The constant \( g \) is a measure of the strength of the
coupling between the angular momentum and circulation. The interpolated connexion
is a solution to the Yang-Mills differential equation with the initial conditions. \( A(0) = g \) and \( dA/du(0) = 0 \), and interpolated current

\[
J^1 = g \left( \frac{2}{a_2 a_3} \right) \left\{ \frac{1}{\cosh^2(u_1)} + \frac{a_2^2 a_3^2}{I_2^R I_3^R} \right\} + g^2 \frac{1}{I_2^R I_3^R} \left\{ -I_1^R A(u_2) A(u_3) + I_2^R A(u_2) (g A(u_3) A(u_1) - A(u_2)) + I_3^R A(u_2) (g A(u_1) A(u_2) - A(u_3)) \right\} .
\] (4.29)

and similarly for \( J^2, J^3 \) by cyclicly permuting 1, 2, 3. For small \( g \), neglecting the \( O(g^2) \) term yields a good approximation to the current.

4.4 Conclusions

The differential-geometric collective model adopts a kinetic energy proportional to the de Rham Laplacian. This Laplacian acts on the quantum Hilbert space of sections of an associated vector bundle. The sections take values in an irreducible representation \( \hat{\rho} \) of the Lie algebra \( \mathfrak{g} \) of the structure or vorticity group. When the connexion is \( A = gA \), [23] shows that the associated bundle’s covariant derivative is

\[
\nabla_\alpha = l_\alpha - gA^\alpha \cdot \hat{\rho}(c_\alpha)
\] (4.30)

and the rotational part of the model’s de Rham Laplacian is

\[
\Delta = \sum_{\alpha=1}^{3} \frac{\nabla^2_\alpha}{\mathcal{I}_\alpha^r} + \ldots
\] (4.31)

where the interpolated moment of inertia depends on the coupling strength. \( r = 1 - g \).

\[
\mathcal{I}_\alpha^r = r^2 \mathcal{I}_\alpha^R + (1-r^2) \mathcal{I}_\alpha^I.
\] (4.32)
and $\mathcal{I}^R$ and $\mathcal{I}^I$ denote the rigid and irrotational flow moments of inertia.

When the interaction strength is about $g = 1/3$, the proposed Yang-Mills collective model attains qualitative agreement between theory and experiment for moments of inertia of deformed isotopes. See Figure 2.3. The quantum model's kinetic energy implies velocity-dependent or magnetic-like forces arising from the square of the covariant derivative, viz., the cross term $-2gA^a \rho(\epsilon_\alpha) l_\alpha$.

The differential-geometric (DG) collective model is but an example of a new class of quantum models in which two different sets of degrees of freedom are unified in one geometric structure. In the DG collective model, the two sets of degrees of freedom are (1) the nuclear ellipsoidal shape and orientation, described by the base manifold, and (2) the vorticity or Kelvin circulation, corresponding to the structure group. The connexion describes quantitatively the interaction between the two sets of degrees of freedom.

This chapter and its predecessor [23] present a paradigm example of this unifying geometrical structure in the field of nuclear collective motion. In subsequent work, other quantum models of emergent phenomena will be investigated using the differential geometric framework.
Chapter 5

The Nuclear Geometric Model

Laplacian for Incompressible Nuclei

N Sparks

Nuclei are highly incompressible. Experiment puts the nuclear compressibility to be within the range of 250-315 MeV[1]. Incompressibility corresponds to a constant volume constraint. This is of particular interest when working with the Bohr-Mottelson geometrical collective model, which treats the nucleus as the surface of a rotating and vibrating ellipsoid for low values of energy [10]. The General Collective Motion group, GCM(3), provides the group structure for the Bohr-Mottelson collective model. GCM(3) is a noncompact 15-dimensional Lie group that contains the 9-dimensional group $P = \text{GL}_+(3, \mathbb{R})$ and a 6-dimensional abelian Lie group generated by the inertia tensor. Any element, $g \in P$, can be written as $g = RAS$, where $R \in SO(3)$. $A$ is a diagonal positive-definite matrix, and $S \in SO(3)$. Here, $R$ corresponds to a rotation of the nucleus. $A$ corresponds to the set of the axes lengths of the ellipsoid, and $S$ corresponds to the intrinsic Kelvin circulation. For these reasons,
$G$ is regarded as the motion group for the model because it is responsible for rotating and deforming the nucleus. The factorization of $P$ in this particular way represents a choice of gauge. It is this particular gauge choice that is most physically illuminating for the physical model. Mathematically, $P$ has the structure of a bundle which is the cornerstone of modern gauge theory.

Wave functions that belong to the model space are sections of the associated bundle $E = P \times_\rho V$ for $\rho$, an irreducible representation (irrep) on the vector space $V$ of the gauge group $G = \text{SO}(3) \subset P$. Each inequivalent irrep is labeled by a non-negative integer $C$ and is $(2C + 1)$ dimensional. For each irrep $\rho$ of the gauge group $\text{SO}(3)$ on the vector space $V$, the Hilbert space $\mathcal{H}_\rho$ of the corresponding irrep $\text{GCM}(3)$ representation $\pi_\rho$ contains functions $\Psi : P \to V$ of type-$\rho$.

$$\Psi(\xi g) = \rho(g^{-1})\Psi(\xi), \forall \xi \in P, g \in \text{SO}(3) \quad (5.1)$$

and $\int ||\Psi(\xi)||^2 d\mu(\xi) < \infty$. Here $d\mu(\xi)$ denotes the Haar measure on $P$. For $x \in P$, its representation is given by

$$(\pi_\rho(x)\Psi)(\xi) = \Psi(x^{-1}\xi), \forall \xi \in P. \quad (5.2)$$

The wave functions are both left and right invariant under the action of the group $\text{SO}(3)$ although the interpretations are different. When $\text{SO}(3)$ acts on the left, this action corresponds to rotation of the nucleus and conserves angular momentum. When $\text{SO}(3)$ acts on the right, this action corresponds to vortex motion and conserves the Kelvin circulation. Because this paper investigates the effects of working with a constant volume constraint, the correct motion group is provided by $\text{SL}_+(3, \mathbb{R})$. A fixed volume constraint corresponds to the determinant of $A$ also being fixed and this value can be normalized to unity. This chapter explores this idea from two different
perspectives. The first perspective fixes the volume in terms of two axes lengths and constructs the tangent space to the corresponding surface. The second perspective calculates the full tangent space and then performs a contraction (interior product) with the normal vector to the subsurface.

To touch base with the nuclear collective model, we express the axes lengths in terms of the collective coordinates $\beta$ and $\gamma$. The deformation parameter $\beta$ describes how much an ellipsoid deviates from being a sphere along the principal $z$-axis. Typical values for $\beta$ fall between 0 and 0.3. The deformation parameter, $\gamma$, measures how close an ellipsoid is to being triaxial, that is, the asymmetry perpendicular to the $z$-axis. When $\beta$ is small, the $\beta - \gamma$ coordinates describe the shape of an ellipsoid. By requiring that the volume of the ellipsoid be made constant, one ellipsoid axis length is expressible in terms of the other two and a correspondence between these remaining axes lengths and the $\beta - \gamma$ coordinates can be made. This is in line with the first perspective of handing the constraint.

The primary goal of this chapter is to elucidate the differential geometric constraint. The physical and mathematical background found in prior work will be discussed briefly in Section 5.2, but the reader is referred to [23] for greater details.

Section 5.1 sets the stage for the paper's goal by calculating all of the vector fields and volume elements relevant for the discussion. First, the constant volume constraint will be treated as a surface in the space of deformations. This will be accomplished by expressing one of the axes lengths in terms of the other two, via the equation for constant volume. For fixed volume, this surface is one dimension lower than the full space. We will construct the tangent vectors for this constrained surface. These will then be used to calculate the volume element for the constrained surface.

Section 5.2 discusses the second method. This will involve making a change of variables for the space of deformations. The primary vector field under consideration
is the normal vector, $N_1$, to the constant volume surface. Performing a contraction with the normal vector reduces the volume form by one degree. The vectors orthogonal to the normal vector necessarily span the same subspace as the constrained vector space from the first method.

Section 5.3 introduces the Hill-Wheeler coordinates, which relate the axes lengths of the ellipsoidal boundary to the usual $\beta - \gamma$ coordinates used by Bohr and Mottelson up to second order in $\beta$. A third set of vector fields are then used which are expressible in terms of sums and differences of the vector fields used in Section 3.2. These vector fields provide a simplification that give rise to a set of coordinates that are highly reminiscent of the usual $\beta - \gamma$ coordinates.

Section 5.4 establishes the goal of the chapter which is to show that by choosing the correct scaling terms for the vector fields in Section 5.2 that the volume elements for the methods are equivalent. Due to the coordinate independence of the Laplacian, these two seemingly different approaches yield the same result.

### 5.1 Background

The full volume element for the base manifold $P/G$ in prior work [23] was given by $\Omega = \sqrt{I_1^{\|F} I_2^{\|F} I_3^{\|F}} l^1 \wedge l^2 \wedge l^3 \wedge da^1 \wedge da^2 \wedge da^3$. Here, $I_i^{\|F}$ corresponds to the irrotational flow moment of inertia and $i = 1, 2, 3$. The base manifold $P/G$ corresponds physically to the orientation and vibrations of the nucleus. This explains why the volume is written in terms of angular momentum variables $l_i$ and axes lengths $da^i$. For the purposes of this paper, we will regard the volume element as consisting of two separate parts, namely a rotational part and the vibrational part given respectively by $\omega_{\text{rot}} = \sqrt{I_1^{\|F} I_2^{\|F} I_3^{\|F}} l^1 \wedge l^2 \wedge l^3$ and $\omega_{\text{vib}} = da^1 \wedge da^2 \wedge da^3$. Integration over the three non-coordinate angular terms is equivalent to integrating over three Euler angles, which produces a result of $8\pi^2$. For the purpose of clarity, this will be
suppressed.

It should be noted that the original Bohr-Mottelson model didn't account for the Kelvin circulation degrees of freedom. Interestingly, we calculate the volume element using the covariant derivative, which includes information about the circulation, but then specialize to the case of the trivial representation and write only the angular terms. Expressing the square root factor in front of the volume element in terms of the $\beta - \gamma$ coordinates gives a similar result to the volume element for the Bohr-Mottelson model volume element.

The focus of this paper will concern the vibrational degrees of freedom of the nucleus. The basis of vector fields that describe vibrational degrees of freedom found in [23] are given by \{ $\frac{\partial}{\partial a_i}$ \} for $i = 1, 2, 3$. These vector fields are an orthonormal set of vectors on $\mathbb{R}^3_+$. Each $a^i$ corresponds to the axes lengths of an ellipsoidal shaped nucleus. Each of these basis vectors has a corresponding dual vector (differential form), $da^i$, that satisfies:

$$da^i(\frac{\partial}{\partial a_j}) = g(\frac{\partial}{\partial a_i}, \frac{\partial}{\partial a_j}) = \delta_{ij}$$  \hspace{1cm} (5.3)

This can be regarded as a contraction between a dual vector and a vector. The original volume element in [3], is given by:

$$\omega_a = h_a \ I^1 \wedge I^2 \wedge I^3 \wedge da^1 \wedge da^2 \wedge da^3$$  \hspace{1cm} (5.4)

where $h_a = \sqrt{I_1^{IF}I_2^{IF}I_3^{IF}}$ and $I_{i}^{IF} = \left(\frac{a^2_i - a^2_j}{a^2_j + a^2_k}\right)^2 I_i^{RR}$ for $i,j,k$ cyclic. It is worth noting that the rigid rotation moment of inertia is defined as $I_i^{RR} = \frac{1}{5} AM(a_j^2 + a_k^2)$ for $i,j,k$ cyclic. For the sake of clarity, we will divide $I^{RR}$ by $\frac{2}{3} AMR^2$. This choice makes all moments of inertias dimensionless which follows the convention found in [33].
A diagram giving the steps taken in this paper:

\[
\begin{array}{c}
gl_+(3, R) \xrightarrow{\text{change of basis}} gl_+(3, R) \\
\downarrow \text{constraint} \quad \quad \downarrow \text{contraction} \\
sl_+(3, R) \xleftarrow{\text{change of basis}} sl_+(3, R)
\end{array}
\]

5.2 Vectorfields and Volume Elements

5.2.1 The Constrained Surface

In this section, we consider vector fields on the constant volume surface given by \( V = a_1 a_2 a_3 \), where \( a_i \) is the axes lengths of the Riemann ellipsoid discussed in [23]. By expressing one of the axis lengths in terms of the other two and using the equation of constraint, we can define new coordinates in the constrained space (sub-manifold).

We denote the change of these coordinates as:

\[
a_1 = \tilde{a}_1, \quad a_2 = \tilde{a}_2, \quad a_3 = \frac{V}{\tilde{a}_1 \tilde{a}_2} \quad (5.5)
\]

By making use of the chain rule, we obtain the following vectors in the subspace denoted with tildes:

\[
\frac{\partial}{\partial \tilde{a}_1} = \frac{\partial a_1}{\partial \tilde{a}_1 \partial a_1} + \frac{\partial a_2}{\partial \tilde{a}_1 \partial a_2} + \frac{\partial a_3}{\partial \tilde{a}_1 \partial a_3} = \frac{\partial}{\partial \tilde{a}_1} - \frac{V}{\tilde{a}_1^2 \tilde{a}_2 \partial a_3} \quad (5.6)
\]

\[
\frac{\partial}{\partial \tilde{a}_2} = \frac{\partial a_1}{\partial \tilde{a}_2 \partial a_1} + \frac{\partial a_2}{\partial \tilde{a}_2 \partial a_2} + \frac{\partial a_3}{\partial \tilde{a}_2 \partial a_3} = \frac{\partial}{\partial \tilde{a}_2} - \frac{V}{\tilde{a}_1 \tilde{a}_2^2 \partial a_3} \quad (5.7)
\]

\[
\frac{\partial}{\partial V} = \frac{\partial a_1}{\partial V \partial a_1} + \frac{\partial a_2}{\partial V \partial a_2} + \frac{\partial a_3}{\partial V \partial a_3} = \frac{1}{\tilde{a}_1 \tilde{a}_2 \partial a_3} \quad (5.8)
\]
Rescaling each of these vectorfields by $a_1$ and $a_2$ respectively gives:

$$
\eta_1 = \hat{a}_1 \frac{\partial}{\partial \hat{a}_1} = a_1 \frac{\partial}{\partial a_1} - a_3 \frac{\partial}{\partial a_3} \quad (5.9)
$$

$$
\eta_2 = \hat{a}_2 \frac{\partial}{\partial \hat{a}_2} = a_2 \frac{\partial}{\partial a_2} - a_3 \frac{\partial}{\partial a_3} \quad (5.10)
$$

$$
\eta_3 = V \frac{\partial}{\partial V} = a_3 \frac{\partial}{\partial a_3} \quad (5.11)
$$

The metric for this constrained surface is inherited from the larger space and is given by:

$$
\eta = \begin{pmatrix}
a_1^2 + a_3^2 & a_2^2 & a_3^2 \\
a_3^2 & a_2^2 + a_3^2 & a_3^2 \\
a_3^2 & a_3^2 & a_3^2
\end{pmatrix} \quad (5.12)
$$

The volume element in this subspace is given by:

$$
\omega_\eta = \sqrt{\det(\eta)} \eta^1 \wedge \eta^2 \wedge \eta^3 \quad (5.13)
$$

$$
= h_\eta \eta^1 \wedge \eta^2 \wedge \eta^3 \quad (5.14)
$$

where $h_\eta = a_1 a_2 a_3 = V$. When working with the constant volume constraint, the wavefunctions can be regarded as functions of only the surface (tilde) variables. To see why this should be the case consider the effects of a constant volume on the potential energy function. The potential energy is a scalar function and so it must remain invariant under a change of variables. Before the constraint is applied, we have that $U = U(a_1, a_2, a_3)$. The potential rewritten in terms of the new variables becomes $U = U(\hat{a}_1, \hat{a}_2, V)$. If $V$ is to be treated as a constant, $V_0$, then we may regard $U = U(\hat{a}_1, \hat{a}_2) \delta(V - V_0)$. This has the effect of making any measurements not directly on the constant volume surface equal to zero. After making this assumption about
the potential energy, we work with only the vectors tangent to the subsurface:

\[ \eta_1 = \frac{\partial}{\partial a_1} = a_1 \frac{\partial}{\partial a_1} - a_3 \frac{\partial}{\partial a_3} \]  

(5.15)

\[ \eta_2 = \frac{\partial}{\partial a_2} = a_2 \frac{\partial}{\partial a_2} - a_3 \frac{\partial}{\partial a_3} \]  

(5.16)

The corresponding metric and volume element are given respectively by:

\[ g_\eta = \begin{pmatrix} a_1^2 + a_3^2 & a_3^2 \\ a_3^2 & a_2^2 + a_3^2 \end{pmatrix} \]  

(5.17)

\[ \omega_\eta = \sqrt{\det(g_\eta)} \eta^1 \wedge \eta^2 \]  

(5.18)

\[ = h_\eta \eta^1 \wedge \eta^2 \]  

(5.19)

Explicitly, \( h_\eta = \sqrt{Z} \) where \( Z = (a_1 a_3)^2 + (a_2 a_3)^2 + (a_1 a_3)^2 \).

5.2.2 The \( \xi - N \) Basis

The normal vector to the constant volume surface is calculated by taking the gradient of the volume, \( V = a_1 a_2 a_3 \):

\[ N_1 = a_2 a_3 \frac{\partial}{\partial a_1} + a_1 a_3 \frac{\partial}{\partial a_2} + a_1 a_2 \frac{\partial}{\partial a_3} \]  

(5.20)

Any multiple of the normal vector will remain normal to the surface. For this reason, we may redefine a new normal vector, \( \tilde{N}_1 = a_3 N_1 \) where \( a_3 \) is a constant to be determined.

To describe the surface orthogonal to this normal vector, we are free to choose any two vectorfields that are orthogonal to the normal vector. The most straightfor-
ward and general choice is given by:

\[
\tilde{\xi}_1 = \alpha_1 \left( a_1 \frac{\partial}{\partial a_1} - a_3 \frac{\partial}{\partial a_3} \right) \\
\tilde{\xi}_2 = \alpha_2 \left( a_2 \frac{\partial}{\partial a_2} - a_3 \frac{\partial}{\partial a_3} \right) 
\]

(5.21)  

(5.22)

where \( \alpha_1 \) and \( \alpha_2 \) are real numbers. This choice is made for three reasons. First, it is trivial to see that these two tangent vectors are orthogonal to the normal vector to the surface, since their inner product with \( N_1 \) is zero. Secondly, this particular choice for the tangent vectors is directly related to the tangent vectors for the constrained surface in the prior section up to a scaling term. Lastly, the inner product of these tangent vectors is proportional to the rigid rotation moments of inertia. The corresponding metric becomes:

\[
\bar{g}_\xi = \begin{pmatrix}
\alpha_1^2 (a_1^2 + a_3^2) & \alpha_1 \alpha_2 a_3^2 & 0 \\
\alpha_1 \alpha_2 a_3^2 & \alpha_2^2 (a_2^2 + a_3^2) & 0 \\
0 & 0 & \alpha_3^2 Z
\end{pmatrix}
\]

(5.23)

The volume form in the \( \xi - N \) basis is given by:

\[
\omega_\xi = h_\xi \tilde{\xi}_1 \wedge \tilde{\xi}_2 \wedge \tilde{N}_1
\]

(5.24)

where \( h_\xi = \sqrt{\det \bar{g}_\xi} = \alpha_1 \alpha_2 \alpha_3 Z \). In the next section, we perform a contraction with the normal vector in order to establish consistency with the method in the prior section.

### 5.2.3 Hill-Wheeler Coordinates

When working with a constant volume constraint, it is natural to work with the Hill-Wheeler coordinates which relate the axes lengths of an ellipsoid to the usual \( \beta - \gamma \) coordinates used by Bohr and Mottelson up to \( O(\beta^2) \). The Hill-Wheeler
coordinates are given explicitly as:

\[ a_k = R \exp \left( \sqrt{\frac{5}{4\pi}} \beta \cos(\gamma - \frac{2\pi k}{3}) \right) \] (5.25)

for \( k = 1, 2, 3 \) cyclic. \( R \) is taken as a dynamical variable. It is clear from the definition of the axes lengths that \( R^3 = a_1 a_2 a_3 \). The vectorfields for the \( \xi - N \) basis become:

\[
\xi_1 = \sqrt{\frac{4\pi}{5}} \left\{ \left( -\cos(\gamma) + \frac{\sqrt{3}}{3} \sin(\gamma) \right) \frac{\partial}{\partial \beta} + \left( \frac{\sqrt{3}}{3} \cos(\gamma) + \sin(\gamma) \right) \frac{1}{\beta} \frac{\partial}{\partial \gamma} \right\} \] (5.26)

\[
\xi_2 = \sqrt{\frac{4\pi}{5}} \left\{ \left( -\cos(\gamma) - \frac{\sqrt{3}}{3} \sin(\gamma) \right) \frac{\partial}{\partial \beta} + \left( -\frac{\sqrt{3}}{3} \cos(\gamma) + \sin(\gamma) \right) \frac{1}{\beta} \frac{\partial}{\partial \gamma} \right\} \] (5.27)

\[
N_1 = R^2 \frac{\partial}{\partial R} \] (5.28)

Rearranging these vector fields gives a new set of vector fields:

\[
\frac{\partial}{\partial \sigma_1} = \frac{\xi_1 + \xi_2}{\sqrt{16\pi/5}} = \cos(\gamma) \frac{\partial}{\partial \beta} - \sin(\gamma) \frac{1}{\beta} \frac{\partial}{\partial \gamma} \] (5.29)

\[
\frac{\partial}{\partial \sigma_2} = \frac{\xi_1 - \xi_2}{\sqrt{16\pi/15}} = \sin(\gamma) \frac{\partial}{\partial \beta} + \cos(\gamma) \frac{1}{\beta} \frac{\partial}{\partial \gamma} \] (5.30)

In terms of coordinates \( \sigma_1 = -\beta \cos(\gamma) \) and \( \sigma_2 = \beta \sin(\gamma) \). These coordinates are highly reminiscent of \( \beta - \gamma \) coordinates.

### 5.3 Contraction with \( \tilde{N}_1 \)

In this section we perform the contraction between \( \omega_\xi \) and \( \tilde{N}_1 \), that is, we wish to calculate \( i_{\tilde{N}_1} \omega_\xi \) where \( i_{\tilde{N}_1} \omega_\xi \) indicates the interior product [5] between the normal vector and the volume element. To accomplish this, we express the full dual
antisymmetric product and contract this with \( \tilde{N}_1 \):

\[
\tilde{\omega}_\xi = \iota_{\tilde{N}_1} \omega_\xi
\]

\[
= \iota_{\tilde{N}_1} (h_\xi \tilde{\xi}^1 \wedge \tilde{\xi}^2 \wedge \tilde{N}^1)
\]

\[
= h_\xi \iota_{\tilde{N}_1} (\tilde{\xi}^1 \wedge \tilde{\xi}^2 \wedge \tilde{N}^1)
\]

\[
= h_\xi \tilde{\xi}^1 \wedge \tilde{\xi}^2
\]

(5.31) 

(5.32) 

(5.33) 

(5.34)

It is clear that by performing the contraction with \( N_1 \) that the resulting volume element matches the dimensionality of the constrained space. In the next section we will choose the constant \( \alpha_3 \) to reach equality of the two volume elements.

5.4 Equivalence of Volume Elements

The volume elements for the surfaces discussed were given explicitly by:

\[
\omega_\eta = \sqrt{Z} \, \eta^1 \wedge \eta^2
\]

and

\[
\tilde{\omega}_\xi = \alpha_1 \alpha_2 \alpha_3 Z \, \tilde{\xi}^1 \wedge \tilde{\xi}^2
\]

(5.35) 

(5.36)

To choose the correct scalars, we must write \( \tilde{\xi}^1 \wedge \tilde{\xi}^2 \) in terms of \( \eta^1 \wedge \eta^2 \). We have have that:

\[
\tilde{\omega}_\xi = \alpha_1 \alpha_2 \alpha_3 Z \, \tilde{\xi}^1 \wedge \tilde{\xi}^2
\]

\[
= \alpha_3 Z \, \eta^1 \wedge \eta^2
\]

\[
= \alpha_3 \sqrt{Z} \, \omega_\eta
\]

(5.37) 

(5.38) 

(5.39)

By choosing \( \alpha_3 = \frac{1}{\sqrt{2}} \) equality is achieved. One interesting consequence of this choice is that it normalizes the gradient vector \( N_1 \). By performing a contraction with
a unit normal vector, we have essentially shown that the two surfaces are identical. This leaves choices for the remaining constants $\alpha_1$ and $\alpha_2$ and we will discuss how to choose them.

5.5 Agreement with Bohr-Mottelson Kinetic Energy

The metric for the full space is given by:

$$\bar{g}_\xi = \begin{pmatrix} I_{1}^{IF} & 0 & 0 & 0 & 0 \\ 0 & I_{2}^{IF} & 0 & 0 & 0 \\ 0 & 0 & I_{3}^{IF} & 0 & 0 \\ 0 & 0 & 0 & a_1^2(a_1^2 + a_3^2) & \alpha_1 \alpha_2 a_3^2 \\ 0 & 0 & 0 & \alpha_1 \alpha_2 a_3^2 & a_3^2(a_1^2 + a_3^2) \end{pmatrix} \quad (5.40)$$

The full volume space for our considerations is given by:

$$\tilde{\Omega}_\xi = \sqrt{I_{1}^{IF}I_{2}^{IF}I_{3}^{IF}} \alpha_1 \alpha_2 \alpha_3 Z \cdot l^1 \wedge l^2 \wedge l^3 \wedge \xi^1 \wedge \xi^2 \quad (5.41)$$

$$= \frac{\sqrt{I_{1}^{IF}I_{2}^{IF}I_{3}^{IF}} \alpha_1 \alpha_2}{\sqrt{Z}} l^1 \wedge l^2 \wedge l^3 \wedge \xi^1 \wedge \xi^2 \quad (5.42)$$

$$= \alpha_1 \alpha_2 \sqrt{\frac{15}{16\pi}} \frac{1}{R^2} \beta^3 \sin(3\gamma) \cdot l^1 \wedge l^2 \wedge l^3 \wedge d\beta \wedge \beta d\gamma \quad (5.43)$$

The Bohr-Mottelson volume element is found in [1] and is given by:

$$\omega_{BM} = 2\beta^3 \sin(3\gamma) \cdot d\beta \wedge \beta d\gamma \quad (5.44)$$

By choosing $\alpha_1 = \alpha_2 = \left(\frac{6\pi}{15}\right)^{\frac{1}{3}} R$ we obtain $\tilde{\Omega}_\xi = \omega_{BM}$. The Laplacian is given
in coordinate free form by:

\[ \nabla^2 \psi = \frac{1}{\sqrt{g}} f_i \left( g^{ij} \sqrt{g(f_j \psi)} \right) \]  \hspace{1cm} (5.45)

Here, \( f_i \) represents an arbitrary vectorfield. Choosing the vectorfields to be angular momentum and the constrained surface variables provides the correct Bohr-Mottelson kinetic energy:

\[ \nabla^2 \psi = \sum_{i,j=1}^{5} \frac{1}{\sqrt{g}} f_i \left( g^{ij} \sqrt{g(f_j \psi)} \right) \]  \hspace{1cm} (5.46)

\[ = \sum_{i=1}^{3} \frac{\dot{\beta}_i^2}{\beta_i^2} + \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \left( \beta^4 \frac{\partial}{\partial \beta} \right) + \frac{1}{\beta^2 \sin(3\gamma)} \frac{\partial}{\partial \gamma} \]  \hspace{1cm} (5.47)

This result agrees with Rowe's derivation [1].

### 5.6 Discussion

We have shown that by imposing a constant volume constraint and by assuming that the representation for the Kelvin circulation be made trivial, that the Bohr-Mottelson Hamiltonian becomes apparent. To see this, make the circulation zero. The first method involved handling the constant volume constraint by finding tangent vectors to a surface of constant volume. The second method involved performing a contraction with the normal vector to the surface. Both give equivalent expressions. The utility in performing the second method is that performing contractions is much simpler than working with possibly multiple variable substitutions. In future work that involves dealing with larger spaces, the differential geometric approach becomes a far more tractable method and will be used.
Chapter 6

Future Directions

One of the fun things about research is that by answering some questions, many more get created along the way. The research presented here has created many questions some of which are discussed here.

6.1 Transverse Electric Form Factors

One of the aspects of the Yang-Mills equation is the source/current term, $J$. The source term is ultimately what is responsible for creating the bundle curvature and hence effects the values that the connection $A$ can take. Knowledge of $J$ is necessary for knowing the correct coupling parameter between the moments of inertias given by $r$. Most importantly, knowledge of the form of $J$ is what allows the theory to be falsified. On the theory end of things, there are many guesses that we can make about the form that the current must take. The work of Rosensteel [30], Carvalho and Rowe [31] help provide possible insight into this current. In each of the works, the transverse electric form factor is discussed and how it relates to the moments of inertia. In principle, measuring these may provide knowledge of $J$. 
6.2 Quantization of the theory

In the standard model, the Yang-Mills equation provides information about how elementary particles and fields interact via connections and covariant derivatives. If these field theories admit gauge bosons that carry information about the fundamental forces, is it also the case that the nuclear geometrical collective model might have a similar structure? We know that the angular momentum and the Kelvin circulation are both quantized, but is there some kind of “mediator” or “gauge boson” that relates the angular momentum to the circulation in some interesting way? This might lead to Feynman type diagrams and a possible paradigm shift in the field of nuclear structure theory.
Chapter 7

Appendices

7.1 Tangent Vectors

Physics doesn’t depend on how it is described. It is for this reason that the techniques of differential geometry naturally lend themselves to areas of physics where we wish frame independent effects to be manifest, namely in the construction of gauge theories and by extension relativity. The language used to describe such ideas is called coordinate independence [28]. We want the form of equations to be independent of the choice of coordinates used to describe them. Such a notion is not without its abstractions and hence learning curve, but the end result is elegant and powerful.

By expressing vectors as first order linear operators, we can show that vectors are independent of the coordinates chosen, even though the components of the vectors are coordinate dependent. To see how this is the case consider the following example. In some abstract sense, if we apply \( \mathbf{\nabla} \), a differential operator, to a set of coordinate functions, say the standard Cartesian basis \((x^1, x^2, x^3) = (x, y, z)\), we have that \( \mathbf{\nabla}(x^r) = V^i \). If we expand the vector into its components, it becomes: \( \mathbf{\nabla} = V^i \frac{\partial}{\partial x^i} \). If we were to choose another basis, say cylindrical basis \((y^1, y^2, y^3) = (r, \theta, z)\), we would again have that \( \mathbf{\nabla}(y^j) = V^j \) and could again expand the vector in terms of it's
components. To see the power of this method, we have shown that:

\[ \tilde{V} = V^i \frac{\partial}{\partial x^i} = V^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} = V^j \frac{\partial}{\partial y^j} \]  

(7.1)

where we have written \( V^j = V^i \frac{\partial y^j}{\partial x^i} \). This is simply the Jacobian transformation. Note that the form of the vector is unchanged between coordinate systems.

Note also that the standard vector space axioms apply to the notion of first order differential operators. For \( \tilde{V} = V^i \frac{\partial}{\partial x^i} \) and \( \tilde{W} = W^i \frac{\partial}{\partial x^i} \), \( f \) is a function, and \( a \) and \( b \) are both scalars, we have that the usual linearity conditions hold, namely

\[ (a\tilde{V} + b\tilde{W})(f) = a(\tilde{V}f) + b(\tilde{W}f). \]

The real power of this method comes when dealing with Lie brackets (commutators when working with vectors). The idea is essentially that the commutators of two vector fields produces another vector field. To see this, we write:

\[
\left[ \tilde{V}, \tilde{W} \right] f = \tilde{V}(\tilde{W}f) - \tilde{W}(\tilde{V}f) 
\]

(7.2)

\[
= V^i \frac{\partial W^j}{\partial x^i} \frac{\partial f}{\partial y^j} - W^i \frac{\partial V^j}{\partial x^i} \frac{\partial f}{\partial y^j} 
\]

(7.3)

\[
= \left[ V^i \frac{\partial W^j}{\partial x^i} \frac{\partial}{\partial y^j} - W^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i} \right] f 
\]

(7.4)

As is demonstrated in the preceding chapters, we make heavy use of a non-coordinate vector basis, namely in the form of the angular momentum and circulation vector fields. Notice that both of these vector fields are non-coordinate because their commutators aren't zero. One could of course use Euler angles instead of a coordinate basis for the group \( SO(3) \). It is much cleaner to work with the vector fields over the rotation group instead. Using the coordinate free formalism becomes an elegant way of handling such an idea.

One of the remarkable things about constructing tangent vectors over a manifold such as \( GL_+(3, R) \) is that depending on the choice of factorization, into a product.
of a rotation, a diagonal matrix, and another rotation is that each matrix has its own tangent vectors. So the idea behind this factorization is that the tangent vectors for each rotation is tangent to the rotation group, \( SO(3) \), and therefore the tangent vector must be a member of the associated Lie algebra \( \mathfrak{so}(3) \). To make this idea more clear, let us express each of the tangent vectors for each part of the factorization we’ve chosen. The first set of tangent vectors is responsible for rotating the nucleus and hence corresponds to the angular momentum:

\[
\mathbf{l}_i = \sum_{lm} (X_i \xi_{lm}) \frac{\partial}{\partial \xi_{lm}}
\]  

(7.5)

This idea is of course made clear in the previous chapters, but it is worth investigating in light of the idea that differential operators can be tangent vectors. Here is the essential idea. Pick some point \( \xi_{lm} \in GL_+(3, R) \). Rotate this to a new point via a left group action such that \( \xi(t) = g(t) \cdot \xi \) where \( g \in SO(3) \). Holding the point \( \xi \) fixed, now vary \( g = \exp(Xt) \) where \( X \in \mathfrak{so}(3) \). When applied to a given fixed member of \( \xi \) and evaluated at \( t = 0 \), the definition above makes more sense. A similar result holds for the right multiplication of \( \xi \) by the gauge group, \( G = SO(3) \), which generates the Kelvin circulation. For the generators of the rotations, we fix \( \xi \) and differentiate the element \( \exp(a_{ti}t) \) acting on the axes lengths. That is for \( \xi = RAS \), we obtain a new point \( \xi(t) = R(\exp(e_{ti}t)A)S \) which leads to the following vectorfields for vibrations:

\[
t_i = \sum_{lm} (Re_{ti} R' \xi_{lm}) \frac{\partial}{\partial \xi_{lm}}
\]  

(7.6)

Here, \( e_{ti} \) is the elementary 3x3 matrix with a single value of one at the intersection of each \( i^{th} \) row and column. If we apply this vector field to the coordinate
\[ A = R^{-1} \xi S^{-1} \], we obtain the following:

\[ t_i = \sum_{lm} (Re_{ii} R^l \xi)_{lm} \frac{\partial}{\partial \xi_{lm}} \]  
\[ = \sum_{lm} (Re_{ii} R^l \xi)_{lm} \frac{\partial A_{ij}}{\partial \xi_{lm}} \frac{\partial}{\partial A_{jj}} \]  
\[ = \sum_{lmjkp} (Re_{ii} R^l AS)_{lm} R^{-1}_{jk} S^{-1}_{mj} \delta_{kl} \delta_{pm} \frac{\partial}{\partial A_{jj}} \]  
\[ = \sum_{lmj} (Re_{ii} AS)_{lm} R^{-1}_{jl} S^{-1}_{mj} \frac{\partial}{\partial A_{jj}} \]  
\[ = \sum_{lnjq} R_{ln} (e_{ii} A)_{nq} S_{qn} R^{-1}_{jl} S^{-1}_{mj} \frac{\partial}{\partial A_{jj}} \]  
\[ = \sum_{lnjq} R^{-1}_{jl} R_{ln} (e_{ii} A)_{nq} S_{qn} S^{-1}_{mj} \frac{\partial}{\partial A_{jj}} \]  
\[ = \sum_{lnjq} \delta_{jn} (e_{ii} A)_{nq} \delta_{qj} \frac{\partial}{\partial A_{jj}} \]  
\[ = a_i \frac{\partial}{\partial a_i} \]  
\[ = \frac{\partial}{\partial x_i} \]  

This simple term is the tangent vector for the vibrational term. We can see first hand how applying the vector to a complicated coordinate function reduces to something very simple!

### 7.2 Appendix: Differential Forms

Differential forms play a central role in modern theoretical physics. In this appendix, we will outline their properties and how calculations can performed with them. The mathematical jargon will be included where necessary, but the primary focus will be on examples of their use and the notation associated with them. The
objective here is to familiarize the reader with the subject and build on their prior knowledge of calculus.

Anyone familiar with multivariable calculus has encountered the differential of a function, \( df \). As an example, suppose that \( f = f(x, y) \) is some real valued function of two variables. Its differential is calculated as:

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]  

(7.17)

This can of course be extended to real valued functions of \( n \) variables as:

\[
df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \, dx^i
\]  

(7.18)

The differential is the starting point for the subject of differential forms as it is one of the simplest examples of a “1-form”. Simply put, a 1-form is any kind of object that has a single copy of “\( dx \)” or any other kind of infinitesimal attached to it. As we will see, this isn’t the case for general 1-forms, but it’s a very good starting point.

### 7.2.1 Construction of Higher Order Differential Forms

One may naturally ask if there are other kinds of forms, such as 0-forms, 2-forms, etc. The answer is YES! A 0-form is simply a function without any kind of infinitesimal attached to it. 0-forms are the absolutely easiest example of a differential form.

Constructing a 1-form from a 0-form can be done in one of two ways. As we saw with the differential, one can simply perform a derivative and multiply it by the form “\( dx \)”. An alternative method is to take the 0-form and multiply it by a 1-form. We think of this a scalar multiplication, as forms can be thought of a vector objects belonging to a “dual” space. This will be explored in greater detail later.
Constructing a 2-form can be done in several ways. The first method is to take a derivative of a 1-form. The second method is to "multiply" two 1-forms. A third (trivial) method is to take a 0-form and multiply it by a 2-form. The focus will be on the first two methods on constructing higher order forms. A nice example will show everything we have discussed simultaneously. Suppose that $\alpha$ is a 1-form and that $f$ is a 0-form:

$$d(f\alpha) = df \wedge \alpha + f d\alpha$$  \hspace{1cm} (7.19)

The object being differentiated on the left is a 1-form, since it is the product of a 0-form with a 1-form. Taking a derivative of a 1-form produces a 2-form. To be consistent, the objects on the left are each 2-forms. The first term is a 1-form multiplied via a "wedge product" by another 1-form producing a 2-form. The second term is simply a 0-form multiplied by the derivative of a 1-form, which produces a 2-form. This example summarizes everything said above.

The wedge product is sometimes called the antisymmetric product. This is because for any two given forms, $\alpha \wedge \beta = -\beta \wedge \alpha$. It can be deduced from this, that for any given forms, $\alpha \wedge \alpha = 0$. These two properties will be used extensively throughout this work.

A 2-form of two infinitesimal lengths can be regarded as an area. To help illustrate this concept, consider the transformation between cartesian and polar coordinates:

$$x = r\cos(\theta)$$ \hspace{1cm} (7.20)

$$y = r\sin(\theta)$$ \hspace{1cm} (7.21)
The 1-forms are given by:

\[ dx = dr\cos(\theta) - r\sin(\theta) d\theta \]  \hspace{1cm} (7.22)
\[ dy = dr\sin(\theta) + r\cos(\theta) d\theta \]  \hspace{1cm} (7.23)

The area element associated with the Cartesian area is given by \( dx dy \). When using forms, however, we write this quantity in the following way: \( dx \wedge dy \). If we calculate this area in terms of the polar coordinates, we obtain:

\[ dx \wedge dy = (dr\cos(\theta) - r\sin(\theta) d\theta) \wedge (dr\sin(\theta) + r\cos(\theta) d\theta) \]  \hspace{1cm} (7.24)
\[ = \cos^2(\theta) dr \wedge dr + r\cos^2(\theta) dr \wedge d\theta \]  \hspace{1cm} (7.25)
\[ - r\sin^2(\theta) d\theta \wedge dr - r^2\sin(\theta)\cos(\theta) d\theta \wedge d\theta \]
\[ = r\cos^2(\theta) dr \wedge d\theta + r\sin^2(\theta) dr \wedge d\theta \]  \hspace{1cm} (7.26)
\[ = rdr \wedge d\theta \]  \hspace{1cm} (7.27)

The properties discussed above have been used to simplify the above expression. The utility of this method really shows itself in this example. Finding area and volume elements becomes simple.

### 7.2.2 Derivatives of Basis Forms

One of the powerful ideas in this work is the notion that some basis forms are differentiable. Before addressing this, let's consider the following examples.

In Cartesian coordinates, the position vector is typically written in the following form: \( \vec{x} = x\hat{i} \). We regard the component as a function of time, that is, \( x = x(t) \). But we think of the basis vector as fixed in time. So a velocity vector would be written as \( \vec{v} = \dot{x}\hat{i} \).

In polar coordinates, the position vector is written as \( \vec{r} = r\hat{\theta} \). Geometrically,
the basis vector always points radially outward. So if some object is moving in say, a circle, the radius is fixed, but the basis vector changes in time. This is a very powerful idea. If we make the following associations:

\[ \hat{r} = \cos(\theta)\hat{x} + \sin(\theta)\hat{y} \]  
\[ \hat{\theta} = -\sin(\theta)\hat{x} + \cos(\theta)\hat{y} \]  

(7.28)  
(7.29)

Here, we have regarded \( \theta = \theta(t) \). Differentiating the vector above and writing all basis vectors in terms of the polar basis vectors, we obtain: \( \ddot{r} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \).

Just as basis vectors can be differentiable, basis forms can be differentiable as well. In general, when a 1-form is differentiated, a linear combination of 2-forms is created. Suppose that \( f^i \) is some basis form belonging to a collection of other basis forms. Its derivative is a linear combination of other members of this basis set. We write this expression as:

\[ df^i = \frac{1}{2} \sum_{jk} T_{jk}^i f^j \wedge f^k \]  

(7.30)

It should be noted that these coefficients are necessarily antisymmetric since the wedge product is also antisymmetric. When the basis forms have a Lie algebra structure, it is the case that this is manifested in the derivatives of these basis forms.

As an example consider the following collection of basis forms: \{l^1, l^2, l^3, da^1, da^2, da^3\} which is found in the thesis work. It will be discussed later how the angular momentum basis forms are constructed, but suffice it to say that angular momentum operators are the generators of the Lie group \( SO(3) \). It is for this reason that the derivatives of the angular momentum basis vectors obey: \( dl^i = -\frac{1}{2} \sum_{jk} \epsilon_{ijk} l^j \wedge l^k \). When compared to the formula above, we see that \( T_{jk}^i = -\epsilon_{ijk} \). All other combinations of \( T_{jk}^i = 0 \) for \( ijk \neq 1.2.3 \).
7.3 Appendix: Hodge Star

In the appendix on differential forms, we discussed how higher order forms were constructed. We will now discuss an operation on forms called the Hodge star operation. The essential idea is that the Hodge star operator takes a $p$-form and maps it to a possible linear combination of $(N - p)$-forms. The ingredients necessary to perform the Hodge star operation are a complete basis of forms and a metric for those forms. The definition used for calculating the Hodge star is given by:

$$\alpha \wedge \star \beta = g(\alpha, \beta) \omega$$  \hspace{1cm} (7.31)

Here, $\alpha$ and $\beta$ are both $p$-forms and $\omega$ is an $N$-form. In fact, $\omega = \sqrt{\det g} \ f^1 \wedge \ldots \wedge f^N$ is called the volume form. Here, $g$ is the metric for the space and the $f$'s refer to the basis of forms for that space. We can immediately see that $\star \beta$ must be an $N - p$-form for consistency. $g(\alpha, \beta)$ is an inner product on forms. It is best to discuss this inner product as the complexity of our examples increases.

Let’s now look at some examples of how the Hodge star works and from there we can discuss a more general framework for using them.

7.3.1 Example: Cartesian Basis

Suppose that $\omega = dx \wedge dy \wedge dz$ is the standard Cartesian basis. The metric for the Cartesian basis is given by: $g = \text{diag}\{1,1,1\}$. Consider how one might go about calculating $\star dx$. To use the formula, we write:

$$dx \wedge \star dx = g(dx, dx) \ dx \wedge dy \wedge dz$$  \hspace{1cm} (7.32)

$$= dx \wedge dy \wedge dz$$  \hspace{1cm} (7.33)

Immediately we can read off the answer by matching each side of the equation:
\[ \star dx = dy \wedge dz. \] This answer illuminates several points. First, the Hodge star of a 1-form does return a 2-form. This is true because the full basis is a 3-form and the object we took the star of was a 1-form, hence the result was a "3-1" form or 2-form. Second, the inner product between the 1-forms in an orthogonal basis simply returns that entry of the metric, in this case, one.

Let's now look at an example of taking the Hodge star operation of a 2-form. Suppose we wish to calculate: \( \star (dy \wedge dz) \). Using the definition:

\[
(dy \wedge dz) \wedge \star (dy \wedge dz) = g(dy \wedge dz, dy \wedge dz) \ dx \wedge dy \wedge dz \tag{7.34}
\]
\[
= (g(dy, dy)g(dz, dz)
- g(dy, dz)g(dz, dy)) \ dx \wedge dy \wedge dz \tag{7.35}
\]
\[
= dx \wedge dy \wedge dz \tag{7.36}
\]
\[
= dy \wedge dz \wedge dx \tag{7.37}
\]

We can read off that \( \star (dy \wedge dz) = dx \). Once again, there are some points for discussion. First, we see that the inner product involves the determinant of a minor of the metric. This point is of paramount importance when calculating the Hodge star operation when the metric is not orthogonal.

### 7.3.2 Example: Special Relativity

We now move onto the case where the metric isn't trivial. In the theory of special relativity, the spacetime metric is given by \( g = \text{diag}\{1, -1, -1, -1\} \). This choice is to ensure that the proper time is positive, but it's merely a convention. The "volume" form is given by \( \omega = dt \wedge dx \wedge dy \wedge dz \). Let's now look at some examples.
starting with $\star dt$. Using the formula above:

\[
dt \land \star dt = g(dt, dt) \ dt \land dx \land dy \land dz
\]

\[
= +1 \ dt \land dx \land dy \land dz
\]  \hspace{1cm} (7.38)

So reading off the result, we find that $\star dt = dx \land dy \land dz$. What happens when we calculate $\star dx$? Let’s see:

\[
dx \land \star dx = g(dx, dx) \ dt \land dx \land dy \land dz
\]

\[
= -1 \ dt \land dx \land dy \land dz
\]  \hspace{1cm} (7.40)

\[
= dx \land dt \land dy \land dz
\]  \hspace{1cm} (7.41)

Reading off this result, we obtain: $\star dx = dt \land dy \land dz$. Now let’s calculate something slightly more difficult. Let’s calculate $\star(dt \land dx)$:

\[
(dt \land dx) \land \star(dt \land dt) = g(dt \land dx, dt \land dx) \ dt \land dx \land dy \land dz
\]

\[
= (g(dt \land dt)g(dx, dx) - g(dt, dx)g(dx, dt)) \ dt \land dx \land dy \land dz
\]  \hspace{1cm} (7.43)

\[
= dx \land dt \land dy \land dz
\]  \hspace{1cm} (7.44)

\[
= -1 \ dt \land dx \land dy \land dz
\]  \hspace{1cm} (7.45)

We find that $\star(dt \land dx) = -dy \land dz$. Comparing this example with the Cartesian case demonstrates that the metric plays a central role in calculating the Hodge star operation.

### 7.3.3 Example: Polar Coordinates

In this example, we will encounter a metric that isn’t as trivial as the former examples. The metric for polar coordinates is given by $g = \text{diag}\{1, r^2\}$. The volume
form is given by \( \omega = r dr \wedge d\theta \). Notice that the volume form consists of the infinitesimals of the polar variables, but also the factor in front is equal to the square root of the determinant of the metric. This will hold in general when calculating volume forms. Let's calculate \(*dr*:\)

\[
\begin{align*}
  dr \wedge *dr &= g(dr,dr) \ rdr \wedge d\theta \\
  &= +1 \ rdr \wedge d\theta 
\end{align*}
\] (7.46) (7.47)

Reading off the result, we find \(*dr = r d\theta\). Special attention must now be taken to the units under consideration. Note that the star operation doesn’t change the units of the infinitesimal that it is operating on. To fully appreciate this fact, note that the metric of forms is actually the inverse for the metric of vectors. We can prove this in general later, but it suffices now to bear in mind that this must be true for the sake of consistent units. We now calculate \(*d\theta*:\)

\[
\begin{align*}
  d\theta \wedge *d\theta &= g(d\theta,d\theta) \ rdr \wedge d\theta \\
  &= \frac{1}{r^2} \ rdr \wedge d\theta \\
  &= -\frac{1}{r} \ d\theta \wedge dr
\end{align*}
\] (7.48) (7.49) (7.50)

We immediately see that \(*d\theta = -\frac{1}{r}dr\). As a final reminder, notice that the units are consistent.

### 7.3.4 Metric of Forms

As mentioned earlier, the metric of forms is actually equal to the inverse of the metric for vectors. Let’s prove why this is the case. Suppose that \(f_i\) and \(f_j\) are basis vectors with some given metric \(g\). Moreover, assume that these basis vectors can be “raised” by contracting with the inverse of the metric in the usual way using
Einstein summation convention. Then,

\[
\begin{align*}
g(f^\alpha, f^\beta) &= g(g^{\alpha i} f_i, g^{\beta j} f_j) \\
&= g^{\alpha i} g^{\beta j} g(f_i, f_j) \\
&= g^{\alpha i} g(f_i, f_j) g^{\beta j} \\
&= g^{\alpha i} g_{ij} g^{\beta j} \\
&= g^{\alpha \beta}
\end{align*}
\]  

(7.51)  

(7.52)  

(7.53)  

(7.54)  

(7.55)

This completes the proof.

### 7.3.5 Forms for Non-orthogonal Metrics

To have a full appreciation for the power of working with forms, we must consider the case when the metric is not necessarily orthogonal. This will lead us into a discussion about how forms and vectors are related in a more general sense. We will begin our discussion with how the metric is constructed. Consider some set of vectors which are not necessarily orthogonal. Written in terms of the standard basis, this means:

\[
\frac{\partial}{\partial u^i} = \sum_j A_{ij} \frac{\partial}{\partial x^j}
\]

(7.56)

The metric for a non-orthogonal set of vectors is based off of the standard
metric. that is:

\[ g_{ij} = g \left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) \]  

(7.57)

\[ = g(A_{ik} \frac{\partial}{\partial x^k}, A_{jl} \frac{\partial}{\partial x^l}) \]  

(7.58)

\[ = A_{ik} A_{jl} \frac{\partial}{\partial x^k} \cdot \frac{\partial}{\partial x^l} \]  

(7.59)

\[ = A_{ik} A_{jl} \delta_{kl} \]  

(7.60)

\[ = A_{ik} A_{jk} \]  

(7.61)

\[ = (AA^T)_{ij} \]  

(7.62)

As was established in the prior section, \( g^{ij} = (AA^T)^{-1} \) for the inner product for forms. One of the interesting consequences of working with a non-orthogonal metric is that the Hodge star operation returns a sum of forms. From the definition:

\[ du^i \wedge *du^j = g (du^i, du^j) \omega \]  

(7.63)

\[ du^i \wedge C_{jl} d\tilde{u}^l = g^{ij} \sqrt{\det g} \ du^1 \wedge \ldots \wedge du^N \]  

(7.64)

\[ C_{jl} du^i \wedge d\tilde{u}^l = g^{ij} \sqrt{\det g} \ (-1)^{i-l} du^i \wedge d\tilde{u}^l \]  

(7.65)

\[ C_{ji} d\tilde{u}^l = g^{ij} \sqrt{\det g} \ (-1)^{i-l} d\tilde{u}^l \]  

(7.66)

\[ C_{ji} = (-1)^{i-l} g^{ij} \sqrt{\det g} \ d\tilde{u}^i \]  

(7.67)

\[ *du^i = \sum_l (-1)^{i-l} g^{ij} \sqrt{\det g} \ d\tilde{u}^l \]  

(7.68)

### 7.3.6 Determining Forms from Vectors

Forms play a central role in differential geometry and knowing how to find them given a set of vectors is of fundamental importance. We will now explore the relationship between forms and vectors. In differential geometry, forms can be thought in some sense as being "dual" to vectors. This is analogous to the relationship
between column vectors and row vectors. Moreover, they transform in some sense "inversely" to vectors in light of "units". Let's explore this idea in more detail.

Above we wrote:

\[
\frac{\partial}{\partial u^i} = \sum_j A_{ij} \frac{\partial}{\partial x^j} \tag{7.69}
\]

Let's assume that some basis of forms transforms linearly:

\[
du^k = \sum_l B_{kl} dx^l \tag{7.70}
\]

Reiterating the point that forms and vectors are dual to one another, this means that for the standard basis: \(dx^i(\frac{\partial}{\partial x^j}) = \delta_{ij}\). Assuming that \(u\) has some units associated with it, it is natural to also require that \(du^k(\frac{\partial}{\partial u^l}) = \delta_{kl}\). As a consequence of this, we find:

\[
\delta_{ki} = du^k \left(\frac{\partial}{\partial u^i}\right) = \sum_l B_{kl} dx^l \left(\sum_j A_{ij} \frac{\partial}{\partial x^j}\right) = \sum_{lj} B_{kl} A_{lj} dx^l \left(\frac{\partial}{\partial x^j}\right) = \sum_{lj} B_{kl} A_{lj} \delta_{lj} = \sum_{lj} B_{kl} A_{lj}^T \tag{7.71-7.75}
\]

It follows that \(B = (A^T)^{-1}\). That is:

\[
du^k = \sum_l (A^T)^{-1}_{kl} dx^l \tag{7.76}
\]

This is truly a remarkable result! We see that forms are dual to vectors hence
the transpose, and that the units transform inversely! As a final check, let’s calculate the metric for our result:

\[
g(du^i, du^j) = g((A^T)^{-1}_{ik} dx^k, (A^T)^{-1}_{jm} dx^m) \quad (7.77)
\]

\[
= (A^T)^{-1}_{ik} (A^T)^{-1}_{jm} g(dx^k, dx^m) \quad (7.78)
\]

\[
= (A^T)^{-1}_{ik} (A^T)^{-1}_{jm} \delta_{km} \quad (7.79)
\]

\[
= (A^T)^{-1}_{ik} (A)^{-1}_{kj} \quad (7.80)
\]

\[
= (A^{-1} A^{-1})_{ij} \quad (7.81)
\]

\[
= (AA^T)^{-1}_{ij} \quad (7.82)
\]

This agrees with what we had calculated earlier!

7.4 Appendix: Bundle Theory

![Figure 7.1: A bundle](34)

Gauge theory is of central importance to modern theoretical physics. At the heart of gauge theory is a mathematical object called a bundle, more specifically a fiber bundle. The two terms in this work will be interchangeable. This appendix explores what a bundle is by giving examples and discussing the language used in describing them.
7.4.1 An Intuitive Approach to Bundles

Simply put, a bundle is a mathematical object that consists locally of a Cartesian product of a "base space" with a "fiber". In many physics applications, the base space can be regarded as spacetime and the fiber can be thought of as a Lie group. This is the basis for the Standard Model. This of course isn't the only application of the bundle idea, but it is certainly the most intuitive and ubiquitous. To further understand this idea, reconsider the idea of a covariant derivative. A derivative is covariant because it consists of a derivative with respect to base space variables and an extra piece that is added that is the derivative of the Lie group (Lie algebra) living above the base space point. So the bundle idea essentially is a geometrical object whose derivative gives the covariant derivative in a natural way: the derivative of the base space variables give the usual derivative and the derivative of the fiber gives the "covariant" piece. Because the two structures are glued together as a Cartesian product, the derivative becomes a sum of derivatives of the two structures.

There are different types of bundles. Bundles are typically categorized by their fibers. The simplest example of a bundle would be a real valued function. We could assign the base space to be the real line and the fiber to be the value of the function that lives above that point. Another example is a vectorfield, where each point in space has a vector assigned to that point. The fiber in this case would be the vector at that point. The standard model mentioned above consists of a base space of spacetime and the fiber is the symmetry group of the Standard Model. This would be an example of a principal bundle since the fiber is group valued.

In this work, the motion group $P = \text{GL}_+(3, \mathbb{R})$ has a bundle structure. If we express an element of this group $\xi \in P$, we can write $\xi = RAS$ where $R \in SO(3)$ corresponds to a rotation of the ellipsoid, $A$ is a matrix of positive real numbers that corresponds to the axes lengths of the ellipsoid, and $S \in G = SO(3)$ corresponds to
rotation of the internal nucleons (Kelvin circulation). To see why this structure is a bundle corresponds nicely to it’s physical interpretation. To make this clear, we write \( q = \xi \xi^t = RA^2R^t \) and we see that the dependence on \( S \) has totally canceled out. Physically, \( q \) corresponds to an inertia tensor since \( q \propto a^2 \). but equally the Kelvin circulation group behaves in an independent way from \( q \), suggesting it’s role as a gauge group. This is the very essence of a bundle! The inertia tensor \( q \) is a base manifold and \( S \) is the fiber above the base manifold.

### 7.4.2 A Formal Approach to Bundles

The language used to formalize the bundle concept is found in [5]. A bundle consists of a total space, \( E \), a base space \( M \), a fiber \( F \), a projection \( \pi : E \to M \), and structure group \( G \) acts on the left of \( F \). A section of the bundle is the inverse image of the projection function: \( \pi^{-1}(p) = F \) where \( p \in M \). A section is therefore essentially thought of as a map that sends a point in the base manifold to a point in the fiber. This then leads to the concept of a local trivialization. Suppose that \( \phi_i \) is a diffeomorphism (essentially a nice function: invertible with continuous derivatives over some neighborhood \( U_i \in M \)). If \( \phi_i : U_i \times F \to \pi^{-1}(U_i) \) is a diffeomorphism and \( \pi \circ \phi_i(p, f) = p \), we say that \( \phi_i \) is a local trivialization.

One of the essential ideas in this work is that wavefunctions for the geometrical collective model are sections of a fiber bundle. Each wavefunction is vector-valued, \( \Psi : P \to V \) and takes arguments from full bundle. When the function satisfies \( \Psi(\xi g) = \rho(g^{-1})\Psi(\xi) \) where \( \xi \in P \) and \( g \in G \). we say that \( \Psi \) is a function of type \( \rho \). This is possible when \( \rho \) is an irreducible representation of the structure group \( G \) on the complex vector space \( V \). If \( \Psi \) is a function of type \( \rho \), this implies that when the gauge group \( G \) acts on \( P \), we have that \( \Psi(\xi g) = \rho(g^{-1})\Psi(\xi) \). This essentially says that \( \Psi(\xi g) = \Psi(RA\xi g) = \Psi(RA(S')) \). Since the action of \( g \) changes the fiber \( S \) to \( S' \) we say that the bundle is an associated bundle.
We note that \((\xi, V) \sim (\xi g, \rho(g^{-1})V)\) forms an equivalence class. The group actions are in different spaces, although play a very similar role when written in the form of a local trivialization. They are equivalent in an abstract sense because locally the group and its representation “cancel out”.

7.4.3 Covariant Derivatives Revisited

In light of the prior discussion on covariant derivatives, a bundle is the natural mathematical object for dealing with covariant derivatives. The bundle consists of a base space and a fiber, just as the covariant derivative consists of a regular derivate with a connection. To make this idea more transparent, the language of the standard model incorporates spacetime as a base space and attached to each of the points is a group element, namely \(SU(3) \times SU(2) \times U(1)\). The covariant derivative used in field theories is expressed as a derivative of spacetime with a derivative of the group element, which we refer to as a connection. The connection this time is Lie algebra valued because the generators of a Lie group, via exponentiation, for a Lie algebra. Mathematically, when we have a derivative in the base space and attach to it the connection, we call this a ‘horizontal lift’. The derivative is ‘lifted’ into the full bundle. The tangent vectors that only involve the fibers are called vertical and the covariant derivative is called horizontal. These two tangent vectors are orthogonal.

Algebraically, we can think of the base space and the full space of the bundle as being written in the following way \(P/G \times G \approx P\). This allows us to think of the bundle in all of its fullness (i.e. \(P\)) or as a base space \((P/G)\) and a fiber \(G\). The term \(P/G\) read as “\(P\) mod \(G\)” is known as a coset or homogeneous space. For the purposes of this work, we considered different ideas involving connections on the bundle \(P\). The first connection was called the Riemannian connection and involved terms involving the circulation \(c\). The second type of connection involved an irrotational flow connection and involved \(\dot{\rho}\).
7.5 Appendix: Laplacian

The Laplacian is a second order differential operator that is ubiquitous in physics. The Laplacian of a function $\psi$ is given in terms of the exterior derivative and the Hodge star by:

$$\nabla^2 \psi = \star d \star d\psi$$

(7.83)

One of the primary advantages of writing the Laplacian in this manner is that it is coordinate free. This means that the Laplacian takes this form independent of the coordinates that are chosen and even in the case of using a basis involving non-coordinate forms.

The goal of this appendix will be to express the Laplacian in the most general way, that is, to write out the expression in terms of general differential operators and a given metric. From here, we will look at examples of this form and the utility of the method will manifest itself.

There are essentially a series of four operations necessary for computing the Laplacian using the differential geometric framework: an exterior derivative, a Hodge star operation, another exterior derivative, followed by a final Hodge star operation. Each of these four steps will be carefully expressed and explained.

The first step is an exterior derivative:

$$d\psi = \sum_i (f_i \psi) f^i$$

(7.84)

Note that this derivative is taken with respect to any of the variables that make up the base manifold chosen. In particular, this could be a derivative taken with respect to an axis length or it could represent an angular momentum type of derivative. The sky is the limit in terms of what kind of operators are allowed. This is part of the
power of the method.

The next step is taking a Hodge star operation:

\[
\star d\psi = \sum_i (f_i \psi) \star f^i = \sum_i (f_i \psi) \frac{h}{B_i} \hat{f}^i
\]

(7.85) \hspace{1cm} (7.86)

Note that as previously explained, the Hodge star maps a 1-form to an \(\cdot X-1\)-form and introduces a factor that involves the metric and the volume element.

We now compute another exterior derivative:

\[
d \star d\psi = \frac{1}{2} \sum_{ij} f_j \left( (f_i \psi) \frac{h}{B_i} \right) f^i \wedge \hat{f}^j + \sum_i (f_i \psi) \frac{h}{B_i} d \hat{f}^i
\]

(7.87)

\[
= \sum_i f_i \left( (f_i \psi) \frac{h}{B_i} \right) f^i \wedge \hat{f}^j - \sum_{ij} (f_i \psi) \frac{h}{B_i} T_{ij}^i f^i \wedge \hat{f}^j
\]

(7.88)

To get the second line above, note that when \(i \neq j\) that the wedge product becomes zero. This extracts only the terms that match. For the second term, the following lemma (which will be proven and explained later) was used:

\[
d \hat{f}^i = - \sum_j T_{ij}^i f^i \wedge \hat{f}^j
\]

(7.89)

Here, \(T_{jk}^i = f^i [f_j, f_k]\). We now take the last remaining Hodge star operation to obtain:

\[
\star d \star d\psi = \sum_i f_i \left( (f_i \psi) \frac{h}{B_i} \right) \star \left( f^i \wedge \hat{f}^i \right) - \sum_{ij} (f_i \psi) \frac{h}{B_i} T_{ij}^i \star \left( f^i \wedge \hat{f}^j \right)
\]

(7.90)

\[
= \sum_i f_i \left( (f_i \psi) \frac{h}{B_i} \right) \frac{1}{h} - \sum_{ij} (f_i \psi) \frac{1}{B_i} T_{ij}^i
\]

(7.91)

\[
= \sum_i f_i \left( (f_i \psi) \frac{h}{B_i} \right) \frac{1}{h}
\]

(7.92)
The second term above vanishes when all of the indices are summed over because the commutator is antisymmetric. Note that \( \omega = dx^1 \wedge ... \wedge dx^N = h f_1 \wedge ... \wedge f^N \). Since \( \ast \omega = 1 \), it follows that \( \ast (f^1 \wedge ... \wedge f^N) = \frac{1}{h} \).

### 7.5.1 Example: Cartesian Coordinates

As an example, consider calculating the Laplacian in three dimensional Cartesian coordinates using the formula. First we write the standard volume element: \( \omega = dx^1 \wedge dx^2 \wedge dx^3 \). This implies that \( f_i = \frac{\partial}{\partial x^i} \). The metric for this particular case is simply the identity matrix in three dimensions. The determinant is one, and the volume prefactor, \( h \), is therefore equal to one as well. We now calculate the Laplacian as:

\[
\ast d \ast d \psi = \sum_i \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial x^i} \psi \right)
\] (7.93)

This is of course the correct Laplacian.

### 7.5.2 Example: Spherical Coordinates

As another example, consider calculating the Laplacian in three dimensional spherical coordinates using the formula. We write the standard volume element: \( \omega = dx^1 \wedge dx^2 \wedge dx^3 \). This implies that \( f_i = \frac{\partial}{\partial x^i} \). Here we make the identification that \( x^1 = r \), \( x^2 = \theta \), and \( x^3 = \phi \). The metric for this particular case is given by:

\[ g = \text{diag}\{1, r^2, r^2\sin(\theta)\} \]

The volume prefactor, \( h = \sqrt{\text{det}(g)} = r^2 \sin(\theta) \). Each of the factors \( B_i \) correspond to the inverse elements of the metric. We now calculate the Laplacian as:

\[
\ast d \ast d \psi = \frac{1}{r^2 \sin(\theta)} \left( \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \psi}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left( \frac{\partial \psi}{\partial \phi} \right)
\]

\[
= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 \psi}{\partial \phi^2}
\] (7.94)
This gives the correct form of the Laplacian.

### 7.5.3 Proof of Lemma

We now wish to show that \( d\hat{f}^i = -\left(\sum_j T_{ji}^j\right) f^1 \wedge \ldots \wedge f^N \) for fixed \( i \). Let \( \omega = f^i \wedge \hat{f}^i \), where \( f^i \) corresponds a 1-form in some basis. Then \( \hat{f}^i = (-1)^{i-1} f^1 \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \wedge f^N \). We assume that the exterior derivative of a 1-form gives back some linear combination of 2-forms. That is, \( df^i = \frac{1}{2} \sum_{jk} T_{jk}^j f^j \wedge f^k \). We now apply the exterior derivative to \( \hat{f}^i \) to obtain:

\[
d\hat{f}^i = (-1)^{i-1} (df^1 \wedge f^2 \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \wedge f^N - f^1 \wedge df^2 \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \wedge f^N \\
+ (-1)^{i-1} f^1 \wedge df^2 \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \wedge f^N + \ldots \\
+ (-1)^i f^1 \wedge df^2 \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \wedge f^{i-1} \wedge df^j \wedge f^{i+1} \wedge \ldots \wedge f^N )
\]

(7.95)

The alternating sign for the exterior derivatives are a result of the formula:

\( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta \) where \( \alpha \) and \( \beta \) are \( p \) and \( q \) forms, respectively. Note that when the iteration crosses over the “i-th” location, an additional minus sign is picked up. Each of the derivatives that now appear in the wedge products can be dramatically simplified. Note that when we sum over \( j \) and \( k \) in the formula for \( df^i \). Only numbers that contribute are the current iteration and the “i-th” slot. Let’s use this to obtain:

\[
d\hat{f}^i = (-1)^{i-1} (T_{ji}^1 f^1 \wedge f^i \wedge f^2 \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \wedge f^N \\
- f^1 \wedge T_{ji}^2 df^2 \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \wedge f^N \\
+ (-1)^{j-1} f^1 \wedge f^2 \wedge \ldots \wedge f^{j-1} \wedge T_{ji}^j f^j \wedge f^i \wedge f^{j+1} \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \wedge f^N + \ldots \\
+ (-1)^{j} f^1 \wedge f^2 \wedge \ldots \wedge f^{j-1} \wedge f^{i+1} \wedge \ldots \wedge f^{j-1} \wedge T_{ji}^j f^j \wedge f^i \wedge f^{j+1} \wedge \ldots \wedge f^N + \ldots)
\]

(7.96)
After moving the “i-th” form into its missing slot, we obtain:

\[ d\hat{f}^i = (-1)^{i-1} \left( \sum_{j=1}^{i-1} (-1)^{i+j+1} T^j_{ji} (-1)^{i-j+1} f^1 \wedge \ldots \wedge f^i \wedge \ldots \wedge f^N \right) \]

\[ = (-1)^{i-1} \left( \sum_{j=1}^{i-1} (-1)^{i+j} T^j_{ji} + \sum_{j=1}^{N} (-1)^{2j-i} T^j_{ji} \right) f^1 \wedge \ldots \wedge f^N \quad (7.97) \]

\[ = (-1)^{i-1} \left( \sum_{j=1}^{N} (-1)^{i+j} T^j_{ji} \right) f^1 \wedge \ldots \wedge f^N \quad (7.98) \]

\[ = - \left( \sum_{j=1}^{N} T^j_{ji} \right) f^1 \wedge \ldots \wedge f^N \quad (7.100) \]

This completes the proof. This is a useful identity because when combined with the sum in the Laplacian, it says that any derivatives of basis forms necessarily cancel out. This gives a clean and familiar formula even in potentially exotic spaces.

### 7.5.4 Covariant Laplacian

The covariant exterior derivative is a modified exterior derivative that includes a connection, \( A \). The connection allows one to compare different tangent spaces on a manifold in a meaningful way. It is the primary recipe used to construct gauge symmetric Lagrangians. When working over manifolds with a Lie group structure or in curved spaces, the covariant derivative must be used in place of the regular derivative. The Laplacian for a non-Euclidean space must reflect this.

The essential idea is that all instances of “\(d\)” must be replaced with “\(d + A\)”. This expression is sometimes written as \(d_V = d + A\). As seen previously, the exterior derivative of a function is a 1-form. This means that the connection “\(A\)” must also be a 1-form. In the context of gauge theory, the connection \( A \) is also Lie algebra valued. Just like before, we will calculate each step and comment on the results.
The covariant exterior derivative of a function $\psi$ is given by:

\[ d_{\nabla} \psi = d\psi + A\psi \]  
\[ = \sum_{\alpha} \left( f_{\alpha} \psi + \sum_i A_{\alpha}^i \psi e_i \right) f^\alpha \]  
\[ = \sum_{\alpha} \nabla_{\alpha} \psi f^\alpha \]

The second term is Lie algebra valued, hence the $e_i$ attached. The significance of this cannot be overstated. This term gives the “correction” for performing derivatives in a curved space, or in another context, in a space with a Lie group structure. In classical electrodynamics, the connection is synonymous with the vector potential.

We now perform a Hodge star operation to get:

\[ \star d_{\nabla} \psi = \sum_{\alpha} \nabla_{\alpha} \psi \star f^\alpha \]  
\[ = \sum_{\alpha} (\nabla_{\alpha} \psi) \sum_{\beta} g^{\alpha\beta} h \widehat{f}^3 \]  
\[ = \sum_{\alpha\beta} (\nabla_{\alpha} \psi) g^{\alpha\beta} h \widehat{f}^3 \]

The Hodge star operation for a 1-form returns an ‘N-1’-form. This is denoted by the hat over the form. In general, if the metric isn’t diagonal, the Hodge star operation will return a sum of ‘N-1’-forms with inverse metric elements $g^{\alpha\beta}$ as coefficients. As usual, the volume element $h$ is attached to the ‘N-1’-form basis elements.
We now perform the final exterior derivative operation:

\[ d\nabla \star d\nabla \psi = \sum_{\gamma} \nabla_{\gamma} \left( \sum_{\alpha\beta} (\nabla_{\alpha} \psi) g^{\alpha\beta} h \hat{f}^\beta \right) \]  
(7.107)

\[ = \sum_{\alpha\beta\gamma} \nabla_{\gamma} \left( (\nabla_{\alpha} \psi) g^{\alpha\beta} h \hat{f}^\beta \right) \]  
(7.108)

\[ = \sum_{\alpha\beta\gamma} \nabla_{\gamma} \left( (\nabla_{\alpha} \psi) g^{\alpha\beta} h \hat{f}^\beta \right) f^\gamma \wedge \hat{f}^\beta + \sum_{\alpha\beta\gamma} \left( (\nabla_{\alpha} \psi) g^{\alpha\beta} h \right) \nabla_{\gamma} \hat{f}^\beta \]  
(7.109)

\[ = \sum_{\alpha\beta} \nabla_{\beta} \left( (\nabla_{\alpha} \psi) g^{\alpha\beta} h \right) f^\beta \wedge \hat{f}^\beta \]  
(7.110)

The sum over gamma collapses to values of where gamma and beta agree.

The second term vanishes due to the Lemma proven above and the part involving the connection vanishes due to the sum over all of the Lie algebra elements. This then gives agreement with what might be expected for a covariant Laplacian.

The last step involves a Hodge star operation:

\[ \star d\nabla \star d\nabla \psi = \sum_{\alpha\beta} \nabla_{\beta} \left( (\nabla_{\alpha} \psi) g^{\alpha\beta} h \right) \star f^\beta \wedge \hat{f}^\beta \]  
(7.111)

\[ = \frac{1}{h} \sum_{\alpha\beta} \nabla_{\beta} \left( (\nabla_{\alpha} \psi) g^{\alpha\beta} h \right) \]  
(7.112)

### 7.6 Appendix: The Yang-Mills Equation

The Yang-Mills equation is given by \( J = \star d\nabla \star F \) where \( F = dA + A \wedge A \). Here, \( A \) is a connection for some bundle, \( F \) is the bundle curvature or "field strength", and \( J \) is the source of the bundle curvature or "current". Originally developed in 1954, the equation has gone on to be of central importance in developing the Standard Model of Particle Physics. The bundle for the Standard Model consists of Minkowski space-time as a base manifold and SU(3) × SU(2) × U(1) as a gauge group. By allowing a general bundle with a Riemannian structure, one can develop more general Yang-Mills
equations. This appendix aims to take the Yang-Mills equations in their coordinate free differential forms and convert them into a more useful coordinate form. By writing them in this manner, some interesting mathematical and physical features become more transparent. This form also allows the equation to be utilized by a computer code.

The derivation begins with considering some connection \( A = A(x) \) where \( x \) represents the variables that make up the base manifold. We assume also that the connection is a Lie algebra valued one form. That is: \( A = A^a_\alpha e_a f^\alpha \). Below, we calculate \( F \):

\[
F = dA + A \wedge A \tag{7.113}
\]

\[
= d(A^a_\alpha e_a f^\alpha) + A^b_\beta e_b f^3 \wedge A^c_\gamma e_c f^\gamma \tag{7.114}
\]

\[
= d(A^a_\alpha e_a \wedge f^\alpha + A^a_\alpha e_a df^\alpha + A^b_\beta A^c_\gamma [e_b, e_c] f^3 \wedge f^\gamma) \tag{7.115}
\]

\[
= f_\beta (A^a_\alpha e_a f^3 \wedge f^\alpha + T^a_{\beta \gamma} A^a_\alpha e_a f^3 \wedge f^\gamma + S^b_{\alpha \beta} A^b_\gamma e_c f^\alpha \wedge f^3) \tag{7.116}
\]

\[
= f_\beta (A^a_\alpha e_a f^3 \wedge f^\alpha + T^a_{\beta \gamma} A^a_\alpha e_a f^3 \wedge f^\gamma + S^b_{\alpha \beta} A^b_\gamma e_c f^\alpha \wedge f^3) \tag{7.117}
\]

\[
= \frac{1}{2} [f_\alpha (A^a_\beta) - f_\beta (A^b_\alpha) + T^a_{\beta \gamma} A^a_\alpha + S^b_{\alpha \beta} A^b_\gamma] e_a f^\alpha \wedge f^3 \tag{7.118}
\]

\[
= \frac{1}{2} F^a_{\alpha \beta} e_a f^\alpha \wedge f^\beta \tag{7.119}
\]

The field strength is a traceless anti-symmetric rank-2 tensor. This agrees with the standard field theory literature. The key feature that separates this field strength from the typical expression is the addition \( T \) term that appears. This arises because non-coordinate forms can be differentiable. This does not occur in the usual cases for field theory, as the forms are taken from the coordinate Minkowski spacetime.

There are several checks that can be performed to ensure the generality of the expression for the field strength. By choosing the case where the gauge group is
U(1) and the base manifold is Minkowski space, the expression for the field strength reduces to the Faraday tensor for electrodynamics. Similarly, when one chooses U(2) as the gauge group and the base manifold to be Minkowski space, one obtains the electroweak theory field strength. We now calculate the expression for the Yang-Mills equation, namely \( J = \ast d_\nabla \ast F \).

We begin by calculating the Hodge star operator of the field strength tensor:

\[
\ast F = \frac{1}{2} F_{\alpha\beta}^a \epsilon_a \ast (f^\alpha \wedge f^\beta) \tag{7.120}
\]

\[
= \frac{1}{2} F_{\alpha\beta}^a g^{\alpha\beta} \epsilon_a f^\alpha \wedge f^\beta \tag{7.121}
\]

\[
= \frac{1}{2} \tilde{F}_{\alpha\beta}^a \epsilon_a f^\alpha \wedge f^\beta \tag{7.122}
\]

The reader is reminded that \( f^\alpha \wedge f^\beta = (-1)^{\alpha+\beta-1} f^1 \wedge f^2 \wedge \ldots \wedge f^{\alpha-1} \wedge f^{\alpha+1} \wedge \ldots \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \wedge f^N \). The tilde notation serves as a reminder that the field strength tensor gets rescaled by \( h \) upon taking the Hodge star operation. Moreover, the inverse of the metric is present, but the labels \( \alpha \) and \( \beta \) aren't raised up in accordance with Einstein notation. This is to avoid overcrowding of the notation, so it is absorbed into the field strength. If it becomes beneficial to reintroduce it, this will be noted.

Next we calculate the exterior covariant derivative:

\[
d_\nabla \ast F = d \left( \frac{1}{2} \tilde{F}_{\alpha\beta}^a \epsilon_a f^\alpha \wedge f^\beta \right) + [A, \frac{1}{2} \tilde{F}_{\alpha\beta}^a \epsilon_a f^\alpha \wedge f^\beta] \tag{7.123}
\]

\[
= \frac{1}{2} f_\gamma (\tilde{F}_{\alpha\beta}^a) \epsilon_a f^\gamma \wedge f^\alpha \wedge f^\beta \tag{7.124}
\]

\[
+ \frac{1}{2} \tilde{F}_{\alpha\beta}^a \epsilon_a d(f^\alpha \wedge f^\beta) + [A_\gamma, \frac{1}{2} \tilde{F}_{\alpha\beta}^a \epsilon_a f^\alpha \wedge f^\beta] \tag{7.125}
\]

\[
= \frac{1}{2} f_\gamma (\tilde{F}_{\alpha\beta}^a) \epsilon_a f^\gamma \wedge f^\alpha \wedge f^\beta \tag{7.124}
\]

\[
+ \frac{1}{2} \tilde{F}_{\alpha\beta}^a \epsilon_a d(f^\alpha \wedge f^\beta) + \frac{1}{2} S_{\alpha\beta}^a A_\gamma \tilde{F}_{\alpha\beta}^a \epsilon_a f^\gamma \wedge f^\alpha \wedge f^\beta \tag{7.126}
\]

\[
= \frac{1}{2} f_\gamma (\tilde{F}_{\alpha\beta}^a) + \frac{1}{2} S_{\alpha\beta}^a A_\gamma \tilde{F}_{\alpha\beta}^a \epsilon_a f^\gamma \wedge f^\alpha \wedge f^\beta + \frac{1}{2} \tilde{F}_{\alpha\beta}^a \epsilon_a d(f^\alpha \wedge f^\beta)(7.126)
\]
Finally, we take the Hodge star operation to obtain $J$:

$$J = \frac{1}{2} (\varepsilon_\gamma (\hat{F}_a^{\alpha \beta}) + S^{a}_{bc} A^b_{\alpha} \hat{F}^c_{\alpha \beta}) \varepsilon_\alpha (f^\gamma \wedge f^\alpha \wedge f^\beta) + \frac{1}{2} \hat{F}^{\alpha \beta}_{\alpha \beta} \varepsilon_\alpha \star d(f^\alpha \wedge f^\beta) \tag{7.127}$$

Since the majority of physical symmetries used will have continuous symmetries (i.e. Lie groups), we can set $T_{\alpha \gamma} = T_{\gamma \beta} = 0$. This then gives the generalized Yang-Mills equation:

$$J = \left( f_\alpha (\hat{F}^{a}_{\alpha \beta}) + S^{a}_{bc} A^b_{\alpha} \hat{F}^c_{\alpha \beta} \right) \varepsilon_\alpha \frac{g_{33}}{\hbar} f^3 + \hat{F}^{\alpha \beta}_{\alpha \beta} \varepsilon_\alpha \sum_{i=1}^{i} T_{\alpha \gamma} \frac{g_{ii}}{\hbar} f^i \tag{7.128}$$

$$= \left( f_\alpha (\hat{F}^{a}_{\alpha \beta}) + S^{a}_{bc} A^b_{\alpha} \hat{F}^c_{\alpha \beta} + T_{\alpha \gamma} \hat{F}^{a}_{\alpha \beta} \right) \varepsilon_\alpha \frac{g_{33}}{\hbar} f^3 \tag{7.129}$$

Note that this equation illuminates each of the ideas present: a covariant derivative and a derivative of the basis forms. The covariant derivative appears to ensure that the gauge condition is satisfied and the basis derivatives appear due to the non-coordinate nature of the choice of base manifold. The factor $\hbar$ also appears in the numerator and denominator and acts as a kind of scaling term.

### 7.7 Lemmas

#### 7.7.1 Lemma #1

Given an $N$-form $\omega$ consisting of basis forms $f^i$ over a Riemannian manifold $M$, the following expression holds:

$$\star (f^\gamma \wedge f^\alpha \wedge f^\beta) = \delta_{\gamma \beta} \frac{g_{33}}{\hbar} f^\alpha - \delta_{\gamma \alpha} \frac{g_{33}}{\hbar} f^3 \tag{7.130}$$

Proof: Let $\omega$ be an $N$-form. Let $f^\alpha$, $f^\beta$, and $f^\gamma$ be one forms that are normalized. For this discussion, Einstein summation is not used and $\alpha$, $\beta$, and $\gamma$ are
held fixed. Then the quantity \( \hat{f}^\alpha \wedge f^3 \) is an \( N-2 \) form, \( f^\alpha \wedge \hat{f}^\alpha \wedge f^3 \) is an \( N-1 \) form, and finally \( * ( f^\alpha \wedge \hat{f}^\alpha \wedge f^3 ) \) is a 1-form. This ensures that the dimensions of the proof work. Next we wish to calculate \( \hat{f}^\alpha \wedge f^3 \). To do this write:

\[
\begin{align*}
\hat{f}^\alpha \wedge f^3 \wedge \hat{f}^\alpha \wedge f^3 &= f^1 \wedge f^2 \wedge ... \wedge f^{\alpha-1} \wedge f^\alpha \\
&\quad \wedge f^{\alpha+1} \wedge ... \wedge f^{3-1} \wedge f^3 \wedge f^{3+1} \wedge ... \wedge f^N \\
&= (-1)^{\alpha+3-1} f^\alpha \wedge f^3 \wedge f^1 \wedge f^2 \wedge ... \wedge f^{\alpha-1} \\
&\quad \wedge f^{\alpha+1} \wedge ... \wedge f^{3-1} \wedge f^{3+1} \wedge ... \wedge f^N
\end{align*}
\]

(7.131)

(7.132)

Canceling the forms on both sides then gives:

\[
\hat{f}^\alpha \wedge f^3 = (-1)^{\alpha+3-1} f^1 \wedge f^2 \wedge ... \wedge f^{\alpha-1} \wedge f^{\alpha+1} \wedge ... \wedge f^{3-1} \wedge f^{3+1} \wedge ... \wedge f^N
\]

(7.133)
We now wish to calculate \( f^\gamma \wedge \widehat{f^\alpha} \wedge f^3 \). To do this write:

\[
f^\gamma \wedge \widehat{f^\alpha} \wedge f^3 = (-1)^{\alpha + \beta - 1} f^\gamma \wedge f^1 \wedge f^2 \\
\wedge \ldots \wedge f^{\alpha-1} \wedge f^{\alpha+1} \\
\wedge \ldots \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \wedge f^N \\
= (-1)^{\alpha + \beta - 1} (-1)^{\alpha - 1} f^1 \wedge f^2 \\
\wedge \ldots \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \wedge f^N \\
+ (-1)^{\alpha + \beta - 1} (-1)^{\beta - 2} f^1 \wedge f^2 \wedge \ldots \\
\wedge f^{\alpha-1} \wedge f^{\alpha+1} \wedge \ldots \wedge f^N \\
= (-1)^{\alpha + \beta - 1} (-1)^{\alpha - 1} f^1 \wedge f^2 \wedge \ldots \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \\
\wedge f^N \\
+ (-1)^{\alpha + \beta - 1} (-1)^{\beta - 2} f^1 \wedge f^2 \wedge \ldots \wedge f^{\alpha-1} \wedge f^{\alpha+1} \\
\wedge \ldots \wedge f^N \\
= (-1)^{\beta} f^1 \wedge f^2 \wedge \ldots \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \wedge f^N \\
+ (-1)^{\alpha - 1} f^1 \wedge f^2 \wedge \ldots \wedge f^{\alpha-1} \wedge f^{\alpha+1} \wedge \ldots \wedge f^N \\
= (-1)^{\beta} (\widehat{f^\gamma} + (-1)^{\alpha - 1} (-1)^{\alpha - 1} \widehat{\widehat{f^\alpha}} \\
= \delta_{\alpha,7} \widehat{f^\alpha} - \delta_{\beta,7} \widehat{f^3} \\
(7.140)

Taking the Hodge star operation of both sides gives:

\[
\star \left( f^\gamma \wedge \widehat{f^\alpha} \wedge f^3 \right) = \delta_{\alpha,7} \star \widehat{f^\alpha} - \delta_{\beta,7} \star \widehat{f^3} \\
(7.141)
\]

We now wish to evaluate \( \widehat{\star f^\alpha} \). Using the definition of the Hodge star operator.
we have that:

\[
\hat{f}^\alpha \wedge \star \hat{f}^\alpha = g(\hat{f}^\alpha, \hat{f}^\alpha) \omega 
\]  
(7.142)

To evaluate this metric, we use the following trick:

\[
g(\omega, \omega) = g(h f^\alpha \wedge \hat{f}^\alpha, h f^\alpha \wedge \hat{f}^\alpha) = h^2 \left[ g(f^\alpha, f^\alpha)g(\hat{f}^\alpha, \hat{f}^\alpha) - g(f^\alpha, \hat{f}^\alpha)^2 \right] 
\]  
(7.144)

This gives an equation for \( g(\hat{f}^\alpha, \hat{f}^\alpha) \):

\[
g(\hat{f}^\alpha, \hat{f}^\alpha) = \frac{\frac{1}{\sqrt{\pi}} g(\omega, \omega) + g(f^\alpha, \hat{f}^\alpha)^2}{g(f^\alpha, f^\alpha)} 
\]  
(7.145)

Note that in all relevant cases for our discussion, we have that \( g(f^\alpha, \hat{f}^\alpha) = 0 \). The only time this term must be considered is when the dimension of the base manifold is two and when the metric is non-orthogonal. So in all other cases:

\[
g(\hat{f}^\alpha, \hat{f}^\alpha) = \frac{g(\omega, \omega)}{h^2 g(f^\alpha, f^\alpha)} = \frac{g_{\alpha\alpha}}{h^2} 
\]  
(7.147)

This then completes our lemma:

\[
\star \left( f^\gamma \wedge f^\alpha \wedge f^\beta \right) = \delta_{\alpha\gamma} \star \hat{f}^\alpha - \delta_{\beta\gamma} \star \hat{f}^\beta = \delta_{\alpha\gamma} \frac{g_{\alpha\alpha}}{h} f^\alpha - \delta_{\beta\gamma} \frac{g_{\beta\beta}}{h} f^\beta 
\]  
(7.149)
7.7.2 Lemma #2

Given an N-form $\omega$ consisting of basis forms $f^i$ (not necessarily coordinate basis forms or orthogonal) over a Riemannian manifold $M$, the following expression holds:

$$ \star d(\widehat{f^\alpha \wedge f^\beta}) = \sum_i T_{\alpha \beta}^i \frac{g_{ii}}{h} f^i $$

(7.150)

where $\alpha$ and $\beta$ are fixed, $\alpha < \beta$, and $df^i = T_{\alpha \beta}^i f^\alpha \wedge f^\beta$.

Proof: Assume that $\alpha$ and $\beta$ are fixed, $\alpha < \beta$, and $df^i = T_{\alpha \beta}^i f^\alpha \wedge f^\beta$. By definition:

$$ \widehat{f^\alpha \wedge f^\beta} = (-1)^{\alpha + \beta - 1} f^1 \wedge ... \wedge f^{\alpha - 1} \wedge f^{\alpha + 1} \wedge ... \wedge f^{\beta - 1} \wedge f^{\beta + 1} \wedge ... \wedge f^N $$

(7.151)

Taking the exterior derivative gives the following three sums:

$$ d(\widehat{f^\alpha \wedge f^\beta}) = (-1)^{\alpha + \beta - 1} \sum_{i < \alpha < \beta} (-1)^{i - 1} f^1 \wedge ... \wedge f^{i - 1} \wedge df^i \wedge f^{i + 1} \wedge ... \wedge f^N $$

$$ + \sum_{\alpha < i < \beta} (-1)^i f^1 \wedge ... \wedge f^{i - 1} \wedge df^i \wedge f^{i + 1} \wedge ... \wedge f^N $$

$$ + \sum_{\alpha < \beta < i} (-1)^{i - 1} f^1 \wedge ... \wedge f^{i - 1} \wedge df^i \wedge f^{i + 1} \wedge ... \wedge f^N $$

(7.152)

The factors of $(-1)^{i - 1}$ are a result of the formula: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$. Since each derivative we are taking is for a 1-form, this introduces a minus sign because $|\alpha| = 1$. As the derivative moves along each successive form, it creates an alternating sign. It’s instructional to look at an example of one of the three sums and simplify it. All three will yield the same expression.
The first sum has the form:

\[
\begin{align*}
& f^1 \wedge \ldots \wedge f^{i-1} \wedge d f^i \wedge f^{i+1} \wedge \ldots \wedge f^{\alpha-1} \wedge f^\alpha + 1 \wedge \ldots \\
& \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \wedge f^N \\
& = f^1 \wedge \ldots \wedge f^{i-1} \wedge T^{i}_{\alpha \beta} f^\alpha \wedge f^\beta \wedge f^{i+1} \wedge \ldots \\
& \wedge f^{\alpha-1} \wedge f^\alpha + 1 \wedge \ldots \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \wedge f^N \\
& = (-1)^{(\alpha-1)-(i+1)+1} f^1 \wedge \ldots \wedge f^{i-1} \wedge T^{i}_{\alpha \beta} f^\beta \wedge f^{i+1} \wedge \ldots \\
& \wedge f^{\alpha-1} \wedge f^\alpha \wedge f^{\alpha+1} \wedge \ldots \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \wedge f^N \\
& = (-1)^{(\alpha-1)-(i+1)+1} f^1 \wedge \ldots \wedge f^{i-1} \wedge T^{i}_{\alpha \beta} f^\beta \wedge f^{i+1} \wedge \ldots \\
& \wedge f^{\beta-1} \wedge f^{\beta+1} \wedge \ldots \wedge f^N \\
& = (-1)^{(\alpha-1)-(i+1)+1} (-1)^{(\beta-1)-(i+1)} T^{i}_{\alpha \beta} f^1 \wedge \ldots \wedge f^{i-1} \wedge f^{i+1} \wedge \ldots \\
& \wedge f^{\beta-1} \wedge f^\beta \wedge f^{\beta+1} \wedge \ldots \wedge f^N \\
& = (-1)^{(\alpha+\beta+1)} T^{i}_{\alpha \beta} \hat{f}^i \\
& = (-1)^{(\alpha+\beta+1)} \sum_{i<\alpha<\beta} (-1)^{i-1} (-1)^{(\alpha+\beta+i)} T^{i}_{\alpha \beta} \hat{f}^i = \sum_{i<\alpha<\beta} T^{i}_{\alpha \beta} \hat{f}^i \\
\end{align*}
\]

(7.153)  (7.154)  (7.155)  (7.156)  (7.157)  (7.158)  (7.159)  (7.160)

Inserting this into the sum:

\[
(-1)^{\alpha+\beta-1} \sum_{i<\alpha<\beta} (-1)^{i-1} (-1)^{(\alpha+\beta+i)} T^{i}_{\alpha \beta} \hat{f}^i = \sum_{i<\alpha<\beta} T^{i}_{\alpha \beta} \hat{f}^i \\
\]

(7.161)

All three of the sums give the same expression. so summing over all i gives:

\[
d(f^\alpha \wedge f^\beta) = \sum_i T^{i}_{\alpha \beta} \hat{f}^i \\
\]

(7.162)
Borrowing our earlier result for taking the Hodge star, we get:

\[ \ast d(f^\alpha \wedge f^\beta) = \sum_i T^i_{\alpha \beta} \ast \hat{f}^i \]  
(7.163)

\[ = T^i_{\alpha \beta} \frac{g_{ii}}{h} f^i \]  
(7.164)

Note that when \( \alpha \) and \( \beta \) are interchanged that the equation still holds. This allows us to freely sum over any values of \( \alpha \) and \( \beta \) in our original Yang-Mills expression.

### 7.8 Appendix: The Wigner-Eckart Theorem, B(E2) Transitions, and Deformation Parameters

The Wigner-Eckart theorem is a central mathematical tool in nuclear structure physics. Simply put, the Wigner-Eckart theorem allows any spherical tensor operator over an angular momentum basis to be expressed as the product of a Clebsch-Gordan coefficient and a reduced matrix element. The utility of this theorem can not be overstated. Mathematically, the theorem is stated as:

\[ \langle jm | T_q^{(k)} | j'm' \rangle = (j'm'kq|jm) \frac{\langle j | T^{(k)} | j' \rangle}{\sqrt{2j+1}} \]  
(7.165)

Here, \( T_q^{(k)} \) is the q-th component of the tensor operator \( T^{(k)} \) of rank k. \( |jm\rangle \) is an eigenstate of the angular momentum operators \( \hat{J}^2 \) and \( \hat{J}_z \). \( (j'm'kq|jm) \) is a Clebsch-Gordan coefficient, and \( \langle j | T^{(k)} | j' \rangle \) is a reduced matrix element. The square root factor is included to give the reduced matrix element an added degree of symmetry. The proof is omitted but many texts on quantum mechanics provide it. To see how the theorem works, let's apply it to the most important case: the quadrupole moment. We begin by writing the expectation of the quadrupole moment over an...
angular momentum basis:

\[ Q_o = e \langle jm = j | (3z^2 - r^2) | jm = j \rangle \]
\[ = e \langle jm = j | (2z^2 - x^2 - y^2) | jm = j \rangle \]
\[ = e \langle jm = j | \sqrt{\frac{16\pi}{5}} r^2 Y_0^2 | jm = j \rangle \]
\[ = e r^2 \langle jm = j | \sqrt{\frac{16\pi}{5}} T_0^2 | jm = j \rangle \]
\[ = e r^2 \sqrt{\frac{16\pi}{5}} (j2j0|jj) \frac{\langle j|\hat{T}^{(2)}|j \rangle}{\sqrt{2j + 1}} \]

This result shows the power of the theorem, namely that the reduced matrix element depends only on \( j \). At times, the reduced matrix element can actually be calculated when the matrix element on the left can be calculated. For the quadrupole moment example above, this can be shown to be:

\[ Q_o = e \langle j'm'k'| (3z^2 - r^2) | jmk \rangle \]
\[ = e \langle j'm'k'| \sqrt{\frac{16\pi}{5}} r^2 Y_0^2 | jmk \rangle \]
\[ = e r^2 \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi \mathcal{D}_{m'k'}^{j'} \mathcal{D}_{k0}^j \mathcal{D}_{mk}^i \sin(\beta) d\beta d\Omega \]
\[ = \frac{1}{5} e r^2 (j'm'jm|20)(j'k'jk|20) \]

Here we’ve used the identity:

\[ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \mathcal{D}_{M,K}^J \mathcal{D}_{m_1,k_1}^{j_1} \mathcal{D}_{m_2,k_2}^{j_2} = \frac{8\pi^2}{2J + 1} (j_1 m_1, j_2 m_2 |JM)(j_1 k_1, j_2 k_2 |JK) \]

An excellent resource for the Wigner d-functions is given by [4] and [1]. Experimentally, the E2 (electric quadrupole) transition rates are what are measured. The Fermi
Golden Rule relates the E2 transition rates directly to the quadrupole moment, which is further related to the deformation parameter, $\beta$. Fermi's Golden Rule [27] states that the transition rate from an initial quantum state to a final quantum state is given by:

$$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} |M_{ij}|^2 \rho_f$$  \hspace{1cm} (7.176)

where $M_{ij}$ is the matrix element associated with the coupling of the initial and final states, and $\rho_f$ is the density of states for the final state. For even-even nuclei, the relationship between the $B(E2)$ transition rates and the angular moment $J$ values is given by:

$$B(E2, J \rightarrow J - 2) = \frac{15}{32\pi} e^2 Q_0^2 \frac{J(J-1)}{(2J+1)(2J-1)}$$ \hspace{1cm} (7.177)

This formula is a direct consequence of the Wigner-Eckart theorem and Fermi's Golden Rule. The rational function involving $J$ comes from square of Clebsch-Gordan coefficients. It allows us to figure out the quadrupole moment within the $K = 0$ band and also acts as a test to determine if a band is rotational. Those that are rotational will conform to this particular formula. Typically, by looking at ratios of the transition rates we determine if a band is rotational. There are similar formulas that exist that could be used for different energy bands with different $K$ values, but this is sufficient for the purposes of this work. As an example, consider the $B(E2)$ transition rates for Erbium-166 found on the Brookhaven nuclear data website [32] and the ratio of the Clebsch-Gordan coefficients:
Table 7.1: Ratio of Experimental values of B(E2) rates to Clebsch-Gordan coefficients squared for Er-166

<table>
<thead>
<tr>
<th>$J^+$</th>
<th>$B(E2, J \rightarrow J - 2)$W.u.</th>
<th>$CG^2$</th>
<th>$B(E2)/CG^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^+$</td>
<td>$217 \pm 5$</td>
<td>0.133</td>
<td>1.630</td>
</tr>
<tr>
<td>$4^+$</td>
<td>$312 \pm 11$</td>
<td>0.190</td>
<td>1.640</td>
</tr>
<tr>
<td>$6^+$</td>
<td>$370 \pm 20$</td>
<td>0.210</td>
<td>1.760</td>
</tr>
<tr>
<td>$8^+$</td>
<td>$373 \pm 14$</td>
<td>0.220</td>
<td>1.700</td>
</tr>
<tr>
<td>$10^+$</td>
<td>$390 \pm 17$</td>
<td>0.226</td>
<td>1.730</td>
</tr>
</tbody>
</table>

Notice that the fourth column doesn't change appreciably across angular momentum values. The ratio is rounded to three significant figures. According to theory, the ratios would all be identical, however, due to various physical effects like 'centrifugal stretching' and other factors the data deviates from the ideal. Notice that the $B(E2)$ transition rates are recorded in Weisskopf units. These units are essentially a dimensionless number that provide a ratio of the experimental $B(E2)$ transition rates from experiment to that of a single particle estimate. They are useful in measuring collectivity in nuclei because for large values of the ratio, the $B(E2)$ transition rates occur due to the effects of numerous nucleons instead of a single nucleon. Closer to closed shells, the $B(E2)$ transition rate is very small, as would be expected due to the small number of nucleons involved in the transitions. It should be noted that according to the shell model, numerous nuclei actually participate in electromagnetic quadrupole transitions, however, due to the fact that the nucleons are indistinguishable the effects are essentially the same as if one nucleon was involved. A Weisskopf unit is based on the Weisskopf estimate:

$$B(E\lambda; J_i \rightarrow 0) = \frac{(1.2)^{2\lambda}}{4\pi} \left( \frac{3}{\lambda + 3} \right)^2 A^{2\lambda/3} e^2 (fm)^{2\lambda} \tag{7.178}$$

To obtain the unit for the electric quadrupole transition, we set $\lambda = 2$, and evaluate the estimate at the number of nucleons $A$ for the nucleus under discussion, which
for the case above is $A = 166$. It is often the case that one must covert square femtometers to barns. We can also use the data in the table to directly measure the deformation parameter by knowing the intrinsic quadrupole moment $Q_0$. This is given by the relationship (see Krane):

$$Q_0 = \frac{3}{\sqrt{5\pi}} R_0^2 Z \beta (1 + 0.16\beta)$$  \hspace{1cm} (7.179)

Here, $R_0 = (1.2A^{1/3} fm)$ and $Z$ is the number of protons. Most values of the deformation parameter $\beta$ are around 0.3 when far away from closed shells. So to first order, $Q_0 = \frac{3}{\sqrt{5\pi}} R_0^2 Z \beta$.

### 7.9 Appendix: Irrotational Moment of Inertia Derivation

![Figure 7.2: Moments of Inertia for each frame](image)

In this appendix, a derivation of the irrotational moment of inertia is derived. This derivation follows (Bohr and Mottelson). By definition, irrotational flow means that for some fluid that its velocity field satisfies $\vec{\nabla} \times \vec{v} = 0$. If we assume that the fluid has constant density (i.e. that the fluid is incompressible), the continuity equation, $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$, implies that $\vec{\nabla} \cdot \vec{v} = 0$. These imply that $\vec{v} = -\vec{\nabla} \chi$ for
some potential field $\chi$ and that $\nabla^2 \chi = 0$. The velocity field in the intrinsic frame (body fixed frame) is given by $\vec{v} = \vec{\tau} - \vec{\omega} \times \vec{r}$, where $\vec{\tau}$ indicates the lab frame. Since the fluid is treated as having a well defined boundary, we can impose the condition that $\vec{v} \cdot \vec{n} = 0$ along the boundary ($r = R$). Next we require that the boundary of the fluid is shaped like an ellipse whose equation is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$  \hfill (7.180)

By calculating the normal vector to the surface of this ellipsoidal shape, we can recover the necessary form that $\chi$ must take on. Set $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ and calculate the gradient to obtain the (non-normalized) normal vector and impose the boundary condition to get:

$$0 = \vec{v} \cdot \nabla F$$  \hfill (7.181)

$$0 = \left( -\nabla \chi - \vec{\omega} \times \vec{r} \right) \cdot \nabla F$$  \hfill (7.182)

$$\nabla \chi \cdot \nabla F = - \left( \vec{\omega} \times \vec{r} \right) \cdot \nabla F$$  \hfill (7.183)

$$\frac{x}{a^2} \frac{\partial \chi}{\partial x} + \frac{y}{b^2} \frac{\partial \chi}{\partial y} + \frac{z}{c^2} \frac{\partial \chi}{\partial z} = \left( \frac{c^2}{a^2 b^2} \right) y z \omega_y + \left( \frac{c^2 - a^2}{c^2 a^2} \right) x z \omega_z + \left( \frac{b^2 - a^2}{b^2 a^2} \right) x y \omega_z$$  \hfill (7.184)

To obtain a solution, we make a guess that the potential $\chi$ must take on the form of a second degree polynomial in the spatial variables:

$$\chi = c_1 x^2 + c_2 y^2 + c_3 z^2 + c_4 y z + c_5 x z + c_6 x y + c_7 x + c_8 y + c_9 z + c_{10}$$  \hfill (7.185)

Applying the incompressibility condition eliminates the first three coefficients and inspection eliminates the last four coefficients. This suggests that $\chi = c_4 y z + c_5 x z + c_6 x y$. Plugging this guess into the lefthand side of the boundary differential equation
gives:

\[
\frac{x}{a^2}(c_5z + c_6y) + \frac{y}{b^2}(c_4z + c_6x) + \frac{z}{c^2}(c_4y + c_5x) = \left(\frac{c^2 - a^2}{c^2b^2}\right)y\omega_x + \left(\frac{c^2 - a^2}{c^2a^2}\right)x\omega_y + \left(\frac{b^2 - a^2}{b^2a^2}\right)x\omega_z
\]  

(7.186)

Coefficient matching leads to the result:

\[
\chi = \left(\frac{c^2 - b^2}{c^2 + b^2}\right)y\omega_x + \left(\frac{c^2 - a^2}{c^2 + a^2}\right)x\omega_y + \left(\frac{b^2 - a^2}{b^2 + a^2}\right)x\omega_z
\]  

(7.187)

The volume of an ellipsoid is given by making the change of variables:

\[
x = arsin(\theta)cos(\phi), \quad y = brsin(\theta)sin(\phi), \quad z = crcos(\theta)
\]  

(7.188)

The volume of an ellipsoid is then given by:

\[
V = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{1} abc r^2sin(\theta) \, dr \, d\theta \, d\phi = \frac{4\pi}{3}abc
\]  

(7.189)

The kinetic energy is given by:

\[
T = \frac{1}{2} \int \rho(\nabla \chi)^2 dV
\]  

(7.190)

The kinetic energy of a rotating body in terms of its moment of inertia is given by

\[
T = \frac{1}{2} I\omega^2
\]  

Setting these two expressions equal to one another gives:

\[
I_c^{IF} = \frac{1}{5} M \left(\frac{b^2 - a^2}{b^2 + a^2}\right)^2, \text{ for a. b. c cyclic}
\]

(7.191)

\[
I_c^{RR} = \left(\frac{b^2 - a^2}{b^2 + a^2}\right) I_c^{RR}
\]

(7.192)

This last expression provides a relationship between the irrotational and rigid rotation.
moments of inertia. It is often conventional to drop coefficients like $\frac{1}{3}M$ for the sake of clarity. This convention was used in prior chapters. As a final note, to touch base with microscopic physics, see [24].

7.10 Mathematica Codes

Here are the codes that were created using Mathematica.

7.10.1 Ellipsoid Animation

The first code gives an animation of how rotation and circulation might look.

\[a_1 = 10; \ a_2 = 30;\]
\[r = .25;(*\text{Irrotational Flow}*)\]
\[\Gamma = r*(2*a_1*a_2)/(a_1^2 + a_2^2) + (1 - r)*(a_1^2 + a_2^2)/(2*a_1*a_2);(*\text{Only around the z axis}\)\]
\[\Omega = 10;\]
\[\Lambda = N[\Gamma*\Omega];\]
\[(*\text{\[\Lambda = 0;\})*}\]
\[R(t) := (\{\]
\[\text{Cos}[\Omega t], -\text{Sin}[\Omega t]\),\]
\[\text{Sin}[\Omega t], \text{Cos}[\Omega t]\}\]
\[);\]
\[A = (\{\]
\[\{a_1, 0\},\]
\[\{0, a_2\}\];\]
\[S(t) := (\{\]
\[\text{Cos}[\Lambda t], \text{Sin}[\Lambda t]\},\]
\[\text{Sin}[\Lambda t], \text{Cos}[\Lambda t]\}\);\]
\{-\text{Sin}[\text{\lambda}\cdot t], \text{Cos}[\text{\lambda}\cdot t]\} \\
);

\text{time} = 10;
\text{step} = 0.01;
\text{data} = \{};
\text{dot} = \{};
\text{dat} = \{};
\text{For}[n = 0, n < 500, 
  n++; \{x = \text{RandomReal}[\{-1, 1\}], y = \text{RandomReal}[\{-1, 1\}];
  \text{If}[x^2 + y^2 <= 1, 
    \text{AppendTo}[\text{data}, 
      \text{Table}[\text{N[R[t].A.S[t].{x, y}], \{t, 0, \text{time, step}\}]]];
  \text{AppendTo}[\text{dot}, \text{Table}[\text{N[R[t].A.S[t].{0, 1}], \{t, 0, \text{time, step}\}]]];
  \text{AppendTo}[\text{dat}, \text{Table}[\text{N[R[t].A.S[t].{0, 0.5}], \{t, 0, \text{time, step}\}]]];
\text{Manipulate[}
\text{ListPlot}[\{\text{data[[1 ;; Length[data]], u]], \{\text{dot[[1]][[u]]}, \{\text{dat[[1]][[u]]}\}, \text{Axes} \to \text{True,} 
\text{PlotRange} \to \{-\text{a2, a2}, \{-\text{a2, a2}\}}, 
\text{PlotStyle} \to \{\text{Blue, PointSize[Large], PointSize[Large]}, 
\text{AspectRatio} \to \text{Automatic}, \{u, 1, \text{N[time/step + 1], 1}\}
\text{7.10.2 Hill-Wheeler Quadrupole Moments}
\text{*Hill-Wheeler Constant Volume*)}
\text{a2} = \text{R*E^-(Sqrt[5/(4 \[Pi])]*[Beta]*Cos[\[\gamma] - 2 \[Pi]/3]);}
\text{b2} = \text{R*E^-(Sqrt[5/(4 \[Pi])]*[Beta]*Cos[\[\gamma] + 2 \[Pi]/3]);}
\text{c2} = \text{R*E^-(Sqrt[5/(4 \[Pi])]*[Beta]*Cos[\[\gamma]));}
FullSimplify[
  Series[a2\[CenterDot]b2\[CenterDot]c2, \{Beta, 0, 2\}\}\{1\}(*Volume*)
]

FullSimplify[
  Series[Sqrt[\Pi/5]*(2*c2\^2 - a2\^2 - b2\^2)/(a2\^2 + b2\^2 + c2\^2), \{Beta, 0, 2\}\}\{1\}(*q_ 0*)
]

FullSimplify[
  Series[Sqrt[3/2]*Sqrt[\Pi/5]*(a2\^2 - b2\^2)/(a2\^2 + b2\^2 + c2\^2), \{Beta, 0, 2\}\}\{1\}(*q_ 2*)
]

### 7.10.3 Bohr-Mottelson Volume Element

This code is what led to the breakthrough for Chapter 5.

FullSimplify[
  Series[Sqrt[(2/5)\^3*((4 \Pi/3)/3)\^3*(a2\^2 - b2\^2)\^2/(a2\^2 + b2\^2)*a2\^2 - c2\^2)\^2/(a2\^2 + c2\^2)*(b2\^2 - c2\^2)\^2/(b2\^2 + c2\^2), \{Beta, 0, 6\}\}\{1\}]
]

SeriesData[\Beta, 0, {2 (R\^6 Sin[3 \Gamma]\^2)\^Rational[1, 2], 0, Rational[-15, 16] Pi\^(-1) (R\^6 Sin[3 \Gamma]\^2)\^Rational[1, 2], 3, 7, 1]
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Biography

Nick Sparks was born in Columbus, Ohio on May 5th, 1984 and grew up in London, Ohio. After graduating London High School in 2002, he attended the Ohio State University where he earned bachelor’s degrees in mathematics and physics in 2007. After attending college, he worked at Harvard Medical School as a lab assistant in the Seidman Lab. From Fall of 2008 until May of 2010, Nick attended Tulane University where he earned a Master’s degree in Applied Math where he wrote a thesis under Dr. Ricardo Cortez called “A New Approach to Correlation Analysis in EEG”. Nick then taught math at Delgado Community College until 2012 when he enrolled as a Ph.D. student at Tulane University. Nick is a fan of science fiction and this had a great deal to do with his love for physics and math.