Theory of the generalized modified Bessel function $K_{z,w}(x)$ and 2-adic valuations of integer sequences.

AN ABSTRACT
SUBMITTED ON THE FIFTH DAY OF DECEMBER, 2017
TO THE DEPARTMENT OF MATHEMATICS
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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
BY

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Abstract

Modular-type transformation formulas are the identities that are invariant under the transformation $\alpha \to 1/\alpha$, and they can be represented as $F(\alpha) = F(\beta)$ where $\alpha\,\beta = 1$. We derive a new transformation formula of the form $F(\alpha, z, w) = F(\beta, z, iw)$ that is a one-variable generalization of the well-known Ramanujan-Guinand identity of the form $F(\alpha, z) = F(\beta, z)$ and a two-variable generalization of Koshliakov's formula of the form $F(\alpha) = F(\beta)$ where $\alpha\,\beta = 1$. The formula is generated by first finding an integral $\mathcal{J}$ that is comprised of an invariance function Z and evaluating the integral to give $F(\alpha, z, w)$ mentioned above. The modified Bessel function $K_z(x)$ appearing in Ramanujan-Guinand identity is generalized to a new function, denoted as $K_{z,w}(x)$, that yields a pair of functions reciprocal in the Koshliakov kernel, which in turn yields the invariance function Z and hence the integral $\mathcal{J}$ and the new formula. The special function $K_{z,w}(x)$, first defined as the inverse Mellin transform of a product of two gamma functions and two confluent hypergeometric functions, is shown to exhibit a rich theory as evidenced by a number of integral and series representations as well as a differential-difference equation.

The second topic of the thesis is 2-adic valuations of integer sequences associated with quadratic polynomials of the form x^2+a. The sequence $\{n^2+a : n \in \mathbb{Z}\}$ contains numbers divisible by any power of 2 if and only if a is of the form $4^m(8l+7)$. Applying this result to the sequences derived from the sums of four or fewer squares when one or more of the squares are kept constant leads to interesting results, that also points

to an inherent connection with the functions $r_k(n)$ that count the number of ways to represent n as sums of k integer squares. Another class of sequences studied is the shifted sequences of the polygonal numbers given by the quadratic formula, for which the most common examples are the triangular numbers and the squares.

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Contents

Chapter 1

Introduction

1.1 Transformation formulas involving a generalization of the modified Bessel function $K_z(x)$

For $\mathrm{Re}(s) > 1$, the Riemann zeta function $\zeta(s)$ is defined by the absolutely convergent Dirichlet series,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and can be analytically continued to the entire complex plane except for a simple pole at $s = 1$ with residue 1. The analytical continuation makes use of the functional equation,

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s). \tag{1.1}$$

Here $\Gamma(s)$ is the gamma function defined for $\mathrm{Re}(s) > 0$ by the integral

$$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}\,dx. \tag{1.2}$$

The integral (1.2) defining the gamma function $\Gamma(s)$ is the Mellin transform of the function e^{-x}. The Mellin transform of a function $f(x)$, used often in the thesis, is defined as

$$\mathcal{F}(s) := \int_0^\infty x^{s-1} f(x)\, dx, \tag{1.3}$$

where s is restricted to the values for which the integral (1.3) converges and the inverse Mellin transform is given by [1, p. 33]

$$f(x) = \frac{1}{2\pi i} \int_{(c)} x^{-s} \mathcal{F}(s)\, ds. \tag{1.4}$$

where $\int_{(c)}$ denotes the line integral $\int_{c-i\infty}^{c+i\infty}$ throughout the thesis.

The Riemann ξ-function, defined by

$$\xi(s) := \tfrac{1}{2} s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{s}{2})\zeta(s),$$

is an entire function satisfying $\xi(s) = \xi(1-s)$, which means it is symmetric with respect to the vertical line $\mathrm{Re}(s) = \frac{1}{2}$ in the complex plane. Riemann's Ξ-function, defined by

$$\Xi(t) := \xi(\tfrac{1}{2} + it),$$

is an even function of t.

The following integral evaluation, comprising of the Riemann Ξ-function, is well-known and was used by Hardy [2] to prove the infinitude of the zeros of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = \frac{1}{2}$:

$$\frac{2}{\pi} \int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2} \cos\left(\frac{1}{2} t \log \alpha\right) dt = \sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum_{n=1}^\infty e^{-\pi\alpha^2 n^2}\right) \tag{1.5}$$

Clearly the integral on the left side of the equation (1.5) is invariant under the transformation $\alpha \to 1/\alpha$. This invariance yields the following formula for complex numbers α and β with $\mathrm{Re}(\alpha^2) > 0$, $\mathrm{Re}(\beta^2) > 0$ satisfying $\alpha\beta = 1$:

$$\sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\right) = \sqrt{\beta}\left(\frac{1}{2\beta} - \sum_{n=1}^{\infty} e^{-\pi\beta^2 n^2}\right), \tag{1.6}$$

The right side of the equation (1.5) relates to the theta function $\theta(x)$ in the following manner:

$$\theta(x) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

The theta function $\theta(x)$ satisfies the following transformation property:

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right) \tag{1.7}$$

Letting $x = \alpha^2$ yields the formula (1.6), so henceforth equation (1.6) will be referred as the theta transformation formula.

The transformation property (1.7) for the theta function $\theta(x)$ is reminiscent of the invariance property of modular transformations under inversion. Indeed, in a slightly different form obtained by a change of variable, the property (1.7) corresponds to the inversion $\tau \to -1/\tau$. On account of this similarity with the modular transformation, the formulas of the form $F(\alpha) = F(\beta)$ for $\alpha\beta = k$, where k is a constant, are known as modular-type transformations. Such formulas can be found in the work of Ramanujan, that have inspired the study of integrals involving Riemann Ξ-function that are invariant under the transformation $\alpha \to 1/\alpha$ to generate the modular-type transformation formulas. One of the elegant formulas found on page 220 in Ramanujan's Lost Notebook [3] is given below.

Theorem 1.1.1. *Define*

$$L(x) := \psi(x) + \frac{1}{2x} - \log x$$

where

$$\psi(z) := \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is the logarithmic derivative of the gamma function. If α *and* β *are positive numbers such that* $\alpha\beta = 1$*, then*

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty} L(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty} L(n\beta)\right\}$$
$$= -\frac{1}{\pi^{\frac{3}{2}}} \int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\, dt. \quad (1.8)$$

The evaluation of the integrals of the type

$$F(\alpha) = \int_0^\infty f(t)\cos\left(\frac{1}{2}t\log\alpha\right)\, dt, \quad (1.9)$$

that are invariant under the transformation $\alpha \to 1/\alpha$, by finding their alternate series or integral representation, is a useful technique to derive modular-type transformation formulas. N.S. Koshlikov, known for his work on modular-type transformation formulas, worked with a kernel function given by a combination of trigonometric and Bessel functions. This kernel, proved to be an essential tool in employing the above mentioned technique to derive some of the results in the thesis, will be referred to as Koshlikov kernel. To introduce the kernel, the definitions of the Bessel functions are presented first. The Bessel functions of the first and second kinds of order z, namely $J_z(x)$ and $Y_z(x)$, are defined by [4, p. 40, 64]

$$J_z(x) := \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+z}}{m!\Gamma(m+1+z)}, \quad ||x|| < \infty, \quad (1.10)$$

and

$$Y_z(x) = \frac{J_z(x)\cos(\pi z) - J_{-z}(x)}{\sin \pi z}$$

respectively. The modified Bessel functions of the first and second kinds of order z are defined by [4, p. 77]

$$I_z(x) = \begin{cases} e^{-\frac{1}{2}\pi zi} J_z(e^{\frac{1}{2}\pi i}x), & \text{if } -\pi < \arg x \leq \frac{\pi}{2}, \\ e^{\frac{3}{2}\pi zi} J_z(e^{-\frac{3}{2}\pi i}x), & \text{if } \frac{\pi}{2} < \arg x \leq \pi, \end{cases} \tag{1.11}$$

and [4, p. 78]

$$K_z(x) := \frac{\pi}{2}\frac{I_{-z}(x) - I_z(x)}{\sin \pi z}.$$

The Koshlikov transform of a function $f(x)$ is

$$2\pi \int_0^\infty f(x) \left(\cos(\pi z) M_{2z}(4\pi\sqrt{x}) - \sin(\pi z) J_{2z}(4\pi\sqrt{x})\right) \, dx, \tag{1.12}$$

whenever the integral converges. A pair of functions (φ, ψ) is said to be reciprocal in the Koshlikov kernel if the two functions φ and ψ are Koshlikov transforms of each other as stated explicitly in (2.1) in Chapter 2. Koshliakov proved that, for $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$,

$$2\pi \int_0^\infty K_z(2\pi t) \left(\cos(\pi z) M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z) J_{2z}(4\pi\sqrt{xt})\right) \, dt = K_z(2\pi x). \tag{1.13}$$

Thus the modified Bessel function of the second kind $K_z(x)$ is self-reciprocal in the Koshliakov kernel. This yields the first known pair of reciprocal functions in the Koshliakov kernel viz. $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$, where $\alpha\beta = 1$.

A large number of the integrals studied for the modular-type transformations comprises of the Riemann Ξ-function or a product of two Ξ-functions in the integrand. In this thesis, we restrict our discussion to such integrals comprising of Riemann Ξ-function(s) that generates the modular-type transformation formulas and can be broadly classified into two categories, both comprising of another function that is invariant under the transformation $\alpha \to 1/\alpha$, called as the invariance function. Henceforth α and β satisfy $\alpha\beta = 1$ throughout the thesis. The first family of integrals, denoted as $\mathcal{I}(h, \nabla; w, \alpha)$, is given by

$$\mathcal{I}(h, \nabla; w, \alpha) := \int_0^\infty h\left(\frac{t}{2}\right) \Xi\left(\frac{t}{2}\right) \nabla\left(\alpha, w, \frac{1+it}{2}\right) dt, \tag{1.14}$$

where $h(t) = |\phi(it)|^2 = \phi(it)\phi(-it)$ is an even function of t, for some analytic function ϕ and the function ∇ satisfying the invariance property $\nabla(\alpha, w, s) = \nabla(\beta, iw, s)$ is the sum of the normalized Mellin transforms of a pair of functions reciprocal in the Fourier cosine transform, defined explicitly in Chapter 2. The evaluation of the integrals $\mathcal{I}(h, \nabla; w, \alpha)$ gives rise to the formulas of the type $F(\alpha, w) = F(\beta, iw)$, where w can be identically equal to zero. When w is identically equal to zero, the invariance function $\nabla\left(\alpha, 0, \frac{1+it}{2}\right)$ reduces to $2\cos\left(\frac{1}{2}t\log\alpha\right)$.

The integral (1.5) associated with the theta transformation formula (1.6) is an example of the first kind $\mathcal{I}(h, \nabla; w, \alpha)$ with $w = 0$, $\phi(t) = \frac{1}{t+\frac{1}{2}}$ so that $h\left(\frac{t}{2}\right) = \frac{4}{1+t^2}$ and $\nabla = 2\cos\left(\frac{1}{2}t\log\alpha\right)$. A variety of integrals of the kind $\mathcal{I}(h, \nabla; w, \alpha)$ are studied by Ramanujan [5], Koshliakov [6], G. H. Hardy [2], W. L. Ferrar [7] and lately by A. Dixit [8]. A transformation of the kind $F(\alpha, w) = F(\beta, iw)$ that is of particular interest in the thesis is the generalization of the theta transformation formula

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2}\nabla\left(\alpha, w, \frac{1+it}{2}\right) dt &= \sqrt{\alpha}\left(\frac{e^{-\frac{w^2}{8}}}{2\alpha} - e^{\frac{w^2}{8}}\sum_{n=1}^\infty e^{-\pi\alpha^2 n^2}\cos(\sqrt{\pi}\alpha n w)\right)\\ &= \sqrt{\beta}\left(\frac{e^{\frac{w^2}{8}}}{2\beta} - e^{-\frac{w^2}{8}}\sum_{n=1}^\infty e^{-\pi\beta^2 n^2}\cos(i\sqrt{\pi}\beta n w)\right),\end{aligned} \tag{1.15}$$

for $w \in \mathbb{C}$, where

$$\begin{aligned}\nabla(\alpha, w, s) &:= \rho(\alpha, w, s) + \rho(\alpha, w, 1-s),\\ \rho(\alpha, w, s) &:= \alpha^{\frac{1}{2}-s} e^{\frac{w^2}{8}} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; -\frac{w^2}{4}\right)\end{aligned} \tag{1.16}$$

where ${}_1F_1(a;c;w)$ denotes the confluent hypergeometric function defined as [9, p. 188]

$$ {}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}, $$

with $(a)_n$ being the Pochhammer symbol given by

$$ (a)_n := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} $$

for $a \in \mathbb{C}$. Here $\rho(\alpha, w, s)$ and $\rho(\alpha, w, 1-s)$ are the normalized Mellin transforms of the functions $\sqrt{\alpha} e^{\frac{w^2}{8}} e^{-\pi\alpha^2 x^2} \cos(\sqrt{\pi}\alpha x w)$ and $\sqrt{\beta} e^{-\frac{w^2}{8}} e^{-\pi\beta^2 x^2} \cos(i\sqrt{\pi}\beta x w)$ respectively. Though the first equality involving the integral in (1.15) is found more recently in [8] by A. Dixit, the second equality in (1.15) is well-known in another version given in terms of Ramanujan's theta function $f(a,b)$:

$$ e^{\frac{w^2}{8}} \sqrt{\alpha} f\left(e^{-\alpha^2 + iz\alpha}, e^{-\alpha^2 - iz\alpha}\right) = e^{-\frac{w^2}{8}} \sqrt{\beta} f\left(e^{-\beta^2 + iz\beta}, e^{-\beta^2 - iz\beta}\right). \tag{1.17} $$

where

$$ f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \ |ab| < 1, \tag{1.18} $$

The second family of integrals, denoted as $\mathcal{J}(h, Z; z, \alpha)$, is obtained by introducing a product of two Riemann Ξ-functions,

$$ \mathcal{J}(h, Z; z, \alpha) := \int_0^{\infty} h\left(z, \frac{t}{2}\right) \Xi\left(\frac{t - iz}{2}\right) \Xi\left(\frac{t + iz}{2}\right) Z\left(\frac{1 + it}{2}, \frac{z}{2}, \alpha\right) dt, \tag{1.19} $$

where $h(z,t)$ is an even function of t of the form

$$ h(z,t) = |\phi(z, it)|^2 = \phi(z, it)\phi(z, -it), \tag{1.20} $$

where the function ϕ is analytic in the complex variable z and in the real variable

t and the function Z satisfying the invariance property $Z(s,z,\alpha)=Z(s,z,\beta)$ is the sum of the normalized Mellin transforms of a pair of functions that is reciprocal in the Koshliakov kernel, defined explicitly in Chapter 2. The evaluation of the integrals $\mathcal{J}(h,w;\alpha)$ generates the formulas of the second kind $F(\alpha,z)=F(\beta,z)$, which is of a different kind than $F(\alpha,w)=F(\beta,iw)$ seen above.

The integral in Theorem (1.1.1) given by Ramanujan is the first example of the integrals of the second kind $\mathcal{J}(h,Z;z,\alpha)$, with the variable z identically equal to zero. A transformation formula derived by N.S. Koshliakov , though it was also proved by Ramanujan ten years earlier[10, p. 253], will be referred to as Koshliakov's formula throughout the thesis. It is obtained from the integrals of the second kind $\mathcal{J}(h,Z;z,\alpha)$ by letting $z=0$ and using $\phi(t)=\frac{1}{(t+\frac{1}{2})(t-\frac{1}{2})}$, so that $h\left(\frac{t}{2}\right)=\frac{16}{(1+t^2)^2}$. For $\alpha,\beta>0$,

$$\frac{32}{\pi}\int_0^\infty\frac{\Xi^2(\frac{t}{2})}{(1+t^2)^2}\cos\left(\frac{1}{2}t\log\alpha\right)dt=\sqrt{\alpha}\left(4\sum_{n=1}^\infty d(n)K_0(2n\pi\alpha)-\frac{\gamma-\log(4\pi\alpha)}{\alpha}\right)$$
$$=\sqrt{\beta}\left(4\sum_{n=1}^\infty d(n)K_0(2n\pi\beta)-\frac{\gamma-\log(4\pi\beta)}{\beta}\right)\quad(1.21)$$

where $d(n)$ is the number of positive divisors of the positive integer n, γ is Euler's constant and $K_0(x)$ is the modified Bessel function of the second kind of order zero.

The well-known Ramanujan-Guinand identity generalizes Koshliakov's formula (1.21) by extending $K_0(x)$ to $K_z(x)$. For $-1<\mathrm{Re}(z)<1$,

$$\frac{32}{\pi}\int_0^\infty\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{((z+1)^2+t^2)((z-1)^2+t^2)}\,dt$$
$$=\sqrt{\alpha}\left(4\sum_{n=1}^\infty\sigma_{-z}(n)n^{\frac{z}{2}}K_{\frac{z}{2}}(2\pi\alpha n)-\alpha^{\frac{z}{2}-1}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)-\alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\right)$$
$$=\sqrt{\beta}\left(4\sum_{n=1}^\infty\sigma_{-z}(n)n^{\frac{z}{2}}K_{\frac{z}{2}}(2\pi\beta n)-\beta^{\frac{z}{2}-1}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)-\beta^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\right)$$
$$(1.22)$$

The second equality is the transformation formula known to Ramanujan (see [10]) and rediscovered by Guinand [11] whereas the first equality is proved recently by A. Dixit [12]. The pair of functions $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$ reciprocal in the Koshliakov kernel is used for the calculation of the invariance function Z for both (1.21) and (1.22), where $z = 0$ for the former. In both the formulas, Z reduce to $\cos\left(\frac{1}{2}t\log\alpha\right)$, as shown in Chapter 2, independent of the variable z.

A number of interesting formulas involving integrals of the kind $\mathcal{J}(h, Z; z, \alpha)$ for which the invariance function Z is restricted to $\cos\left(\frac{1}{2}t\log\alpha\right)$ are known, see for example [12–15]. The paper [16] has developed a unified theory for the integrals of the kind $\mathcal{J}(h, Z; z, \alpha)$ for a specific $h = \frac{1}{(t^2+(z+1)^2)(t^2+(z-1)^2)}$. The integrals involving both the Koshlikov's formula and Ramanujan-Guinand formula are examples for this particular h. The results in the paper [16] serves as a template for generating new transformation formulas that can possibly involve an invariance functions Z other than $\cos\left(\frac{1}{2}t\log\alpha\right)$. To derive a transformation formula that uses a non-trivial invariance function Z, the trick is to find a quintessential pair of functions reciprocal in the Koshliakov kernel for which Z does not reduce to $\cos\left(\frac{1}{2}t\log\alpha\right)$, while being not too complicated so that we are still able to find an alternate representation for the integral $\mathcal{J}(h, Z; z, \alpha)$. The difficulty lies in generating the reciprocal pairs in the Koshliakov kernel as it is rare for a function to yield an exact evaluation in the Koshliakov transform. In course of searching the literature for finding a reciprocal pair in three variables x, z and w, that reduces to $K_z(x)$ when $w = 0$, only one useful equation in Koshliakov's paper [17] was found, though it proved to be inapplicable for calculating Z. To the best of the author's knowledge, the transformation formula derived in the thesis is the first one obtained from the integrals of the kind $\mathcal{J}(h, Z; z, \alpha)$ in which the invariance function Z do not reduce to $\cos\left(\frac{1}{2}t\log\alpha\right)$. It is to be noted that this is not the case with the invariance function ∇ for the integrals of the first kind $\mathcal{I}(h, w; \alpha)$, as seen above from the generalization of the theta transformation formula (1.15) of the kind $F(\alpha, w) = F(\beta, iw)$.

Evidently the integrals evaluating the theta transformation formula (1.6) and Koshliakov's formula (1.21), that are of the first and second kind respectively, comprise of the same functions, though all of the functions, except the invariance function cosine, are squared in the latter. The two formulas exhibit other similarities as well that is elaborated as follows. The theta transformation formula (1.6) is equivalent to the functional equation (1.1) of the Riemann zeta function $\zeta(s)$ whereas Koshliakov's formula (1.21) is shown to be equivalent to that of $\zeta^2(s)$ as shown by W. L. Ferrar [7] and F. Oberhettinger and K. L. Soni [18]. The functions appearing in the sums of the respective formulas, that is e^{-x^2} and $K_0(x)$ exhibit similarity in the following manner.

The Mellin transforms for the functions e^{-x^2} and $K_0(x)$ are well-known [19]. For $\mathrm{Re}(s) > 0$,

$$\int_0^\infty x^{s-1} e^{-x^2}\, dx = \frac{1}{2}\Gamma\left(\frac{s}{2}\right), \tag{1.23}$$

and again, for $\mathrm{Re}(s) > 0$,

$$\int_0^\infty x^{s-1} K_0(x)\, dx = 2^{s-2}\Gamma^2\left(\frac{s}{2}\right) \tag{1.24}$$

Thus, up to constant factors, the functions e^{-x^2} and $K_0(x)$ are the inverse Mellin transforms of $\Gamma\left(\frac{s}{2}\right)$ and $\Gamma^2\left(\frac{s}{2}\right)$ respectively.

Ramanujan-Guinand formula is a one-variable generalization of Koshliakov's formula of the kind $F(\alpha, z) = F(\beta, z)$. This generalization is of a different kind than the one seen above for the generalization of the theta transformation formula, that is $F(\alpha, w) = F(\beta, iw)$. One of the goals of the thesis is to find a one-variable generalization of the Ramanujan-Guinand formula and hence, a two-variable generalization of Koshliakov's formula. This formula is of the form $F(\alpha, z, w) = F(\beta, z, iw)$. To reach that goal, we first discover the elusive reciprocal pair by defining a new special function $K_{z,w}(x)$, that is first introduced as an inverse Mellin transform as follows. For $z, w \in \mathbb{C}$, $x \in \mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$, and $\mathrm{Re}(s) > \pm\, \mathrm{Re}(z)$, we define

$$K_{z,w}(x) := \frac{1}{2\pi i}\times \int_{(c)} \Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right) {}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right) {}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right) 2^{s-2}x^{-s}ds, \tag{1.25}$$

The special function $K_{z,w}(x)$ can be considered as a generalization of the modified Bessel function of second kind $K_z(x)$. For $w = 0$

$$K_{z,0}(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)2^{s-2}x^{-s}ds = K_z(x). \tag{1.26}$$

The integral in (1.26) is an inverse Mellin transform representation of $K_z(x)$, as given in [19, p. 115, formula 11.1], hence the second equality.

The new function $K_{z,w}(x)$ yields a non-trivial invariance function Z leading to the desired generalization of the Ramanujan-Guinand formula of the kind $F(\alpha, z, w) = F(\beta, z, iw)$ involving two variables z and w as follows, unlike previous formulas which belong strictly to one of the two kinds, with either one or no variable. Let $w \in \mathbb{C}$, $z \in \mathbb{C}\backslash\{-1, 1\}$ and $\alpha, \beta > 0$,

$$\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\mathcal{Z}\left(\frac{1+it}{2},\frac{z}{2},w\right)\frac{dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)} =$$
$$\sqrt{\alpha}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{-\frac{w^2}{4}}K_{\frac{z}{2},iw}(2n\pi\alpha)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)$$
$$-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)\Bigg) =$$
$$\sqrt{\beta}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2n\pi\beta)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\beta^{\frac{z}{2}-1}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right)$$
$$-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\beta^{-\frac{z}{2}-1}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)\Bigg) \quad (1.27)$$

where

$$\mathcal{Z}(\alpha,z,w,s) := \mathscr{Z}(\alpha,z,w,s)+\mathscr{Z}(\alpha,z,w,1-s),$$

$$\mathscr{Z}(\alpha,z,w,s) := \alpha^{\frac{1}{2}-s}{}_1F_1\left(\frac{1-s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{1-s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right).$$

Here $\mathscr{Z}(\alpha,z,w,s)$ and $\mathscr{Z}(\alpha,z,w,1-s)$ are the normalized Mellin transforms of the functions $\sqrt{\alpha}e^{-\frac{w^2}{4}}K_{z,w}(2\pi\alpha x)$ and $\sqrt{\beta}e^{\frac{w^2}{4}}K_{z,iw}(2\pi\beta x)$ respectively.

Ramanujan-Guinand formula is a special case of (1.27) when $w=0$ whereas Koshliakov's formula is a special case of (1.27) when both $w=0$ and $z=0$. As a special case when only $z=0$ in (1.27), we obtain another previously unknown formula, that is one-variable generalization of Koshliakov's formula of the kind $F(\alpha,w)=F(\beta,iw)$. Let $w\in\mathbb{C}$ and $\alpha,\beta>0$,

$$\frac{16}{\pi}\int_0^{\infty}\frac{\Xi\left(\frac{t}{2}\right)^2}{(t^2+1)^2}\left(\alpha^{-\frac{it}{2}}\,{}_1F_1^2\left(\frac{1-it}{4};\frac{1}{2};-\frac{w^2}{4}\right)+\alpha^{\frac{it}{2}}\,{}_1F_1^2\left(\frac{1+it}{4};\frac{1}{2};-\frac{w^2}{4}\right)\right)dt$$
$$=\sqrt{\alpha}e^{-\frac{w^2}{4}}\left(4\sum_{n=1}^{\infty}d(n)e^{-\frac{w^2}{4}}K_{0,w}(2n\pi\alpha)-\frac{\gamma-\log(4\pi\alpha)}{\alpha}\left(1-\frac{w^2}{4}\right)+\frac{w^2}{2\alpha}\right)$$
$$=\sqrt{\beta}e^{-\frac{w^2}{4}}\left(4\sum_{n=1}^{\infty}d(n)e^{-\frac{w^2}{4}}K_{0,iw}(2n\pi\beta)-\frac{\gamma-\log(4\pi\beta)}{\beta}\left(1+\frac{w^2}{4}\right)-\frac{w^2}{2\beta}\right).$$

Note that this transformation formula can be given in the form $F(\alpha, w) = F(\beta, iw)$, like most formulas associated with the integrals of the first kind $\mathcal{I}(h, \nabla; w, \alpha)$, nevertheless, the integral equal to it is of the second kind $\mathcal{J}(h, Z; z, \alpha)$, as reflected by the squares or the products of the functions, for example Ξ-function, in the integrand.

The generalization of the theta transformation formula led to the motivation behind the discovery of the function $K_{z,w}(x)$, as explained in Chapter 2. The function involved in the theta transformula formula (of the type $F(\alpha) = F(\beta)$), that is e^{-x^2}, is extended to an elementary function $e^{-x^2}\cos(wx)$ appearing in its generalization of the kind $F(\alpha, w) = F(\beta, iw)$. On the other hand, the special function of $K_z(x)$, that appears in Ramanujan-Guinand formula of the kind $F(\alpha, z) = F(\beta, z)$, is extended to a *new* special function, namely $K_{z,w}(x)$ that yields the generalized formula of the type $F(\alpha, z, w) = F(\beta, z, iw)$. A significant portion of the thesis is devoted to developing the theory of $K_{z,w}(x)$ as much as possible with the intention it might be useful in other places. Hence, a large part of the analysis of $K_{z,w}(x)$, given in Chapter 5, is not integral to the discovery of the desired modular-type transformation but is independent of it. A number of integral and series representations of $K_{z,w}(x)$ are derived in Chapter 5, motivated by the corresponding representations of the modified Bessel function $K_z(x)$, and they point to a rich theory of the special function $K_{z,w}(x)$.

For example, the function $K_{z,w}(x)$ exhibits the following simple integral representation. For $z, w \in \mathbb{C}$ and $|\arg x| < \frac{\pi}{4}$,

$$K_{z,w}(2x) = x^{-z} \int_0^\infty e^{-t^2 - \frac{x^2}{t^2}} \cos(wt) \cos\left(\frac{wx}{t}\right) t^{2z-1}\, dt. \tag{1.28}$$

We note that from the definition of $K_{z,w}(x)$ itself, it is evident that it is an even function in both the variables z and w. From the above integral representation (1.28) as well, it is clear that $K_{z,w}(x)$ is an even function of w and by the change of variable $t \to x/t$, it follows that it is also an even function in z.

Another integral representation for the function $K_{0,w}(x)$ involving an exponential and two cosine functions is as follows. For $|\arg x| < \frac{1}{4}\pi$ and $w \in \mathbb{C}$, we have

$$K_{0,w}(x) = \int_0^\infty \exp\left(-\frac{w^2x^2}{2(x^2+u^2)}\right) \cos\left(\frac{w^2xu}{2(x^2+u^2)}\right) \frac{\cos u\, du}{\sqrt{x^2+u^2}}. \tag{1.29}$$

This is a generalization of the well-known Basset's formula for the modified Bessel function of the second kind $K_z(x)$, given in [4, p. 172],

$$K_z(xy) = \frac{\Gamma\left(z+\frac{1}{2}\right)(2x)^z}{y^z\Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos(yu)\, du}{(x^2+u^2)^{z+\frac{1}{2}}}, \tag{1.30}$$

for $\mathrm{Re}(z) > -\frac{1}{2}, y > 0$, and $|\arg x| < \frac{1}{2}\pi$ in the case $z = 0$.

The function $K_{z,w}(x)$ relates to the modified Bessel function $K_z(x)$ in the following manner. For $z, w \in \mathbb{C}$ and $|\arg x| < \frac{\pi}{4}$,

$$K_{z,w}(2x) = \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-w^2x)^{n+m}}{(2n)!(2m)!} K_{n-m+z}(2x). \tag{1.31}$$

The special function $K_{z,w}(x)$ also exhibits a series representation, involving the three different Bessel functions $J_z(x)$, $I_z(x)$ and $K_z(x)$ as follows. For $-\frac{1}{2} < \mathrm{Re}(z) <$

$\frac{1}{2}$, $w \in \mathbb{C}$, and $|\arg x| < \frac{\pi}{4}$,

$$K_{z,w}(2x) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n K_{z+n}(2x) \left(I_{2n}(2w\sqrt{x}) + J_{2n}(2w\sqrt{x}) \right). \tag{1.32}$$

When $w = 0$, the above result reduces to the trivial relation $K_z(2x) = K_z(2x)$ since $I_0(0) = J_0(0) = 1$ and $I_n(0) = J_n(0) = 0$ for $n \neq 0$. It is to be noted that in both the series (1.31) and (1.32) representing $K_{z,w}(x)$, the variable z appears only in the order of the modified Bessel function $K_z(x)$ whereas the variable w appears either in the arguments of the Bessel functions $J_z(x)$ and $I_z(x)$, as in (1.32) or seperately as in (1.32). Moreover the Bessel functions themselves are defined as series, see (1.10), more specially the modified Bessel function $K_z(x)$ is given by

$$K_z(2x) = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{x^{2k}}{k! \sin(\pi z)} \left(\frac{x^{-z}}{\Gamma(k-z+1)} - \frac{x^z}{\Gamma(k+z+1)} \right) \tag{1.33}$$

Hence the function $K_{z,w}(x)$ is effectively a triple series, one representation of which is derived using (1.31) and (1.33) and is given by

$$\begin{aligned} K_{z,w}(2x) = \frac{\pi}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{w^{2(n+m)} x^{2k}}{(2n)!\ (2m)!\ k! \sin(\pi z)} \times \\ \left(\frac{x^{2m-z}}{\Gamma(k+m-n-z+1)} - \frac{x^{2n+z}}{\Gamma(k+n-m+z+1)} \right), \end{aligned} \tag{1.34}$$

for $w \in \mathbb{C}$, $z \notin \mathbb{Z}$ and $|\arg x| < \frac{\pi}{4}$. Another representation for $K_{z,w}(x)$ is an infinite series of Laplace transform of a special function involving ${}_0F_2$ as given below. For $w \in \mathbb{C}$, $\mathrm{Re}(z) > -\frac{1}{2}$ and $|\arg x| < \frac{\pi}{4}$,

$$\begin{aligned} K_{z,w}(x) = \frac{(2x)^{z+\frac{1}{2}}}{\Gamma\left(z+\frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{\left(-\frac{w^2 x}{2}\right)^n}{(2n)!} \int_0^{\infty} t^{z-\frac{1}{2}} (t+1)^{z-\frac{1}{2}} (2t+1)^{-n+\frac{1}{2}} K_{n+\frac{1}{2}}(x(2t+1)) \\ \times {}_0F_2\left(-; \frac{1}{2}, \frac{1}{2}+z; -\frac{w^2 x^2 t(t+1)}{4}\right) dt. \end{aligned} \tag{1.35}$$

This is the generalization of the well-known integral [20, p. 236][1] for $K_z(x)$, where it is expressed as a Laplace transform of an elementary function, that is, for $\mathrm{Re}(z) > -\frac{1}{2}$ and $|\arg x| < \frac{\pi}{2}$,

$$K_z(x) = \frac{\sqrt{\pi}(2x)^z e^{-x}}{\Gamma\left(z+\frac{1}{2}\right)} \int_0^\infty e^{-2xt} t^{z-\frac{1}{2}} (t+1)^{z-\frac{1}{2}}\, dt. \tag{1.36}$$

The integral in the above representation for $K_{z,w}(x)$ is indeed a Laplace transform as [21, p. 934, formula **8.468**]

$$K_{n+\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2y}} e^{-y} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2y)^k}. \tag{1.37}$$

In an attempt to derive the asymptotic expansion of the function $K_{z,w}(x)$, the following double integral representation is discovered by chance. For $w \in \mathbb{C}$, $\mathrm{Re}(z) > -1$ and $|\arg x| < \frac{\pi}{4}$,

$$\begin{aligned} K_{z,w}(x) = \frac{1}{2\Gamma(1+z)} \int_0^\infty \int_0^\infty \frac{y^z t^{-1/2}}{\sqrt{y+\frac{x}{2}}} \exp\left(-2\sqrt{\left(t+\frac{x}{2}\right)\left(y+\frac{x}{2}\right)}\right) \\ \times {}_0F_2\left(-;\frac{1}{2},1+z;-\frac{w^2xy}{8}\right) {}_0F_2\left(-;\frac{1}{2},\frac{1}{2};-\frac{w^2xt}{8}\right) dt\, dy. \end{aligned} \tag{1.38}$$

Letting $w = 0$ in the above equation (1.38) gives the following double integral representation for the modified Bessel function $K_z(x)$ which the author could not find in the literature. For $\mathrm{Re}(z) > -1$ and $|\arg x| < \frac{\pi}{4}$,

$$K_z(x) = \frac{1}{2\Gamma(1+z)} \int_0^\infty \int_0^\infty \frac{y^z t^{-1/2}}{\sqrt{y+\frac{x}{2}}} \exp\left(-2\sqrt{\left(t+\frac{x}{2}\right)\left(y+\frac{x}{2}\right)}\right) dt\, dy.$$

The asymptotic expansion of $K_{z,w}(x)$ for large values of $|x|$ obtained by Nico M. Temme is stated in Theorem 5.4.1 in Chapter 5. Its proof is given in the Appendix of

[1] There is minor misprint in that $(2x)^z$ is typed as $(2/x)^z$.

[22], the paper that also contains most of the results from the first topic of the thesis. For small values of $|x|$, the asymptotic expansion of $K_{z,w}(x)$ is as follows.
(i) Let $w \in \mathbb{C}$ be fixed. Consider a fixed z such that $\mathrm{Re}(z) > 0$. Let $\mathfrak{D} = \{x \in \mathbb{C} : |\arg x| < \frac{\pi}{4}\}$. Then as $x \to 0$ along any path in $\mathfrak{D}$, we have

$$K_{z,w}(x) \sim \frac{1}{2}\Gamma(z)\left(\frac{x}{2}\right)^{-z} {}_1F_1\left(z; \frac{1}{2}; \frac{-w^2}{4}\right). \tag{1.39}$$

(ii) Let $w \in \mathbb{C}$ be fixed. Let $|\arg x| < \frac{\pi}{4}$. As $x \to 0$,

$$K_{0,w}(x) \sim -\log x - \frac{w^2}{2} {}_2F_2\left(1, 1; \frac{3}{2}, 2; -\frac{w^2}{4}\right). \tag{1.40}$$

The result in (1.40) shows that, similar to the modified Bessel function $K_0(x)$, the function $K_{0,w}(x)$ also has a logarithmic singularity at $x = 0$. Note that when $w = 0$, the above two results agree with the corresponding ones for $K_z(x)$ given below.

$$K_z(x) \sim \begin{cases} \frac{1}{2}\Gamma(z)\left(\frac{x}{2}\right)^{-z}, \text{if Re } z > 0, \\ -\log x, \qquad \text{if } z = 0. \end{cases} \tag{1.41}$$

The function satisfies the differential-difference equation

$$\frac{d^4}{dw^4}K_{z,w}(2x) + 2x\left(\frac{d^2}{dw^2}K_{z+1,w}(2x) + \frac{d^2}{dw^2}K_{z-1,w}(2x)\right) + x^2\left(K_{z+2,w}(2x) - 2K_{z,w}(2x) + K_{z-2,w}(2x)\right) = 0.$$

for $z, w \in \mathbb{C}$ and $|\arg x| < \frac{\pi}{4}$.

The first topic in the thesis, described above, is organized as follows. Chapter 2 presents the motivation behind the definition of the new function $K_{z,w}(x)$ as the inverse Mellin transform by inspecting the analogy between Ramanujan-Guinand formula and the theta transformation formula. In Chapter 3, the desired pair of functions that is reciprocal in the Koshliakov kernel is derived using the function $K_{z,w}(x)$. In Chapter 4, the generalization of the Ramanujan-Guinand transformation formula of the kind $F(\alpha, z, w) = F(\beta, z, iw)$ is obtained along with the integral associated with it. In the last chapter of this topic, that is Chapter 5, the theory of the function $K_{z,w}(x)$ is developed. The introduction to the second topic in the thesis is given below.

1.2 2-adic valuations of integer sequences associated with quadratic polynomials

The second part of the thesis deals with the 2-adic valuations of the integer sequences. The p-adic valuation, also known as p-order and denoted by $\nu_p(n)$, of a number n is the exponent of the highest power of p that divides n. A sequence (s_n) is said to possess infinte p-adic valuation, denoted as $\nu_p(s_n) \to \infty$ if for any power of p, say p^v, there is a term in the sequence that is divisible by p^v.

The sequences studied in the thesis can all be traced back to those generated by the quadratic polynomials with integer coefficients and no linear term, for example the sequence $\{n^2 + 7 : n \in \mathbb{N}\}$. The p-adic valuations, for any prime p, of such sequences are inherently related to the existence of roots of the corresponding polynomials in the respective p-adic field $\mathbb{F}_p$. The Hensel's lemma is a useful tool to study the roots of polynomials in $\mathbb{F}_p$, but it fails to hold in case of the combination of the prime $p = 2$ and the polynomials of the kind $p(x) = x^2 + k$ since the derivative of $p(x)$ equals $0 \pmod 2$. An interesting result that links the distinct sequences studied in the later part of the thesis is that the sequences generated by the integer polynomials $\{x^2 + k : x \in N\}$ contain numbers divisible by any high power of 2 if and only if k is of the form $4^m(8l + 7)$. To sum it up, $\nu_2(n^2 + k) \to \infty$ and only if k is of the form $4^m(8l + 7)$.

The integers of the form $4^m(8l+7)$ mentioned above are special in the sense that they are the only integers that cannot be represented as the sum of three or less squares, a result known as Legendre's three-square theorem. This leads to the study of the sequences given by the sums of four integer squares that yields the following counterintuitive results.

Let the set S be defined as all the possible sums of four integers squares, that is $S = \{a^2 + b^2 + c^2 + d^2 : a, b, c, d \in \mathbb{Z}\}$. By Lagrange's four-square theorem, every number can be written as a sum of four integer squares. Thus the set S is essentially the set of natural numbers, repeated a certain number of times, which in turn implies that S has numbers divisible by any power of 2. Three different classes of sequences can be derived from the set S by fixing either one, two or three of the squares in the sum and letting the rest vary. For example, fixing one of the squares would yield the integer sequences given by $S(a) = \{a^2+b^2+c^2+d^2 : b, c, d \in \mathbb{Z}\}$ where a is fixed and b, c, d takes all integer values. The sets $S(a)$ are the first class of sequences and the other two classes are given by $S(a, b) = \{a^2 + b^2 + c^2 + d^2 : c, d \in \mathbb{Z}\}$ and $S(a, b, c) = \{a^2+b^2+c^2+d^2 : d \in \mathbb{Z}\}$ fixing two and three squares respectively. For every integer a, it turns out that the set $S(a)$ contains *no* numbers divisible by high enough power of 2 (see Theorem 6.2.1). For example, $S(3) = \{9 + b^2 + c^2 + d^2 : b, c, d \in \mathbb{Z}\}$ has *no* numbers divisible by 8. Moreover the highest power of 2 that divides numbers in the set $S(a)$ is proportional to the highest power of 2 that divides a, in fact it is equal to $2\nu_2(a) + 2$ (see Theorem 6.2.1).

Similar results holds true for the sets obtained by fixing two and three squares viz. $S(a, b) = \{a^2 + b^2 + c^2 + d^2 : c, d \in \mathbb{Z}\}$ (see Theorem 6.2.4) and $S(a, b, c) = \{a^2 + b^2 + c^2 + d^2 : d \in \mathbb{Z}\}$ (see Theorem 6.2.7). In a nutshell, for sets of the type $S(a)$, $S(a, b)$ and $S(a, b, c)$, there is an exponent v, specific to that particular set, such that no number in the set is divisible by powers of 2 higher than v. On the contrary, another question is, given a fixed expoenent v, what proportion of the sets in either one of the three classes, say $S(a,b)$, contains no numbers divisible by 2^v? For example, only half of the sets of the class $S(a,b)$ have numbers that are divisible by 4. In contrast with the natural numbers for which every fourth number is divisible by 4, half of the sets of the class $S(a,b)$ are devoid of any multiples of 4 whereas for the other half the multliples of 4 are fewer and inherently related to the sequence of triangular numbers defined below.

The function $r_k(n)$, that counts the number of ways n can be represented as sums of k integer squares, allows zeros and distinguishes between signs and order in the representations. The collection of the sets $S(a)$ forms a disjoint partition of the set S if the numbers equal in value but coming from different representations as sums of squares are considered different. Similarly, the collections of the sets of both the classes $S(a,b)$ and $S(a,b,c)$ always gives respective partitions of the set S. The connection of this analysis with the function $r_3(n)$ is evident in Theorem 6.2.9 in Chapter 6 that says the proportion of subsets of the type $S(a,b,c)$ that has numbers divisible by 2^{2v} is $1/2^{3v}$. This can be compared with the asymptotic behavior of the function $r_3(n)$ (see [23]):

$$\sum_{n\leq x} r_3(n) \sim \frac{4}{3}\pi x^{\frac{3}{2}}$$

which implies the following identity satisfied asymptotically by the function $r_3(n)$:

$$\frac{1}{2^{3v}}\sum_{n=1}^{2^{2v}k} r_3(n) \sim \sum_{n=1}^{k} r_3(n)$$

It is elaborated in the last section of Chapter 6 that the combination of prime $p=2$ and the sums of four squares is special and the results cannot be generalized if the problem is even slightly modified. For example if any other prime p is considered, then the above property will no longer hold true. For example, the set $\{1+1+0+n^2 : n \in \mathbb{Z}\}$ contains numbers divisible by all powers of 3. Similarly, if prime $p=2$ but sum of more than four squares are considered, then again some sets will contain numbers divisible by any power of 2, for example the set $\{1+1+1+4+n^2 : n \in \mathbb{Z}\}$. This is because of the beautiful confluence of the conditions for which Hensel's lemma fails to hold, an integer is a sum of three squares as given by the Legendre's three-square theorem and the polynomials n^2+k yields finite 2-adic valuations.

The sequence arising from the polynomials n^2+k can be considered as the translations of the sequence of squares in the number line. It is an interesting observation that, much more often than not, when an integer is added to each square, the resulting sequence contains no number is divisible by high enough powers of 2. Like squares, there is another type of figurate numbers called triangular numbers given by the formula $n(n+1)/2$. The sequences of triangular numbers $\{1, 3, 6, 10, 15, 21, 28, 36, 45, \dots\}$ contains infinitely many numbers divisible by any power of 2, similar to the sequence of squares. However unlike squares, when shifted by adding any integer the resulting sequence always contains infitinely many numbers divisible by any power of 2 no matter what integer is added uniformly to the sequence. Both squares and traingular numbers are examples of polygonal numbers that are given by a quadratic formula and hence it is natural to ask what is the pattern exhibited in general by polygonal numbers. It turns out that the behaviour of squares is truly unique whereas the behaviour of the remaining polygonal numbers can be put into two categories, that is summed up as follows.

1. For $s = 4$, the sequence obtained by adding an integer k to s-gonal numbers (that is squares) have numbers divisible by all powers of 2 if and only if k is of the form $4^m(8l+7)$.

2. For $s \not\equiv 0 \pmod 4$, the sequence obtained by adding an integer k to s-gonal numbers have numbers divisible by all powers of 2 for any k.

3. For $s \neq 4$ and $s \equiv 0 \pmod 4$, let $\nu_2(s-4) = u$, then the sequence obtained by adding an integer k to s-gonal numbers have numbers divisible by all powers of

2 if and only if $k \equiv 0 \pmod{2^{2u-1}}$.

Chapter 2

Analogy between the theta transformation formula and Ramanujan-Guinand formula

The motivation behind the definition of special function $K_{z,w}(x)$ can be clearly seen by understanding the analogy between the theta transformation formula and its generalization on one hand and Koshliakov's formula and Ramanujan-Guinand formula on the other. The former are the transformation formulas associated with the integrals of the first kind $\mathcal{I}(h, \nabla; w, \alpha)$ while the latter are the ones associated with the integrals of the second kind $\mathcal{J}(h, Z; z, \alpha)$. The pair of functions involved in the generalization of the theta transformation formula gives a clue about the corresponding pair needed to find the generalization of Ramanujan-Guinand formula. In view of this, the mechanism behind deriving the formulas along with the associated integrals using pairs of functions is demonstrated for the above known formulas of both the kinds. A comparison is made pointing the similarities and the differences in the respective mechanisms for the two distinct kinds of formulas mentioned above leading to the definition of the generalized modified Bessel function $K_{z,w}(x)$.

2.1 Transformation formulas associated with the integrals of the second kind $\mathcal{J}(h, Z; z, \alpha)$

A pair of functions φ and ψ is said be reciprocal in the Koshliakov kernel if the pair satisfies the following.

$$\varphi(x, z, w) = 2\pi \int_0^\infty \psi(t, z, w) \left(\cos(\pi z)\, M_{2z}(4\sqrt{tx}) - \sin(\pi z)\, J_{2z}(4\sqrt{tx}) \right) dt,$$
$$\psi(x, z, w) = 2\pi \int_0^\infty \varphi(t, z, w) \left(\cos(\pi z)\, M_{2z}(4\sqrt{tx}) - \sin(\pi z)\, J_{2z}(4\sqrt{tx}) \right) dt. \quad (2.1)$$

The pair of functions that is reciprocal in the Koshliakov kernel (φ, ψ) yields invariance function $Z\left(\frac{1+it}{2}, \frac{z}{2}, w\right)$, an example of which is the function $\cos\left(\frac{1}{2} t \log \alpha\right)$. More specifically, the invariance function Z is the sum of the normalized Mellin transforms of the functions φ and ψ from the reciprocal pair.

The normalized Mellin transforms $Z_1(s, z, w)$ and $Z_2(s, z, w)$ of the functions $\varphi(x, z, w)$ and $\psi(x, z, w)$ are given by

$$\pi^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right) Z_1(s, z, w) = \int_0^\infty x^{s-1}\varphi(x, z, w)\, dx, \quad (2.2)$$
$$\pi^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right) Z_2(s, z, w) = \int_0^\infty x^{s-1}\psi(x, z, w)\, dx, \quad (2.3)$$

where each equation is valid in a specific vertical strip in the complex s-plane. The function $Z(s, z, w)$ is defined as

$$Z(s, z, w) := Z_1(s, z, w) + Z_2(s, z, w), \quad (2.4)$$

so that

$$\pi^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)Z(s,z,w)=\int_0^\infty x^{s-1}\Theta(x,z,w)\,dx,$$

for values of s in the intersection of the above two vertical strips, where

$$\Theta(x,z,w):=\varphi(x,z,w)+\psi(x,z,w). \tag{2.5}$$

For a pair of functions φ and ψ reciprocal in the Koshliakov kernel to eventually generate a transformation formula, the functions φ and ψ must satisfy additional conditions. The $\Diamond_{\eta,\omega}$ class, that was originally defined in [16] for functions in two variables $\varphi(s,z)$ and $\psi(s,z)$, is slightly modified below to be used for $\varphi(s,z,w)$ and $\psi(s,z,w)$.

Definition 2.1.1. Let $0<\omega\leq\pi$ and $\eta>0$. For fixed z and w, if $u(s,z,w)$ is such that

(i) $u(s,z,w)$ is an analytic function of $s=re^{i\theta}$ regular in the angle defined by $r>0$, $|\theta|<\omega$,

(ii) $u(s,z,w)$ satisfies the bounds

$$u(s,z,w)=\begin{cases}O_{z,w}(|s|^{-\delta}) & \text{if } |s|\leq 1,\\ O_{z,w}(|s|^{-\eta-1-|\mathrm{Re}(z)|}) & \text{if } |s|>1,\end{cases} \tag{2.6}$$

for every positive δ and uniformly in any angle $|\theta|<\omega$, then we say that u belongs to the class $\Diamond_{\eta,\omega}$ and write $u(s,z,w)\in\Diamond_{\eta,\omega}$.

For the second family of integrals of the kind $\mathcal{J}(h,Z;z,\alpha)$ and the specific function $h=((t^2+(z+1)^2)(t^2+(z-1)^2))^{-1}$, the following result is given in [16, Theorem 1.2]

Theorem 2.1.2. *Let $\eta > 1/4$ and $0 < \omega \leq \pi$. Suppose that $\varphi, \psi \in \Diamond_{\eta,\omega}$, are reciprocal in the Koshliakov kernel, and that $-1/2 < \mathrm{Re}(z) < 1/2$. Let $Z(s,z,w)$ and $\Theta(x,z,w)$ be defined in (2.4) and (2.5). Let $\sigma_{-z}(n) = \sum_{d|n} d^{-z}$. Then*

$$\begin{aligned}
&\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2},w\right)\frac{dt}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\\
&=\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}\Theta\left(\pi n,\frac{z}{2},w\right)-R(z,w),
\end{aligned} \tag{2.7}$$

where

$$R(z,w) := \pi^{z/2}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)Z\left(1+\frac{z}{2},\frac{z}{2},w\right)+\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)Z\left(1-\frac{z}{2},\frac{z}{2},w\right). \tag{2.8}$$

The rigth side of the equation (2.7) comprises of $\Theta(x,z,w)$ in the summation, that is itself the sum of the functions $\varphi(s,z,w)$ and $\psi(s,z,w)$ in the reciprocal pair, as well as the invariance function $Z(s,z,w,\alpha)$ that satisfies $Z(s,z,w,\alpha) = Z(s,z,iw,\beta)$. When Theorem 2.1.2 is applied to a pair of functions, the integral on the rigth side of the equation (2.7), say $\mathcal{J}$, is equal to a function that can be broken into two parts in the following manner.

$$\mathcal{J} = \frac{1}{2}\left(F(\alpha,z,w)+F(\beta,z,iw)\right). \tag{2.9}$$

The intermediate step is to prove that the two parts are equal to this each other, that is

$$F(\alpha,z,w) = F(\beta,z,iw), \tag{2.10}$$

which implies

$$\mathcal{J} = F(\alpha,z,w) \tag{2.11}$$

The equation (2.10) is the modular-type transformation formula associated with the integral $\mathcal{J}$. Thus the application of Theorem 2.1.2 to a pair of functions generates a transformation formula, without proving it. The lemma 4.0.1 by Guinand [11, equation (1)] stated in Chapter 4 is a useful tool in the proof of such formulas. To summarize, Theorem 2.1.2 not only helps in finding modular-type transformation formulas but also gives an integral that is equal to the known or newly found transformation formulas. Finding new formulas is the trickier part than proving them in most cases, however with the help of Theorem 2.1.2, the task boils down to finding an approriate pair of functions reciprocal in the Koshliakov kernel.

One of the goal of the thesis is to obtain the transformation formula of the kind $F(\alpha, z, w) = F(\beta, z, iw)$ that is a one-variable generalization of the Ramanujan-Guinand formula (1.21) and hence, a two-variable generalization of Koshliakov's formula (1.21). We first demonstrate the application of Theorem 2.1.2 in these two formulas using the reciprocial pairs in both cases so as to motivate the definition of the new special function $K_{z,w}(x)$ that gives the reciprocal pair leading to the desired generalization.

For Koshliakov's formula

$$\sqrt{a}\left(4\sum_{n=1}^{\infty} d(n)K_0(2n\pi\alpha) - \frac{\gamma - \log(4\pi\alpha)}{\alpha}\right) = \sqrt{b}\left(4\sum_{n=1}^{\infty} d(n)K_0(2n\pi\beta) - \frac{\gamma - \log(4\pi\beta)}{\beta}\right), \tag{2.12}$$

for $\alpha, \beta > 0$, the integral associated with the formula is

$$\frac{32}{\pi}\int_0^{\infty} \frac{\Xi^2(\frac{t}{2})}{(1+t^2)^2} \cos\left(\frac{1}{2}t\log\alpha\right) dt, \tag{2.13}$$

whereas for the Ramanujan-Guinand identity of the kind $F(\alpha, z) = F(\beta, z)$, which is essentially the generalization of Koshliakov's formula,

$$\sqrt{\alpha}\left(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}K_{\frac{z}{2}}(2\pi\alpha n)+\alpha^{\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)+\alpha^{-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)\right)$$
$$=\sqrt{\beta}\left(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}K_{\frac{z}{2}}(2\pi\beta n)+\beta^{\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)+\beta^{-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)\right), \tag{2.14}$$

for $-1 < \mathrm{Re}(z) < -1$, the integral associated with it is obtained by introducing variable z in (2.13) in the following manner [16]

$$\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{((z+1)^2+t^2)((z-1)^2+t^2)}\,dt. \tag{2.15}$$

Both the formulas (2.12) and (2.14) comprises of the modified Bessel function of the second kind $K_z(x)$, that is self-reciprocal in the Koshliakov kernel and yeilds the reciprocal pair as shown below. The invariance function, derived from the reciprocal pairs, in the integral of both the formulas is $\cos\left(\frac{1}{2}t\log\alpha\right)$ which turns out to be independent of z, even though the reciprocal pair in case of Ramanujan-Guinand formula is not independent of z, as demonstrated below.

The modified Bessel function of the second kind $K_z(x)$ is reciprocal in the Koshliakove kernel, as proved in [17],

$$2\pi\int_0^{\infty}K_z(2\pi t)\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt})-\sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dt=K_z(2\pi x). \tag{2.16}$$

for $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$. The reciprocal pair $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$ is derived using the self-reciprocity of $K_z(2\pi x)$ by change of variables as follows.

$$2\pi\int_0^{\infty}\sqrt{\beta}K_z(2\pi\beta t)\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt})-\sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dt=\sqrt{\alpha}K_z(2\pi\alpha x),$$
$$2\pi\int_0^{\infty}\sqrt{\alpha}K_z(2\pi\alpha t)\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt})-\sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dt=\sqrt{\beta}K_z(2\pi\beta x).$$

The Mellin transform for $K_z(x)$ [19, p. 115, formula 11.1] is

$$\int_0^\infty x^{s-1} K_z(x)\,dx = 2^{s-2}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right), \tag{2.17}$$

for $\mathrm{Re}(s) > \pm\,\mathrm{Re}(z)$. Using change of variables in (2.17), the Mellin transforms derived for the functions in the reciprocal pair $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$ are given below.

$$\int_0^\infty x^{s-1}\sqrt{\alpha}K_z(2\pi\alpha x)\,dx = \pi^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)\frac{\alpha^{\frac{1}{2}-s}}{4},$$
$$\int_0^\infty x^{s-1}\sqrt{\beta}K_z(2\pi\beta x)\,dx = \pi^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)\frac{\beta^{\frac{1}{2}-s}}{4}.$$

By definition (2.2), the normalized Mellin transforms are

$$Z_1(\alpha, s, z) = \frac{\alpha^{\frac{1}{2}-s}}{4},$$
$$Z_2(\beta, s, z) = \frac{\beta^{\frac{1}{2}-s}}{4}. \tag{2.18}$$

Hence the invariance function is

$$Z\left(\frac{1+it}{2}, \frac{z}{2}\right) = \frac{\alpha^{\frac{-it}{2}} + \alpha^{\frac{it}{2}}}{4} = \frac{1}{2}\cos\left(\frac{1}{2}t\log\alpha\right).$$

as expected. Note that $Z_1(\alpha, s, z)$ and $Z_2(\beta, s, z)$ are independent of the variable z and satisfy $Z_1(\alpha, s, z) = Z_2(\beta, 1-s, z)$. For Koshliakov's formula, that is a special case of Ramanujan-Guinand formula when $z = 0$, the pair of reciprocal functions $(\sqrt{\alpha}K_0(2\pi\alpha x), \sqrt{\beta}K_0(2\pi\beta x))$ is used instead of $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$ and all the calculations above are the same. Since the invariance function $Z(\alpha, s, z)$ for the pair $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$ turned out to be independent of z, it is the exactly the same for both the reciprocal pairs.

We expect the generalized reciprocal pair we are looking for to appear in the generalization of Ramanujan-Guinand formula, in the same manner that the reciprocal pairs $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$ and $(\sqrt{\alpha}K_0(2\pi\alpha x), \sqrt{\beta}K_0(2\pi\beta x))$ appeared in the summations of the Ramanujan-Guinand formula and Koshlikov formula respectively. The desired pair of functions in three variables x, z and w must also reduce to $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$ when $w = 0$. Since $K_z(x)$ is itself a special function, so must be the function that is used in the generalization.

For a function to be a part of the reciprocal pair, it is to be integrated with the Koshliakov kernel and the evaluation must give the other function in the pair and vice versa. An extensive search for functions involving $K_z(x)$ that have exact evaluation in the Koshliakov transform lead to only one identity in [24], that is not helpful for this purpose. The desired generalization formula require introduction of a new function, namely $K_{z,w}(x)$, that eluded discovery by trial and error. The function $K_{z,w}(x)$ is intuitively found by observing the analogy between the theta transformation formula and Koshliakov's formula. The generalization of the theta transformation formula of the kind $F(\alpha, w) = F(\beta, iw)$ is known and a close inspection of this generalization leads to defining $K_{z,w}(x)$ as the inverse Mellin transform of the product of two gamma and two confluent hypergeometric functions as follows.

2.2 Transformation formulas associated with the integrals of the first kind $\mathcal{I}(h, \nabla; w, \alpha)$

For the theta transformation formula

$$\sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\right) = \sqrt{\beta}\left(\frac{1}{2\beta} - \sum_{n=1}^{\infty} e^{-\pi\beta^2 n^2}\right), \tag{2.19}$$

for $\mathrm{Re}(\alpha^2) > 0$ and $\mathrm{Re}(\beta^2) > 0$, the integral associated with the formula is

$$\int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2} \cos\left(\frac{1}{2} t \log \alpha\right) dt. \tag{2.20}$$

The integrals for the theta transformation formula and its generalization are of the kind $\mathcal{I}(h, \nabla; w, \alpha)$ whereas Theorem 2.1.2 is applicable to those of a different kind that is $\mathcal{J}(h, Z; z, \alpha)$. However, Theorem 2.1.2 covers the integral for which the h function is essentially the square of the h function used for these two formulas. It turns out the mechanism is inherently the similar to that of the two formulas discussed above except that the Koshliakov kernel is replaced with another kernel as explained below. Laplace's integral gives

$$e^{-\pi x^2} = 2 \int_0^\infty e^{-\pi t^2} \cos(2\pi x t)\, dt. \tag{2.21}$$

Thus e^{-x^2} is self reciprocal in the Fourier cosine transform. So the natural kernel to work with for choosing the reciprocal pair (φ, ψ) is the Fourier cosine transform. The invariance function ∇ is given by $\nabla(\alpha, w, s) := \rho_1(\alpha, w, s) + \rho_2(\alpha, w, 1-s)$ where the normalization in the normalized Mellin transforms $\nabla_1(\alpha, w, s)$ and $\nabla_2(\beta, w, s)$ are adjusted accordingly to a single gamma factor devoid of the variable w as follows.

$$\pi^{-s}\Gamma\left(\frac{s}{2}\right) \nabla_1(\alpha, w, s) = \int_0^\infty x^{s-1}\varphi(x, w, \alpha)\, dx,$$
$$\pi^{-s}\Gamma\left(\frac{s}{2}\right) \nabla_2(\beta, w, s) = \int_0^\infty x^{s-1}\psi(x, w, \beta)\, dx,$$

where each equation is valid in a specific vertical strip in the complex s-plane.

The self-reciprocity of the function e^{-x^2} yields the pair $(\sqrt{\alpha} e^{-\pi\alpha^2 x^2}, \sqrt{\beta} e^{-\pi\beta^2 x^2})$

reciprocal in the Fourier cosine transform as follows.

$$\sqrt{\alpha}e^{-\pi\alpha^2x^2} = 2\int_0^\infty \sqrt{\beta}e^{-\pi\beta^2t^2}\cos(2\pi xt)\,dt,$$
$$\sqrt{\beta}e^{-\pi\beta^2x^2} = 2\int_0^\infty \sqrt{\alpha}e^{-\pi\alpha^2t^2}\cos(2\pi xt)\,dt.$$

The Mellin transform of the function e^{-x^2} derived from the definition of the gamma function (1.2) by a change of variable is

$$\int_0^\infty x^{s-1}e^{-\pi x^2}\,dx = \pi^{-\frac{s}{2}}\frac{1}{2}\Gamma\left(\frac{s}{2}\right), \tag{2.22}$$

for $\mathrm{Re}(s) > 0$. The respective Mellin transform for the reciprocal pair are as follows.

$$\int_0^\infty x^{s-1}\sqrt{\alpha}e^{-\pi\alpha^2x^2}\,dx = \pi^{-\frac{s}{2}}\frac{\alpha^{\frac{1}{2}-s}}{2}\Gamma\left(\frac{s}{2}\right), \tag{2.23}$$

$$\int_0^\infty x^{s-1}\sqrt{\beta}\beta e^{-\pi\beta^2x^2}\,dx = \pi^{-\frac{s}{2}}\frac{\beta^{\frac{1}{2}-s}}{2}\Gamma\left(\frac{s}{2}\right). \tag{2.24}$$

The equations (2.23) and (2.24) implies that the following normalized Mellin transforms are exactly the same as those obtained for both Koshliakov's formula and Ramanujan-Guinand formula, up to a factor by a constant.

$$\rho_1(\alpha, w, s) = \frac{\alpha^{\frac{1}{2}-s}}{2}, \tag{2.25}$$
$$\rho_2(\beta, w, s) = \frac{\beta^{\frac{1}{2}-s}}{2}. \tag{2.26}$$

Hence the invariance function is also the same as that of the above two formulas

$$\nabla\left(\alpha, w, \frac{1+it}{2}\right) = \cos\left(\frac{1}{2}t\log\alpha\right).$$

For all the three formulas above, the invariance function is the same $\cos\left(\frac{1}{2}t\log\alpha\right)$.

To generalize the Ramanujan-Guinand formula of the kind $F(\alpha, z) = F(\beta, z)$ to that of the kind $F(\alpha, z, w) = F(\beta, z, iw)$, it requires a non-trivial invariance function that is achieved by introducing a generalization of the function $K_z(x)$ by adding another variable, say w. The invariance function in the integral associated with the generalization of theta transformation formula is non-trivial and it is helpful to study the mechanism for this formula since the function h is essentially similar, except for the being squared in case of Koshliakov's formula. The generalized theta transformation formula of the kind $F(\alpha, w) = F(\beta, iw)$ is

$$\sqrt{\alpha}\left(\frac{e^{-\frac{w^2}{8}}}{2\alpha} - e^{\frac{w^2}{8}}\sum_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\cos(\sqrt{\pi}\alpha n w)\right) = \sqrt{\beta}\left(\frac{e^{\frac{w^2}{8}}}{2\beta} - e^{-\frac{w^2}{8}}\sum_{n=1}^{\infty} e^{-\pi\beta^2 n^2}\cos(i\sqrt{\pi}\beta n w)\right) \quad (2.27)$$

for $w \in \mathbb{C}$, for which the associated integral [8] given by

$$e^{\frac{w^2}{8}}\int_0^{\infty}\frac{\Xi(\frac{t}{2})}{1+t^2}\left(\alpha^{-\frac{it}{2}}{}_1F_1\left(\frac{1-it}{4};\frac{1}{2};-\frac{w^2}{4}\right) + \alpha^{\frac{it}{2}}{}_1F_1\left(\frac{1+it}{4};\frac{1}{2};-\frac{w^2}{4}\right)\right)dt,$$

as demonstrated below. The generalized theta transformation formula is inherently related to the pair $(\sqrt{\alpha}e^{\frac{w^2}{8}}e^{-\pi\alpha^2x^2}\cos(\sqrt{\pi}\alpha x w), \sqrt{\beta}e^{-\frac{w^2}{8}}e^{-\pi\beta^2x^2}\cos(i\sqrt{\pi}\beta x w))$ that appeared in the summation of the formula and is reciprocal in the Fourier cosine transform [21, p. 527, Formula **4.133.2**] as stated below.

$$\sqrt{\alpha}e^{\frac{w^2}{8}}e^{-\pi\alpha^2x^2}\cos(\sqrt{\pi}\alpha x w) = 2\int_0^{\infty}\sqrt{\beta}e^{-\frac{w^2}{8}}e^{-\pi\beta^2t^2}\cos(i\sqrt{\pi}\beta t w))\cos(2\pi x t)\,dt,$$
$$\sqrt{\beta}e^{-\frac{w^2}{8}}e^{-\pi\beta^2x^2}\cos(i\sqrt{\pi}\beta x w)) = 2\int_0^{\infty}\sqrt{\alpha}e^{\frac{w^2}{8}}e^{-\pi\alpha^2t^2}\cos(\sqrt{\pi}\alpha t w)\cos(2\pi x t)\,dt.$$

Note that the above pair is a one-variable generalization of the reciprocal pair $(\sqrt{\alpha}e^{-\pi\alpha^2x^2}, \sqrt{\beta}e^{-\pi\beta^2x^2})$ obtained by introducing the cosine function.

For $0 < \mathrm{Re}(s) < 1$, the Mellin transform for the functions in the pair [19] is

$$\int_0^\infty x^{s-1}\sqrt{\alpha}e^{\frac{w^2}{8}}e^{-\pi\alpha^2x^2}\cos(\sqrt{\pi}\alpha xw)\,dx = \pi^{-\frac{s}{2}}\frac{\alpha^{\frac{1}{2}-s}}{2}\Gamma\left(\frac{s}{2}\right)e^{-\frac{w^2}{8}}\,{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};\frac{w^2}{4}\right), \tag{2.28}$$

$$\int_0^\infty x^{s-1}\sqrt{\beta}e^{-\frac{w^2}{8}}e^{-\pi\beta^2x^2}\cos(i\sqrt{\pi}\beta xw))\,dx$$
$$= \pi^{-\frac{s}{2}}\frac{\beta^{\frac{1}{2}-s}}{2}\Gamma\left(\frac{s}{2}\right)e^{\frac{w^2}{8}}\,{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};-\frac{w^2}{4}\right). \tag{2.29}$$

Hence the normalized Mellin transform is given by

$$\rho_1(\alpha,w,s) := \frac{\alpha^{\frac{1}{2}-s}}{2}e^{-\frac{w^2}{8}}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};\frac{w^2}{4}\right), \tag{2.30}$$

$$\rho_2(\beta,w,s) := \frac{\beta^{\frac{1}{2}-s}}{2}e^{\frac{w^2}{8}}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};-\frac{w^2}{4}\right). \tag{2.31}$$

Kummer's first transformaion for the confluent hypergeometric function [9, p. 191, Equation (4.1.11)] is

$${}_1F_1(a;c;-z) = e^{-z}{}_1F_1(c-a;c;z). \tag{2.32}$$

Using (2.32) in the second equality below, we get

$$\begin{aligned}\rho_2(\beta,w,s) &= \frac{\beta^{\frac{1}{2}-s}}{2}e^{\frac{w^2}{8}}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};-\frac{w^2}{4}\right)\\ &= \frac{\alpha^{s-\frac{1}{2}}}{2}e^{-\frac{w^2}{8}}{}_1F_1\left(\frac{s}{2};\frac{1}{2};\frac{w^2}{4}\right)\\ &= \rho_1(\alpha,w,1-s). \end{aligned} \tag{2.33}$$

Hence the invariance function is given by

$$\begin{aligned}\nabla(\alpha, w, s) &= \rho_1(\alpha, w, s) + \rho_2(\beta, w, s) \\ &= \rho_1(\alpha, w, s) + \rho_1(\alpha, w, 1-s) \\ &= e^{-\frac{w^2}{8}} \left(\frac{\alpha^{\frac{1}{2}-s}}{2} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{w^2}{4}\right) + \frac{\alpha^{s-\frac{1}{2}}}{2} {}_1F_1\left(\frac{s}{2}; \frac{1}{2}; \frac{w^2}{4}\right)\right).\end{aligned}$$

We check that the function ∇ is invariant under the transformation $\alpha \to 1/\alpha$ and $w \to iw$, that is $\nabla(\alpha, w, s) = \nabla(\beta, iw, s)$, using (2.32) in the second equality below.

$$\begin{aligned}\nabla(\beta, iw, s) &= e^{\frac{w^2}{8}} \left(\frac{\beta^{\frac{1}{2}-s}}{2} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; -\frac{w^2}{4}\right) + \frac{\beta^{s-\frac{1}{2}}}{2} {}_1F_1\left(\frac{s}{2}; \frac{1}{2}; -\frac{w^2}{4}\right)\right) \\ &= e^{-\frac{w^2}{8}} \left(\frac{\alpha^{s-\frac{1}{2}}}{2} {}_1F_1\left(\frac{s}{2}; \frac{1}{2}; \frac{w^2}{4}\right) + \frac{\alpha^{\frac{1}{2}-s}}{2} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{w^2}{4}\right)\right) \\ &= \nabla(\alpha, w, s).\end{aligned}$$

This implies that the following integral (2.34) is invariant under the transformation $\alpha \to 1/\alpha$.

$$\frac{e^{-\frac{w^2}{8}}}{2} \int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2} \left(\alpha^{-\frac{it}{2}} {}_1F_1\left(\frac{1-it}{4}; \frac{1}{2}; \frac{w^2}{4}\right) + \alpha^{\frac{it}{2}} {}_1F_1\left(\frac{1+it}{4}; \frac{1}{2}; \frac{w^2}{4}\right)\right) dt. \quad (2.34)$$

The rigorous proof that integral (2.34) is equal to the generalization of the theta transformation formula can be found in [8]. In the paper [8], this integral is first converted into a line integral and then evaluated using the residue calculus and the theory of Mellin transforms.

For $w = 0$

$$\nabla\left(\alpha, 0, \frac{1+it}{2}\right) = \frac{1}{2}\left(\alpha^{-\frac{it}{2}} + \alpha^{\frac{it}{2}}\right) = \cos\left(\frac{1}{2} t \log \alpha\right)$$

as expected since the integral is obtained from (2.20) by generalizing the function

$\cos\left(\frac{1}{2}t\log\alpha\right)$ to $\nabla\left(\alpha, w, \frac{1+it}{2}\right)$.

2.3 Rationale behind the definition of the generalized modified Bessel function $K_{z,w}$

For finding the function $K_{z,w}(x)$ that leads to the pair $(\varphi(\alpha,x,z,w), \psi(\alpha,x,z,w))$ reciprocal in the Koshliakov kernel, we first start with the normalized Mellin transform motivated from the ones (2.30) used for the generalization of the theta transformation formula. We replace the ${}_1F_1$ in these normalized Mellin transforms with a product of two ${}_1F_1$ along with introducing another variable z as follows.

$$\begin{aligned} Z_1(\alpha,s,z,w) &:= \frac{\alpha^{\frac{1}{2}-s}}{4}e^{-\frac{w^2}{4}}{}_1F_1\left(\frac{1-s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{1-s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right) \\ &= \frac{\alpha^{\frac{1}{2}-s}}{4}e^{-\frac{w^2}{4}}{}_1F_1\left(\frac{s+z}{2};\frac{1}{2};\frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};\frac{w^2}{4}\right), \end{aligned} \tag{2.35}$$

$$Z_2(\beta,s,z,w) := \frac{\beta^{\frac{1}{2}-s}}{4}e^{\frac{w^2}{4}}{}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right). \tag{2.36}$$

Using (2.32) in the second equality below, we get

$$\begin{aligned} Z_2(\beta,s,z,w) &:= \frac{\beta^{\frac{1}{2}-s}}{4}e^{\frac{w^2}{4}}{}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right) \\ &= \frac{\alpha^{\frac{1}{2}-s}}{4}e^{-\frac{w^2}{4}}{}_1F_1\left(\frac{s+z}{2};\frac{1}{2};\frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};\frac{w^2}{4}\right) \\ &= Z_1(\alpha,1-s,z,w). \end{aligned}$$

Similarly, it is easy to check the invariance function Z given by

$$\begin{aligned} Z(s,z,w) &= Z_1(\alpha,s,z,w) + Z_2(\beta,s,z,w) \\ &= Z_1(\alpha,s,z,w) + Z_1(\alpha,1-s,z,w), \end{aligned}$$

is invariant under the transformation $\alpha \to 1/\alpha$ and $w \to iw$, that is $Z(\alpha, s, z, w) = Z(\beta, s, z, iw)$.

The normalized Mellin transform $Z_1(\alpha, s, z, w)$ and $Z_2(\beta, s, z, w)$ yields the Mellin transform of the reciprocal pair $(\varphi(\alpha, x, z, w), \psi(\alpha, x, z, w))$ as follows.

$$\int_0^\infty x^{s-1}\varphi(\alpha, x, z, w)\,dx = \pi^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)\frac{\alpha^{\frac{1}{2}-s}}{4}e^{-\frac{w^2}{4}}\times \\ {}_1F_1\left(\frac{s+z}{2};\frac{1}{2};\frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};\frac{w^2}{4}\right) \quad (2.37)$$

$$\int_0^\infty x^{s-1}\psi(\beta, x, z, w)\,dx = \pi^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)\frac{\beta^{\frac{1}{2}-s}}{4}e^{\frac{w^2}{4}}\times \\ {}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right) \quad (2.38)$$

This implies that the functions in the reciprocal pair $(\varphi(\alpha, x, z, w), \psi(\alpha, x, z, w))$ are given by the following inverse Mellin transforms.

$$\varphi(\alpha, x, z, w) = \frac{1}{2\pi i}\int_{(c)}\pi^{-s}\frac{\alpha^{\frac{1}{2}-s}}{4}e^{-\frac{w^2}{4}}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)\times \\ {}_1F_1\left(\frac{s-z}{2};\frac{1}{2};\frac{w^2}{4}\right){}_1F_1\left(\frac{s+z}{2};\frac{1}{2};\frac{w^2}{4}\right)x^{-s}ds, \quad (2.39)$$

and

$$\psi(\beta, x, z, w) = \frac{1}{2\pi i}\int_{(c)}\pi^{-s}\frac{\beta^{\frac{1}{2}-s}}{4}e^{\frac{w^2}{4}}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)\times \\ {}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)x^{-s}ds. \quad (2.40)$$

This in turn leads to the definition of the new function $K_{z,w}(x)$ as follows. For

$z, w \in \mathbb{C}$, $x \in \mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$, and $\mathrm{Re}(s) > \pm\,\mathrm{Re}(z)$,

$$K_{z,w}(2\pi x) := \frac{1}{2\pi i}\times \int_{(c)} \frac{\pi^{-s}}{4}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right) {}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right) {}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)x^{-s}ds. \tag{2.41}$$

The pair of functions given by $\varphi(\alpha, x, z, w) = \sqrt{\alpha}e^{\frac{w^2}{4}}K_{z,w}(2\pi\alpha x)$ and $\psi(\beta, x, z, w) = \sqrt{\beta}e^{-\frac{w^2}{4}}K_{z,iw}(2\pi\beta x)$ reduces to the pair $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$, seen above in relation to the Ramanujan-Guinand formula, when $w = 0$. Note that the pair itself is invariant under the transformations $\alpha \to 1/\alpha$ and $w \to iw$, that is $\psi(\beta, x, z, iw) = \varphi(\alpha, x, z, w)$. In the next chapter, we prove that this pair (φ, ψ) is reciprocal in the Koshliakov kernel and then proceed to apply Theorem 2.1.2 for this pair to find the desired modular-type transformation formula of the kind $F(\alpha, z, w) = F(\beta, z, iw)$ in Chapter 4.

Chapter 3

A pair of functions reciprocal in the Koshliakov kernel

The new function $K_{z,w}(x)$ given as an inverse Mellin transform can be considered as a generalization of the modified Bessel function of the second kind function $K_z(x)$, in the sense that the function $K_{z,w}(x)$ reduces to $K_z(x)$ when $w = 0$. To the best of the author's knowledge, the modified Bessel function $K_z(x)$ is the only known function that is self-reciprocal in the Koshlikov kernel. The self-reciprocity of $K_z(x)$ leads to the pair $(\sqrt{\alpha}K_z(2\pi\alpha x), \sqrt{\beta}K_z(2\pi\beta x))$ reciprocal in the Koshliakov kernel, that is central to Koshliakov's formula and Ramanujan-Guinand formula, as seen in the previous chapter. It is to be noted that there are pairs of functions, not involving $K_z(x)$, known to be reciprocal in the Koshliakov kernel that are not derived from a self-reciprocal function.

The function $K_{z,w}(x)$ is introduced with an intention to generate a pair reciprocal in the Koshliakov kernel that can yield a transformula formula. It turns out that the function $K_{z,w}(x)$ is not self-reciprocal in the Koshlikov kernel like the modified Bessel function $K_z(x)$ and its definition cannot be easily tweaked to yield a self-reciprocal function. This is because the extra variable w introduced in the confluent hypergeometric functions ${}_1F_1$ in the definition of the function $K_{z,w}(x)$ do not play along well. However, the Koshliakov transform of a slightly modified function $e^{\frac{w^2}{4}}K_{z,w}(x)$ yields $e^{-\frac{w^2}{4}}K_{z,iw}(x)$ and vice versa. The pair of functions are almost the same except for the change in variable $w \to iw$. Thus, even though $K_{z,w}(x)$ itself is not self-reciprocal in the Koshliakov kernel, it conveniently yields a pair of functions $(\sqrt{\alpha}e^{\frac{w^2}{4}}K_{z,w}(2\pi\alpha x), \sqrt{\beta}e^{-\frac{w^2}{4}}K_{z,iw}(2\pi\beta x))$ that is reciprocal in the Koshliakov kernel exhibiting the desired symmetry.

To show the reciprocity of the pair, the Koshliakov transform of the function $e^{\frac{w^2}{4}}K_{z,w}(x)$ is taken and first shown to be convergent and then evaluated to give back the $e^{-\frac{w^2}{4}}K_{z,iw}(x)$. For the convergence of the integral, the asymptotics of the function $K_{z,w}(x)$ is required, that is stated in Theorem 5.4.1 in Chapter 5. The derivation of the asymptotics do not use the reciprocity of the functions involving $K_{z,w}(x)$ in the Koshliakov kernel and hence, it can be used in the proof of reciprocity given below. On a side note, the results of Chapter 5 establishing the theory of the function $K_{z,w}(x)$ are studied independently of the goal of proving the reciprocity in the Koshliakov kernel and the transformation formula.

Let the Koshlikov kernel be denoted as $g(x)$, that is

$$g(x) := \cos(\pi z)M_{2z}(4\pi\sqrt{x}) - \sin(\pi z)J_{2z}(4\pi\sqrt{x}) \tag{3.1}$$

Then the Koshlikov transform of a function $f(x)$ is given by

$$2\pi\int_0^\infty f(x)\left(\cos(\pi z)M_{2z}(4\pi\sqrt{x}) - \sin(\pi z)J_{2z}(4\pi\sqrt{x})\right)\,dx = 2\pi\int_0^\infty f(x)g(x)\,dx \tag{3.2}$$

To evaluate the Koshlikov transform of any function $f(x)$ in general, a very useful tool is the Parseval identity given below. If $\mathfrak{F}$ and $\mathfrak{G}$ respectively denote the Mellin transforms of g and h satisfying appropriate conditions, and if the line $\mathrm{Re}(s) = c$ lies in the common strip of analyticity of $\mathfrak{F}(1-s)$ and $\mathfrak{G}(s)$, then Parseval's identity

[25, p. 82, Equation (3.1.11)] gives

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{(c)} \mathfrak{F}(1-s)\mathfrak{G}(s)\,ds. \tag{3.3}$$

A variant of this formula is [25, p. 83, Equation (3.1.13)]

$$\int_0^\infty f(x)g\left(\frac{t}{x}\right)\frac{dx}{x} = \frac{1}{2\pi i}\int_{(\sigma)} \mathfrak{F}(s)\mathfrak{G}(s)t^{-s}ds. \tag{3.4}$$

Thus to evaluate the integral (3.2), we need the Mellin transforms for both the functions $f(x)$, which is $K_{z,w}(x)$ in our case, and $g(x)$, that is the Koshlikov kernel. The Mellin transform for the function $K_{z,w}(x)$ follows directly from its definition whereas the Mellin transform the Koshliakov kernel is derived below by first using the Mellin transforms of the Bessel functions in its definition.

Lemma 3.0.1. *For* $\pm Re(z) < Re(s) < \frac{3}{4}$ *and* $t > 0$, *the Mellin transform of the Koshlikove kernel is*

$$\int_0^\infty x^{s-1}\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dx$$
$$= \frac{1}{2^{2s}\pi^{1+2s}t^s}\Gamma(s-z)\Gamma(s+z)\left(\cos(\pi z)+\cos(\pi s)\right) \tag{3.5}$$

Proof. The Mellin transform

$$\int_0^\infty x^{s-1}\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dx, \tag{3.6}$$

where the function $M_z(x)$ is defined by

$$M_z(x) := \frac{2}{\pi}K_z(x) - Y_z(x),$$

can be evaluated by considering the mellin transforms for the Bessel functions appearing in the Koshliakov kernel. The Mellin transform of the modified Bessel function $K_z(x)$, as given in [19, p. 115, formula 11.1], is

$$\int_0^\infty x^{s-1} K_z(ax)\, dx = 2^{s-2} a^{-s} \Gamma\left(\frac{s-z}{2}\right) \Gamma\left(\frac{s+z}{2}\right), \tag{3.7}$$

for $\mathrm{Re}(s) > \pm\,\mathrm{Re}(z)$ and $a > 0$. This implies

$$\int_0^\infty x^{s-1} \frac{2}{\pi} K_{2z}(4\pi\sqrt{xt})\, dx = 2^{-2s} \pi^{-1-2s} t^{-s} \Gamma(s-z)\Gamma(s+z), \tag{3.8}$$

for $\mathrm{Re}(s) > \pm\,\mathrm{Re}(z)$ and $t > 0$. Similarly, the Mellin transform of the Bessel function of the second kind $Y_z(x)$, as given in [19, p. 93, formula 10.2], is

$$\int_0^\infty x^{s-1} Y_z(ax)\, dx = -\frac{1}{\pi} 2^{s-1} a^{-s} \cos\left(\frac{\pi}{2}(s-z)\right) \Gamma\left(\frac{s-z}{2}\right) \Gamma\left(\frac{s+z}{2}\right), \tag{3.9}$$

for $\pm\mathrm{Re}(z) < \mathrm{Re}(s) < \frac{3}{2}$. This implies

$$\int_0^\infty x^{s-1} Y_{2z}(4\pi\sqrt{xt})\, dx = 2^{-2s} \pi^{-1-2s} t^{-s} \cos\left(\pi(s-z)\right) \Gamma(s-z)\Gamma(s+z), \tag{3.10}$$

for $\pm\mathrm{Re}(z) < \mathrm{Re}(s) < \frac{3}{4}$. This in turn implies

$$\int_0^\infty x^{s-1} M_{2z}(4\pi\sqrt{xt})\, dx = 2^{-2s} \pi^{-1-2s} t^{-s} \left(1 + \cos\left(\pi(s-z)\right)\right) \Gamma(s-z)\Gamma(s+z) \tag{3.11}$$

for $\pm\mathrm{Re}(z) < \mathrm{Re}(s) < \frac{3}{4}$. On the other hand, the Mellin transform of the Bessel function of the first kind $J_z(x)$, as given in [19, p. 93, formula 10.1], is

$$\int_0^\infty x^{s-1} J_z(ax)\, dx = -\frac{1}{2} \left(\frac{a}{2}\right)^{-s} \frac{\Gamma\left(\frac{s+z}{2}\right)}{\Gamma\left(1 + \frac{z-s}{2}\right)} \tag{3.12}$$

for $-\mathrm{Re}(z) < \mathrm{Re}(s) < \frac{3}{2}$. This implies

$$\int_0^\infty x^{s-1} J_{2z}(4\pi\sqrt{xt})\, dx = 2^{-2s}\pi^{-2s}t^{-s}\frac{\Gamma(s+z)}{\Gamma(1-s+z)} \tag{3.13}$$

for $-\mathrm{Re}(z) < \mathrm{Re}(s) < \frac{3}{4}$. Using the reflection formula for the gamma function [20, p.46]

$$\Gamma(\omega)\Gamma(1-\omega) = \frac{\pi}{\sin(\pi\omega)} \tag{3.14}$$

the equation (3.13) yields

$$\int_0^\infty x^{s-1} J_{2z}(4\pi\sqrt{xt})\, dx = 2^{-2s}\pi^{-1-2s}t^{-s}\sin\left(\pi(s-z)\right)\Gamma(s-z)\Gamma(s+z) \tag{3.15}$$

Combining the evaluations (3.11) and (3.15), we get

$$\begin{aligned}
&\int_0^\infty x^{s-1}\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\, dx\\
&= \frac{1}{2^{2s}\pi^{1+2s}t^s}\Gamma(s-z)\Gamma(s+z)\left(\cos(\pi z) + \cos\left(\pi(s-z)\right)\cos(\pi z) - \sin\left(\pi(s-z)\right)\sin(\pi z)\right)\\
&= \frac{1}{2^{2s}\pi^{1+2s}t^s}\Gamma(s-z)\Gamma(s+z)\left(\cos(\pi z) + \cos(\pi s)\right).
\end{aligned}$$

Hence proved. □

Lemma 3.0.2. *The Mellin transform of the generalization of the modified Bessel function $K_{z,w}(x)$, for $z, w \in \mathbb{C}$, and Re$(s) > \pm$ Re(z), is*

$$\begin{aligned}
&\int_0^\infty x^{s-1}K_{z,w}(2\pi x)\, dx\\
&\quad = \frac{\pi^{-s}}{4}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right).
\end{aligned} \tag{3.16}$$

Proof. Follows directly from the definition (2.41) of the function $K_{z,w}(x)$ as the inverse Mellin transform of the right side of equation (3.16). □

The reciprocal behavior of the generalized modified Bessel function $K_{z,w}(x)$ in the

Ksohliakov kernel is stated and proved as follows.

Theorem 3.0.3. *Let* $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$. *Let* $w \in \mathbb{C}$ *and* $x > 0$.

$$2\pi \int_0^\infty e^{\frac{w^2}{4}} K_{z,w}(2\pi x) \left(\cos(\pi z) M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z) J_{2z}(4\pi\sqrt{xt}) \right) dx \\ = e^{-\frac{w^2}{4}} K_{z,iw}(2\pi t). \quad (3.17)$$

Proof. Note that from Theorems 5.4.1, 5.4.2 and from the bound [16, Eqn. (2.11)]

$$\left| \cos(\pi z)\, M_{2z}(4\pi\sqrt{tx}) - \sin(\pi z)\, J_{2z}(4\pi\sqrt{tx}) \right| \ll_z \begin{cases} 1 + |\log(tx)|, & \text{if } z = 0, 0 \le tx \le 1, \\ (tx)^{-|\mathrm{Re}(z)|}, & \text{if } z \ne 0, 0 \le tx \le 1, \\ (tx)^{-1/4}, & \text{if } tx \ge 1, \end{cases}$$

we see that the integrals in Theorem 3.0.4 indeed converge for $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$. From Lemma 3.0.1, we have

$$\int_0^\infty x^{s-1} \left(\cos(\pi z) M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z) J_{2z}(4\pi\sqrt{xt}) \right) dx \\ = \frac{1}{2^{2s}\pi^{2s+1}t^s} \Gamma(s-z)\Gamma(s+z) \left(\cos(\pi z) + \cos(\pi s) \right), \quad (3.18)$$

for $\pm\mathrm{Re}(z) < \mathrm{Re}(s) < 3/4$ and $t > 0$. From Lemma 3.0.2, we have

$$\int_0^\infty x^{s-1} K_{z,w}(2\pi x)\, dx \\ = \frac{\pi^{-s}}{4} \Gamma\left(\frac{s-z}{2}\right) \Gamma\left(\frac{s+z}{2}\right) {}_1F_1\left(\frac{s-z}{2}; \frac{1}{2}; -\frac{w^2}{4}\right) {}_1F_1\left(\frac{s+z}{2}; \frac{1}{2}; -\frac{w^2}{4}\right), \quad (3.19)$$

for $\mathrm{Re}(s) > \pm\mathrm{Re}(z)$. Note that by the hypothesis, we have $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$ so that $\pm\mathrm{Re}(z) < 1 \pm \mathrm{Re}(z)$. Then by Parseval's identity (3.3), and the evaluations (3.18) and

(3.19), we have

$$
\begin{aligned}
&2\pi \int_0^\infty e^{\frac{w^2}{4}} K_{z,w}(2\pi t) \left(\cos(\pi z) M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z) J_{2z}(4\pi\sqrt{xt})\right) \, dt \\
&= \frac{1}{2\pi i} \int_{(c)} e^{\frac{w^2}{4}} \Gamma\left(\frac{1-s-z}{2}\right) \Gamma\left(\frac{1-s+z}{2}\right) \frac{\Gamma(s-z)\Gamma(s+z)}{2^{2s+1}\pi^{s+1}t^s} \left(\cos(\pi z) + \cos(\pi s)\right) \times \\
&\qquad {}_1F_1\left(\frac{1-s-z}{2}; \frac{1}{2}; -\frac{w^2}{4}\right) {}_1F_1\left(\frac{1-s+z}{2}; \frac{1}{2}; -\frac{w^2}{4}\right) ds, \qquad (3.20)
\end{aligned}
$$

for $\pm\mathrm{Re}(z) < c = \mathrm{Re}(s) < \min\left(\frac{3}{4}, 1 \pm \mathrm{Re}(z)\right)$, by an application of (2.32). Legrendre's duplication formula for the gamma function is

$$
\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right). \qquad (3.21)
$$

Hence, we have

$$
\begin{aligned}
\Gamma(s-z) &= \frac{2^{s-z-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-z}{2}\right) \Gamma\left(\frac{s-z+1}{2}\right), \\
\Gamma(s+z) &= \frac{2^{s+z-1}}{\sqrt{\pi}} \Gamma\left(\frac{s+z}{2}\right) \Gamma\left(\frac{s+z+1}{2}\right).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\Gamma\left(\frac{1-s-z}{2}\right) \Gamma\left(\frac{1-s+z}{2}\right) \Gamma(s-z)\Gamma(s+z) &= \frac{2^{2s-2}}{\pi} \Gamma\left(\frac{s-z}{2}\right) \Gamma\left(\frac{s+z}{2}\right) \times \\
&\quad \Gamma\left(\frac{s-z+1}{2}\right) \Gamma\left(\frac{1-s+z}{2}\right) \Gamma\left(\frac{s+z+1}{2}\right) \Gamma\left(\frac{1-s-z}{2}\right) \\
&= \frac{2^{2s-2}\pi}{\sin(\pi\left(\frac{s-z+1}{2}\right)) \sin(\pi\left(\frac{s+z+1}{2}\right))} \Gamma\left(\frac{s-z}{2}\right) \Gamma\left(\frac{s+z}{2}\right), \qquad (3.22)
\end{aligned}
$$

where the last equality is obtained by using the reflection formula (3.14) for the gamma function. Using the trigonometric identity for the product of the sine functions yields

$$\Gamma\left(\frac{1-s-z}{2}\right)\Gamma\left(\frac{1-s+z}{2}\right)\Gamma(s-z)\Gamma(s+z)\left(\cos(\pi z)+\cos(\pi s)\right)$$
$$=\pi 2^{2s-1}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right). \quad (3.23)$$

Thus from (3.20) and (3.23), we have

$$2\pi\int_0^\infty e^{\frac{w^2}{4}}K_{z,w}(2\pi t)\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt})-\sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dt$$
$$=\frac{1}{2\pi i}\int_{(c)}\frac{\pi^{-s}}{4}e^{\frac{w^2}{4}}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)\times$$
$${}_1F_1\left(\frac{1-s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{1-s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)t^{-s}\,ds.$$

Using Kummer's first transformationn 2.32 for ${}_1F_1$, we get

$$2\pi\int_0^\infty e^{\frac{w^2}{4}}K_{z,w}(2\pi t)\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt})-\sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dt$$
$$=\frac{1}{2\pi i}\int_{(c)}\frac{\pi^{-s}}{4}e^{-\frac{w^2}{4}}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right)\times$$
$${}_1F_1\left(\frac{s-z}{2};\frac{1}{2};\frac{w^2}{4}\right){}_1F_1\left(\frac{s+z}{2};\frac{1}{2};\frac{w^2}{4}\right)t^{-s}\,ds$$
$$=e^{-\frac{w^2}{4}}K_{z,iw}(2\pi t),$$

where the last step follows from the definition (2.41) of $K_{z,iw}(x)$. Hence proved. □

Replacing w by iw on both sides of (3.17) in Theorem 3.0.3 above proves the reciprocity of the pair of function $(e^{\frac{w^2}{4}}K_{z,w}(2\pi x), e^{-\frac{w^2}{4}}K_{z,iw}(2\pi x))$ in the Koshliakov kernel. The technique used to prove the reciprocity can be summarized as first deriving the Mellin tranform of one of the functions in the pair say φ, then applying Parseval's identity (3.3) using the Mellin transforms of function φ as well as that of the Koshliakov kernel, and then evaluating the integral to give back the other function in the pair say ψ, along with showing the convergence of the Koshliakov transform integral and then repeating the process with the other function ψ. This technique is one of the reasons that starting with a function defined as an inverse Mellin transform has worked so well in generating the transformation formula, as shown below in Chapter 4. to derive the transformation formulas.

Theorem 3.0.4. *Let* $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$, $w \in \mathbb{C}$, $x > 0$ *and* $\alpha, \beta > 0$. *The functions* $\sqrt{\alpha}e^{\frac{w^2}{4}}K_{z,w}(2\pi\alpha x)$ *and* $\sqrt{\beta}e^{-\frac{w^2}{4}}K_{z,iw}(2\pi\beta x)$ *form a pair of reciprocal functions in the Koshliakov kernel, that is*

$$2\pi \int_0^\infty \sqrt{\alpha}e^{\frac{w^2}{4}}K_{z,w}(2\pi\alpha x)\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dx$$
$$= \sqrt{\beta}e^{-\frac{w^2}{4}}K_{z,iw}(2\pi\beta t), \quad (3.24)$$

$$2\pi \int_0^\infty \sqrt{\beta}e^{-\frac{w^2}{4}}K_{z,iw}(2\pi\beta x)\left(\cos(\pi z)M_{2z}(4\pi\sqrt{xt}) - \sin(\pi z)J_{2z}(4\pi\sqrt{xt})\right)\,dx$$
$$= \sqrt{\alpha}e^{\frac{w^2}{4}}K_{z,w}(2\pi\alpha t). \quad (3.25)$$

Proof. The equation (3.24) follows from Theorem 3.0.3 by the change of variables $x \to \alpha x$ and $t \to \beta t$. Similarly the equation (3.25) follows from Theorem 3.0.3 by the change of variables $x \to \beta x$ and $t \to \alpha t$. □

This gives the desired reciprocal pair $(\sqrt{\alpha}e^{\frac{w^2}{4}}K_{z,w}(2\pi\alpha x), \sqrt{\beta}e^{-\frac{w^2}{4}}K_{z,iw}(2\pi\beta x))$ that is used below in Chapter 4 in deriving the generalization of the Ramanujan-Guinand formula, that is of the form $F(\alpha, z, w) = F(\beta, z, iw)$.

Chapter 4

A generalization of Ramanujan-Guinand formula of the form $F(\alpha, z, w) = F(\beta, z, iw)$

The transformation formula of the kind $F(\alpha, z, w) = F(\beta, z, iw)$ given in 4.0.2 is the desired generalization of Ramanujan-Guinand formula. Theorem 4.0.4 gives the integral associated with this formula. The formula is first proved independently using Lemma 4.0.1 by Guinand [11, equation (1)] given below and then the formula itself is later used as an intermediate step in the proof of Theorem 4.0.4.

Lemma 4.0.1. *If $f(x)$ and $f'(x)$ are integrals, f tends to zero as $x \to \infty$, $f(x), xf'(x)$, and $x^2 f''(x)$ belong to $L^2(0,\infty)$, and*

$$g(x) = 2\pi \int_0^\infty f(t) \left(\cos\left(\frac{\pi z}{2}\right) M_z(4\pi\sqrt{xt}) - \sin\left(\frac{\pi z}{2}\right) J_z(4\pi\sqrt{xt}) \right) dt, \tag{4.1}$$

then the following identity holds true:

$$\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}f(n)-\zeta(1+z)\int_0^{\infty}x^{\frac{z}{2}}f(x)\,dx-\zeta(1-z)\int_0^{\infty}x^{-\frac{z}{2}}f(x)\,dx$$
$$=\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}g(n)-\zeta(1+z)\int_0^{\infty}x^{\frac{z}{2}}g(x)\,dx-\zeta(1-z)\int_0^{\infty}x^{-\frac{z}{2}}g(x)\,dx$$

Theorem 4.0.2 (Generalization of Ramanujan-Guinand identity)**.** *Let* $w\in\mathbb{C}$, $z\in\mathbb{C}\backslash\{-1,1\}$. *For* $\alpha,\beta>0$,

$$\sqrt{\alpha}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2n\pi\alpha)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)$$
$$-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)\Bigg)$$
$$=\sqrt{\beta}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{-\frac{w^2}{4}}K_{\frac{z}{2},iw}(2n\pi\beta)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\beta^{\frac{z}{2}-1}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right)$$
$$-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\beta^{-\frac{z}{2}-1}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)\Bigg). \quad (4.2)$$

Proof. We first prove the result for a fixed z such that $-1<\mathrm{Re}(z)<1$ and then extend it by analytic continuation. Let $f(x)=\sqrt{\alpha}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2\pi\alpha x)$ and $g(x)=\sqrt{\beta}e^{-\frac{w^2}{4}}K_{\frac{z}{2},iw}(2\pi\beta x)$. Then from Theorem 3.0.4 $f(x)$ and $g(x)$ satisfy (4.1) when $-1<\mathrm{Re}(z)<1$. Now (2.41) implies that for $\mathrm{Re}(s)>\pm\mathrm{Re}(\frac{z}{2})$,

$$\int_0^{\infty}x^{s-1}f(x)\,dx=\frac{\sqrt{\alpha}e^{\frac{w^2}{4}}}{4(\pi\alpha)^s}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\times$$
$${}_1F_1\left(\frac{s}{2}-\frac{z}{4};\frac{1}{2};-\frac{w^2}{4}\right){}_1F_1\left(\frac{s}{2}+\frac{z}{4};\frac{1}{2};-\frac{w^2}{4}\right). \quad (4.3)$$

Since $\mathrm{Re}(z)>-1$, we can let $s=1+\frac{z}{2}$ in the above equation so that

$$\int_0^{\infty}x^{\frac{z}{2}}f(x)\,dx=\frac{1}{4}(\pi\alpha)^{-\frac{(1+z)}{2}}\Gamma\left(\frac{1+z}{2}\right){}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right), \quad (4.4)$$

since ${}_1F_1\left(\frac{1}{2};\frac{1}{2};-\frac{w^2}{4}\right)=e^{-\frac{w^2}{4}}$. Since $\mathrm{Re}(z)<1$, we can let $s=1-\frac{z}{2}$ in (4.3) whence

$$\int_0^\infty x^{-\frac{z}{2}}f(x)\,dx=\frac{1}{4}(\pi\alpha)^{-\frac{(1-z)}{2}}\Gamma\left(\frac{1-z}{2}\right){}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right). \tag{4.5}$$

Furthermore, for $\mathrm{Re}(s)>\pm\mathrm{Re}(\frac{z}{2})$,

$$\int_0^\infty x^{s-1}g(x)\,dx=\frac{\sqrt{\beta}e^{-\frac{w^2}{4}}}{4(\pi\beta)^s}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\times$$
$$ {}_1F_1\left(\frac{s}{2}-\frac{z}{4};\frac{1}{2};\frac{w^2}{4}\right){}_1F_1\left(\frac{s}{2}+\frac{z}{4};\frac{1}{2};\frac{w^2}{4}\right),$$

so that

$$\int_0^\infty x^{\pm\frac{z}{2}}g(x)\,dx=\frac{1}{4}(\pi\beta)^{-\frac{1\pm z}{2}}\Gamma\left(\frac{1\pm z}{2}\right){}_1F_1\left(\frac{1\pm z}{2};\frac{1}{2};\frac{w^2}{4}\right). \tag{4.6}$$

Hence from (4.4), (4.5) and (4.6) and Lemma 4.0.1, we see that

$$\sum_{n=1}^\infty \sigma_{-z}(n)n^{\frac{z}{2}}\sqrt{\alpha}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2\pi\alpha n)-\frac{(\pi\alpha)^{-\frac{(1+z)}{2}}}{4}\Gamma\left(\frac{1+z}{2}\right)\zeta(1+z)\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)$$
$$-\frac{(\pi\alpha)^{-\frac{(1-z)}{2}}}{4}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)=$$
$$\sum_{n=1}^\infty \sigma_{-z}(n)n^{\frac{z}{2}}\sqrt{\beta}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2\pi\underline{n})-\frac{(\pi\beta)^{-\frac{(1+z)}{2}}}{4}\Gamma\left(\frac{1+z}{2}\right)\zeta(1+z)\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)$$
$$-\frac{(\pi\beta)^{-\frac{(1-z)}{2}}}{4}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right).$$

Using the functional equation of the Riemann $\zeta(s)$ in the following forms

$$\pi^{-\frac{(1+z)}{2}}\Gamma\left(\frac{1+z}{2}\right)\zeta(1+z)=\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z),$$
$$\pi^{-\frac{(1-z)}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)=\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z),$$

replacing α by β^{-1} and β by α^{-1} and multliplying by 4 on both sides, we get

$$\begin{aligned}
&4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}\sqrt{\alpha}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2\pi\alpha n)-\alpha^{\frac{(1+z)}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)\\
&\qquad\qquad-\alpha^{\frac{(1-z)}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)\\
&=4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}\sqrt{\beta}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2\pi\underline{\mathrm{n}})-\beta^{\frac{(1+z)}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)\\
&\qquad\qquad-\beta^{\frac{(1-z)}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right).
\end{aligned}$$

This simplifies to (4.2) completing the proof of Theorem 4.0.2 for $-1<\mathrm{Re}(z)<1$. Note that both sides are analytic, as functions of z, in $\mathbb{C}\backslash\{-1,1\}$ since the poles of $\Gamma\left(\pm\frac{z}{2}\right)$ at $z=\mp 2,\mp 4,\cdots$ are the trivial zeros of $\zeta(\pm z)$. Hence the result holds in $\mathbb{C}\backslash\{-1,1\}$ by analytic continuation. □

Remark: In [26, p. 60], it is shown that the Ramanujan-Guinand formula is equivalent to the functional equation of the non-holomorphic Eisenstein series on $SL_2(\mathbb{Z})$. (See also [10, p. 23] for discussion on this topic.) The generalization of the Ramanujan-Guinand formula that we have obtained in Theorem 4.0.2 now poses a very interesting question - is this generalization equivalent to the functional equation of some generalization of the non-holomorphic Eisenstein series on $SL_2(\mathbb{Z})$?

The following corollary is obtained by letting $z\to 0$ in Theorem 4.0.2.

Corollary 4.0.3. *Let $w\in\mathbb{C}$. For $\alpha,\beta>0$,*

$$\begin{aligned}
&\sqrt{\alpha}\left\{4\sum_{n=1}^{\infty}d(n)e^{-\frac{w^2}{4}}K_{0,iw}(2n\pi\alpha)-\frac{1}{\alpha}\left((\gamma-\log(4\pi\alpha))\left(1-\frac{w^2}{4}\right)+\frac{w^2}{2}\right)\right\}\\
&=\sqrt{\beta}\left\{4\sum_{n=1}^{\infty}d(n)e^{\frac{w^2}{4}}K_{0,w}(2n\pi\beta)-\frac{1}{\beta}\left((\gamma-\log(4\pi\beta))\left(1+\frac{w^2}{4}\right)-\frac{w^2}{2}\right)\right\}. \qquad (4.7)
\end{aligned}$$

Proof. The Laurent series expansion of the gamma function is given by [21, p. 903, formula **8.321**, no. 1]

$$\Gamma(z) = \frac{1}{z} - \gamma + \cdots , \tag{4.8}$$

where as the power series expansion of $\zeta(z)$ around $z = 0$ is given by [1, p. 19-20, Equations (2.4.3), (2.4.5)]

$$\zeta(z) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)z + \cdots , \tag{4.9}$$

Also,

$$\left(\frac{\alpha}{\pi}\right)^{\frac{z}{2}} = 1 + \frac{z}{2}\log\left(\frac{\alpha}{\pi}\right) + \cdots , \tag{4.10}$$

and

$${}_1F_1\left(\frac{1-z}{2}; \frac{1}{2}; \frac{w^2}{4}\right) = 1 + \frac{w^2}{4} - \frac{w^2}{4}z + \cdots . \tag{4.11}$$

From (4.8), (4.9), (4.10) and (4.11),

$$\begin{aligned}
&\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}\,{}_1F_1\left(\frac{1-z}{2}; \frac{1}{2}; \frac{w^2}{4}\right) \\
&= \frac{1}{\alpha}\left(\frac{2}{z} - \gamma + \cdots\right)\left(-\frac{1}{2} - \frac{1}{2}\log(2\pi)z + \cdots\right)\left(1 + \frac{z}{2}\log\left(\frac{\alpha}{\pi}\right) + \cdots\right)\times \\
&\qquad\qquad\left(1 + \frac{w^2}{4} - \frac{w^2}{4}z + \cdots\right) \\
&= \frac{1}{\alpha}\Bigg[-\frac{1}{z}\left(1 + \frac{w^2}{4}\right) + \left(1 + \frac{w^2}{4}\right)\left(\frac{\gamma}{2} - \log(2\sqrt{\pi\alpha})\right) + \frac{w^2}{4} \\
&\qquad\qquad + \text{terms with positive powers of } z\Bigg].
\end{aligned} \tag{4.12}$$

Similarly,

$$\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)$$
$$=\frac{1}{\alpha}\left[\frac{1}{z}\left(1+\frac{w^2}{4}\right)+\left(1+\frac{w^2}{4}\right)\left(\frac{\gamma}{2}-\log(2\sqrt{\pi\alpha})\right)+\frac{w^2}{4}\right.$$
$$\left.+\text{ terms with positive powers of } z\right]. \tag{4.13}$$

From (4.12) and (4.13),

$$\lim_{z\to0}\left[\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)+\right.$$
$$\left.\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)\right]$$
$$=\frac{1}{\alpha}\left[\left(1+\frac{w^2}{4}\right)(\gamma-\log(4\pi\alpha))+\frac{w^2}{2}\right]. \tag{4.14}$$

Letting $z\to0$ in (4.2) and using (4.14) for the left side of (4.2), and then replacing α by β and w by iw in (4.14) and using it for the right side of (4.2), gives (4.7) upon simplification. □

Theorem 4.0.4. *Let $w\in\mathbb{C}$, $-1<\mathrm{Re}(z)<1$ and $\alpha,\beta>0$*

$$\frac{16}{\pi}\int_0^\infty\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\mathcal{Z}\left(\alpha,\frac{1+it}{2},\frac{z}{2},w\right)\,dt}{(t^2+(z+1)^2)\,(t^2+(z-1)^2)}$$
$$=e^{-\frac{w^2}{4}}\sqrt{\alpha}\left\{4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{-\frac{w^2}{4}}K_{\frac{z}{2},iw}(2n\pi\alpha)-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)\right.$$
$$\left.-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)\right\}. \tag{4.15}$$

where

$$\mathcal{Z}\left(\alpha, \frac{1+it}{2}, \frac{z}{2}, w\right) = e^{-\frac{w^2}{4}}\left(\alpha^{-\frac{it}{2}}{}_1F_1\left(\frac{1+z+it}{2}; \frac{1}{2}; \frac{w^2}{4}\right){}_1F_1\left(\frac{1-z+it}{2}; \frac{1}{2}; \frac{w^2}{4}\right)\right.$$
$$\left. + \alpha^{\frac{it}{2}}{}_1F_1\left(\frac{1+z-it}{2}; \frac{1}{2}; \frac{w^2}{4}\right){}_1F_1\left(\frac{1-z-it}{2}; \frac{1}{2}; \frac{w^2}{4}\right)\right). \tag{4.16}$$

Proof. We show that Theorem 4.0.4 follows from (2.7) upon choosing the pair (φ, ψ) of functions reciprocal in the Koshliakov kernel to be $(\sqrt{\alpha}e^{\frac{w^2}{4}}K_{z,w}(2\pi\alpha x), \sqrt{\beta}e^{\frac{w^2}{4}}K_{z,w}(2\pi\beta x))$. The reciprocal property for this choice of the pair follows from Theorem 3.0.4. First we show that these two functions are in the diamond class $\Diamond_{\eta,\omega}$ defined in the introduction. It suffices to show only $K_{z,w}(x)$ as a member of the class. To that end, note that Theorem 2.3 from [20, p. 30-31] implies that $K_{z,w}(x)$ defined by the integral in Theorem 5.1.1 is analytic in x in $|\arg x| < \frac{\pi}{4}$, so the ω in the definition of $\Diamond_{\eta,\omega}$ can be taken to be $\pi/4$. Now Theorem 5.4.2 implies that the first bound in (2.6) is satisfied where as Theorem 5.4.1 implies that $K_{z,w}(x)$ satisfies the second bound as well. This prove that $K_{z,w}(x) \in \Diamond_{\eta,\omega}$.

The normalized Mellin transforms (2.35) and (2.36) for the above pair (φ, ψ) from Chapter 2 are

$$Z_1(\alpha, s, z, w) = \frac{\alpha^{\frac{1}{2}-s}}{4}e^{-\frac{w^2}{4}}{}_1F_1\left(\frac{s+z}{2}; \frac{1}{2}; \frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2}; \frac{1}{2}; \frac{w^2}{4}\right),$$
$$Z_2(\beta, s, z, w) = \frac{\beta^{\frac{1}{2}-s}}{4}e^{\frac{w^2}{4}}{}_1F_1\left(\frac{s+z}{2}; \frac{1}{2}; -\frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2}; \frac{1}{2}; -\frac{w^2}{4}\right),$$

so that

$$Z(s, z, w) = \frac{\alpha^{\frac{1}{2}-s}}{4}e^{-\frac{w^2}{4}}{}_1F_1\left(\frac{s+z}{2}; \frac{1}{2}; \frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2}; \frac{1}{2}; \frac{w^2}{4}\right)$$
$$+ \frac{\beta^{\frac{1}{2}-s}}{4}e^{\frac{w^2}{4}}{}_1F_1\left(\frac{s+z}{2}; \frac{1}{2}; -\frac{w^2}{4}\right){}_1F_1\left(\frac{s-z}{2}; \frac{1}{2}; -\frac{w^2}{4}\right). \tag{4.17}$$

Using Kummer's first transformation (2.32) for ${}_1F_1$ in the second equality below:

$$\begin{aligned} Z_2(\beta,s,z,w) &= \frac{\beta^{\frac{1}{2}-s}}{4} e^{\frac{w^2}{4}} {}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right) {}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right) \\ &= \frac{\alpha^{s-\frac{1}{2}}}{4} e^{-\frac{w^2}{4}} {}_1F_1\left(\frac{1-s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right) {}_1F_1\left(\frac{1-s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right) \\ &= Z_1(\alpha,1-s,z,w). \end{aligned}$$

This implies

$$\begin{aligned} Z(s,z,w) &= Z_1(\alpha,s,z,w) + Z_2(\beta,s,z,w), \\ Z(\alpha,s,z,w) &= Z_1(\alpha,s,z,w) + Z_1(\alpha,1-s,z,w). \end{aligned}$$

Hence

$$\begin{aligned} Z\left(\alpha,\frac{1+it}{2},\frac{z}{2},w\right) = \frac{e^{-\frac{w^2}{4}}}{4}\Bigg(&\alpha^{-\frac{it}{2}} {}_1F_1\left(\frac{1+z+it}{2};\frac{1}{2};\frac{w^2}{4}\right) {}_1F_1\left(\frac{1-z+it}{2};\frac{1}{2};\frac{w^2}{4}\right) \\ &+ \alpha^{\frac{it}{2}} {}_1F_1\left(\frac{1+z-it}{2};\frac{1}{2};\frac{w^2}{4}\right) {}_1F_1\left(\frac{1-z-it}{2};\frac{1}{2};\frac{w^2}{4}\right)\Bigg). \end{aligned} \tag{4.18}$$

Now

$$\Theta\left(\pi n,\frac{z}{2},w\right) = \sqrt{\alpha} e^{\frac{w^2}{4}} K_{\frac{z}{2},w}(2\pi\alpha n) + \sqrt{\beta} e^{\frac{w^2}{4}} K_{\frac{z}{2},w}(2\pi\beta n). \tag{4.19}$$

To compute $R(z,w)$, we first compute

$$\begin{aligned} Z\left(1+\frac{z}{2},\frac{z}{2},w\right) &= \frac{\alpha^{-\frac{1+z}{2}}}{4} {}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right) + \frac{\beta^{-\frac{1+z}{2}}}{4} {}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right), \\ Z\left(1-\frac{z}{2},\frac{z}{2},w\right) &= \frac{\alpha^{-\frac{1-z}{2}}}{4} {}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right) + \frac{\beta^{-\frac{1-z}{2}}}{4} {}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right). \end{aligned}$$

From the definition of $R(z,w)$ given by (2.8) in Chapter 2,

$$R(z,w) = \frac{\pi^{-\frac{z}{2}}}{4}\Gamma\left(\frac{z}{2}\right)\zeta(z)\left(\alpha^{-\frac{1-z}{2}}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)+\beta^{-\frac{1-z}{2}}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};-\frac{w^2}{4}\right)\right)$$
$$+\frac{\pi^{\frac{z}{2}}}{4}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\left(\alpha^{-\frac{1+z}{2}}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)+\beta^{-\frac{1+z}{2}}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)\right). \tag{4.20}$$

Thus substituting (4.18), (4.19) and (4.20) in Theorem 2.1.2, we get

$$\frac{32}{\pi}\int_0^\infty \Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\mathcal{Z}\left(\alpha,\frac{1+it}{2},\frac{z}{2},w\right)\,dt}{(t^2+(z+1)^2)\,(t^2+(z-1)^2)}$$
$$=\mathfrak{F}(z,w,\alpha)+\mathfrak{F}(z,iw,\beta), \tag{4.21}$$

where $\mathcal{Z}\left(\alpha,\frac{1+it}{2},\frac{z}{2},w\right)$ is given by (4.16) and

$$\mathfrak{F}(z,w,\alpha)=\sqrt{\alpha}\Bigg(4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{\frac{z}{2}}e^{\frac{w^2}{4}}K_{\frac{z}{2},w}(2n\pi\alpha)$$
$$-\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-\frac{z}{2}}\alpha^{\frac{z}{2}-1}\,{}_1F_1\left(\frac{1-z}{2};\frac{1}{2};\frac{w^2}{4}\right)$$
$$-\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\pi^{\frac{z}{2}}\alpha^{-\frac{z}{2}-1}\,{}_1F_1\left(\frac{1+z}{2};\frac{1}{2};\frac{w^2}{4}\right)\Bigg).$$

However, Theorem 4.0.2 implies that $\mathfrak{F}(z,w,\alpha)=\mathfrak{F}(z,iw,\beta)$, which simplifies (4.21) to (4.15). □

If we let $z\to 0$ in Theorem 4.0.4 and note that Corollary 4.0.3 is the special case when $z\to 0$ of Theorem 4.0.2, we readily obtain the corollary given below.

Corollary 4.0.5. *For $w \in \mathbb{C}$,*

$$\frac{16}{\pi}\int_0^\infty \frac{\Xi\left(\frac{t}{2}\right)^2}{\left(t^2+1\right)^2}\left(\alpha^{-\frac{it}{2}}\,{}_1F_1^2\left(\frac{1-it}{4};\frac{1}{2};-\frac{w^2}{4}\right)+\alpha^{\frac{it}{2}}\,{}_1F_1^2\left(\frac{1+it}{4};\frac{1}{2};-\frac{w^2}{4}\right)\right)dt$$
$$=\sqrt{\alpha}e^{-\frac{w^2}{4}}\left(4\sum_{n=1}^\infty d(n)e^{-\frac{w^2}{4}}K_{0,iw}(2n\pi\alpha)-\frac{\gamma-\log(4\pi\alpha)}{\alpha}\left(1-\frac{w^2}{4}\right)+\frac{w^2}{2\alpha}\right).$$

When $w = 0$, the above corollary gives the integral associated with Koshliakov's formula [27, Equation (17)] (see also [28, p. 169]).

Chapter 5

Properties of the generalized modified Bessel function $K_{z,w}(x)$

5.1 Some integral representations for the function $K_{z,w}(x)$

The following integral representation for the generalized modified Bessel function $K_{z,w}(x)$ involving the product of the exponential and two cosine functions is very useful. It is used in the derivation of the asymptotic expansion $K_{z,w}(x)$ for small as well as large values of x, the differential-difference equation for $K_{z,w}(x)$, and a few interesting representations for $K_{z,w}(x)$, for example the series representation (5.24) involving $K_z(x)$ and the representation given by an infinite series of Laplace transforms of a special function (5.39).

Theorem 5.1.1. *For $z, w \in \mathbb{C}$ and $|\arg x| < \frac{\pi}{4}$, we have*

$$K_{z,w}(2x) = x^{-z} \int_0^\infty e^{-t^2 - \frac{x^2}{t^2}} \cos(wt) \cos\left(\frac{wx}{t}\right) t^{2z-1}\, dt. \tag{5.1}$$

Proof. For $c = \text{Re } s > 0$ and $\text{Re}(a) > 0$, we have [19, p. 47, Equation 5.30]

$$e^{-at^2}\cos bt = \frac{1}{2\pi i}\int_{(c)} \frac{1}{2}a^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)e^{-\frac{b^2}{4a}}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};\frac{b^2}{4a}\right)t^{-s}\,ds, \tag{5.2}$$

Using Kummer's first transformation for the confluent hypergeometric function (2.32) in (5.2) and then using the resultant with s replaced by $s - z$, $a = 1$ and $b = w$, we obtain

$$\frac{1}{2\pi i}\int_{(c)} \frac{1}{2}\Gamma\left(\frac{s-z}{2}\right){}_1F_1\left(\frac{s-z}{2};\frac{1}{2};-\frac{w^2}{4}\right)t^{-s}\,ds = t^{-z}e^{-t^2}\cos(wt),$$

for $c = \text{Re}(s) > \text{Re}(z)$. Similarly for $c = \text{Re}(s) > -\text{Re}(z)$,

$$\frac{1}{2\pi i}\int_{(c)} \frac{1}{2}\Gamma\left(\frac{s+z}{2}\right){}_1F_1\left(\frac{s+z}{2};\frac{1}{2};-\frac{w^2}{4}\right)t^{-s}\,ds = t^{z}e^{-t^2}\cos(wt).$$

So for $c = \text{Re}(s) > \pm\text{Re}(z)$ and $\text{Re}(x^2) > 0$, that is, for $|\arg x| < \frac{\pi}{4}$, the above two equations along with Parseval's identity (3.4) and the definition of $K_{z,w}(x)$ (2.41) imply

$$\begin{aligned} K_{z,w}(2x) &= \int_0^\infty t^z e^{-t^2}\cos(wt)\left(\frac{x}{t}\right)^{-z}e^{-\frac{x^2}{t^2}}\cos\left(\frac{wx}{t}\right)\frac{dt}{t} \\ &= \int_0^\infty t^{2z-1}e^{-t^2-\frac{x^2}{t^2}}\cos(wt)\cos\left(\frac{wx}{t}\right)\,dt. \end{aligned}$$

□

The following lemma is needed to prove the next integral representation for $K_{z,w}(x)$.

Lemma 5.1.2. *For* $|\arg x| < \frac{\pi}{4}$ *and* $w \in \mathbb{C}$,

$$\int_0^\infty e^{-t^2-x^2/t^2}\cos(wt)\frac{dt}{t} = \int_0^\infty \exp\left(-\frac{w^2x^2}{4(x^2+t^2)}\right)\frac{\cos(2t)}{\sqrt{x^2+t^2}}\,dt. \tag{5.3}$$

Proof. Consider the left-hand side. Expanding $\cos(wt)$ into its Taylor series and then interchanging the order of summation and integration because of absolute convergence, we see that

$$\begin{aligned}\int_0^\infty e^{-t^2-x^2/t^2}\cos(wt)\frac{dt}{t} &= \sum_{n=0}^\infty \frac{(-w^2)^n}{(2n)!}\int_0^\infty t^{2n-1}e^{-t^2-x^2/t^2}\,dt\\ &= \frac{1}{2}\sum_{n=0}^\infty \frac{(-w^2)^n}{(2n)!}\int_0^\infty u^{n-1}e^{-u-x^2/u}\,du.\end{aligned}$$

From [29, p. 344, Formula **2.3.16.1**], for $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$,

$$\int_0^\infty y^{s-1}e^{-py-q/y}\,dy = 2\left(\frac{q}{p}\right)^{s/2}K_s(2\sqrt{pq}). \tag{5.4}$$

This gives for $\mathrm{Re}(x^2) > 0$, that is, for $|\arg x| < \frac{\pi}{4}$,

$$\int_0^\infty e^{-t^2-x^2/t^2}\cos(wt)\frac{dt}{t} = \sum_{n=0}^\infty \frac{(-w^2x)^n}{(2n)!}K_n(2x). \tag{5.5}$$

On the other hand, expanding the exponential function in the integrand on the right side of (5.3), separating the $n = 0$ term and then interchanging the order of summation and integration because of absolute convergence [20, p. 30, Thm. 2.1], we find that

$$\begin{aligned}&\int_0^\infty \exp\left(-\frac{w^2x^2}{4(x^2+t^2)}\right)\frac{\cos(2t)}{\sqrt{x^2+t^2}}\,dt\\ &\qquad= \int_0^\infty \frac{\cos(2t)}{\sqrt{x^2+t^2}}\,dt + \sum_{n=1}^\infty \frac{(-w^2x^2)^n}{n!4^n}\int_0^\infty \frac{\cos(2t)}{(x^2+t^2)^{n+\frac{1}{2}}}\,dt.\end{aligned}$$

It is to be noted that the first integral on the above right-hand side is not absolutely convergent which is why we need to first separate it before interchanging the order.

Employing (5.7) and making use of the fact that $(\frac{1}{2})_n = (2n)!/(n!4^n)$, we arrive at

$$\int_0^\infty \exp\left(-\frac{w^2x^2}{4(x^2+t^2)}\right)\frac{\cos(2t)}{\sqrt{x^2+t^2}}\,dt = K_0(2x) + \sum_{n=1}^\infty \frac{(-w^2x)^n}{(2n)!}K_n(2x). \tag{5.6}$$

The identity in the lemma follows immediately from (5.5) and (5.6). □

The following theorem gives the generalization of Basset's formula for the modified Bessel function of the second kind [4, p. 172]

$$K_z(xy) = \frac{\Gamma\left(z+\frac{1}{2}\right)(2x)^z}{y^z\Gamma(\frac{1}{2})}\int_0^\infty \frac{\cos(yu)\,du}{(x^2+u^2)^{z+\frac{1}{2}}}, \tag{5.7}$$

valid for $\mathrm{Re}(z) > -\frac{1}{2}, y > 0$, and $|\arg x| < \frac{1}{2}\pi$.

Theorem 5.1.3. *For* $|\arg x| < \frac{1}{4}\pi$ *and* $w \in \mathbb{C}$, *we have*

$$K_{0,w}(x) = \int_0^\infty \exp\left(-\frac{w^2x^2}{2(x^2+u^2)}\right)\cos\left(\frac{w^2xu}{2(x^2+u^2)}\right)\frac{\cos u\,du}{\sqrt{x^2+u^2}}. \tag{5.8}$$

Proof. From [30, p. 121, Eqn. (43)], we have

$${}_1F_1(a;c;u){}_1F_1(a;c;v) = \sum_{n=0}^\infty \frac{(a)_n(c-a)_n}{n!(c)_n(c)_{2n}}(-uv)^n{}_1F_1(a+n;c+2n;u+v). \tag{5.9}$$

First assume $x > 0$. Let

$$I(x,w) := \frac{1}{2\pi i}\int_{(c)}\Gamma^2\left(\frac{s}{2}\right){}_1F_1^2\left(\frac{s}{2};\frac{1}{2};\frac{-w^2}{4}\right)x^{-s}ds$$

so that from (2.41),

$$K_{0,w}(x) = \frac{1}{4}I\left(\frac{x}{2},w\right). \tag{5.10}$$

Now from (5.9), we have

$$
\begin{aligned}
I(x,w) = \frac{1}{2\pi i}\times \\
&\int_{(c)} \Gamma^2\left(\frac{s}{2}\right)\sum_{n=0}^{\infty}\frac{\left(\frac{s}{2}\right)_n\left(\frac{1-s}{2}\right)_n}{n!\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{2n}}\left(-\frac{w^4}{16}\right)^n {}_1F_1\left(\frac{s}{2}+n;\frac{1}{2}+2n;-\frac{w^2}{2}\right)x^{-s}\,ds \\
&=: I_1(x,w)+I_2(x,w),
\end{aligned}
$$

where

$$I_1(x,w) := \frac{1}{2\pi i}\int_{(c)}\Gamma^2\left(\frac{s}{2}\right){}_1F_1\left(\frac{s}{2};\frac{1}{2};-\frac{w^2}{2}\right)x^{-s}\,ds, \tag{5.11}$$

$$
\begin{aligned}
I_2(x,w) := \frac{1}{2\pi i}\int_{(c)}\Gamma^2\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{\left(\frac{s}{2}\right)_n\left(\frac{1-s}{2}\right)_n}{n!\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{2n}}\left(-\frac{w^4}{16}\right)^n\times \\
{}_1F_1\left(\frac{s}{2}+n;\frac{1}{2}+2n;-\frac{w^2}{2}\right)x^{-s}\,ds.
\end{aligned}
\tag{5.12}
$$

We first evaluate $I_1(x,w)$. First employing (2.32) in (5.2), and then using the resultant with $a=2$ and $b=2w$ and t replaced by $t/\sqrt{2}$, we find that for $c=\mathrm{Re}(s)>0$,

$$\frac{1}{2\pi i}\int_{(c)}\Gamma\left(\frac{s}{2}\right){}_1F_1\left(\frac{s}{2};\frac{1}{2};-\frac{w^2}{2}\right)t^{-s}\,ds = 2e^{-t^2}\cos(\sqrt{2}wt). \tag{5.13}$$

Also for $c=\mathrm{Re}(s)>0$,

$$\frac{1}{2\pi i}\int_{(c)}\Gamma\left(\frac{s}{2}\right)t^{-s}\,ds = 2e^{-t^2}. \tag{5.14}$$

Hence from (5.13), (5.14) and Parseval's identity (3.4), we see that for $0<c=\mathrm{Re}(s)<1$,

$$I_1(x,w) = 4\int_0^{\infty} e^{-t^2-\frac{x^2}{t^2}}\cos(\sqrt{2}wt)\frac{dt}{t}. \tag{5.15}$$

We now evaluate $I_2(x,w)$. Let $0<c=\mathrm{Re}(s)<1$. Stirling's formula for $\Gamma(s)$,

$s = \sigma + it$, in a vertical strip $\alpha \le \sigma \le \beta$ is given by

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}} |t|^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi|t|} \left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{5.16}$$

as $|t| \to \infty$. The exponential decay of the gamma function, seen from (5.16), allows us to interchange the order of summation and integration on the right side of (5.12). Hence

$$I_2(x, w) = \sum_{n=1}^{\infty} \frac{(-w^4/16)^n}{n! \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{2n}} A_n(x, w),$$

where

$$A_n(x, w) := \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+n\right)\Gamma\left(\frac{1-s}{2}+n\right)}{\Gamma\left(\frac{1-s}{2}\right)} {}_1F_1\left(\frac{s}{2}+n; \frac{1}{2}+2n; -\frac{w^2}{2}\right) x^{-s}\, ds.$$

Now write the ${}_1F_1$ in the above equation in the form of series and again interchange the order of summation and integration using (5.16) to arrive at

$$A_n(x, w) = \Gamma\left(\frac{1}{2} + 2n\right) \sum_{m=0}^{\infty} \frac{(-w^2/2)^m}{m!} B_{n,m}(x), \tag{5.17}$$

where

$$B_{n,m}(x) := \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \frac{\Gamma\left(\frac{1-s}{2}+n\right)\Gamma\left(\frac{s}{2}+n+m\right)}{\Gamma\left(\frac{1}{2}+2n+m\right)} x^{-s}\, ds. \tag{5.18}$$

We now evaluate $B_{n,m}(x)$. Using elementary properties of the gamma function, one can show [31, p. 73] that

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \pi^{-\frac{1}{2}} 2^{1-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right).$$

Hence for $0 < c = \mathrm{Re}(s) < 1$,

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} x^{-s}\, ds = \frac{2}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{(c)} \Gamma(s)\cos\left(\frac{\pi s}{2}\right)(2x)^{-s}\, ds$$
$$= \frac{2}{\sqrt{\pi}}\cos(2x), \tag{5.19}$$

as can be seen from [19, p. 42, Eqn. 5.2]. Next, Euler's beta integral gives for $0 < d = \mathrm{Re}(s) < \mathrm{Re}(z)$,

$$\frac{1}{2\pi i}\int_{(d)} \frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)} x^{-s}\, ds = \frac{1}{(1+x)^z},$$

so that for $-2n-2m < c = \mathrm{Re}(s) < 1+2n,\ n, \in \mathbb{N}, m \in \mathbb{N}\cup\{0\}$,

$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma\left(\frac{1-s}{2}+n\right)\Gamma\left(\frac{s}{2}+n+m\right)}{\Gamma\left(\frac{1}{2}+2n+m\right)} x^{-s}\, ds = \frac{2x^{2n+2m}}{(1+x^2)^{\frac{1}{2}+2n+m}}. \tag{5.20}$$

From (5.19), (5.20), (5.18) and (3.4), we deduce that for $0 < \mathrm{Re}(s) < 1$,

$$B_{n,m}(x) = \frac{4x^{2n+2m}}{\sqrt{\pi}}\int_0^\infty \frac{t^{2n}\cos(2t)\, dt}{(x^2+t^2)^{\frac{1}{2}+2n+m}},$$

which implies through (5.17),

$$A_n(x,w) = \frac{4x^{2n}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+2n\right)\sum_{m=0}^{\infty}\frac{(-w^2x^2/2)^m}{m!}\int_0^\infty \frac{t^{2n}\cos(2t)\, dt}{(x^2+t^2)^{\frac{1}{2}+2n+m}}. \tag{5.21}$$

Note that $\sum_{m=0}^{\infty}\frac{(-w^2x^2/2)^m}{m!(x^2+t^2)^{\frac{1}{2}+m}}$ converges uniformly on any compact interval of $(0,\infty)$ to $\exp\left(-\frac{w^2x^2}{2(x^2+t^2)}\right)$. Moreover, it is easy to see that

$$\sum_{m=0}^{\infty}\int_0^\infty \left|\frac{t^{2n}\cos(2t)(-w^2x^2/2)^m}{m!(x^2+t^2)^{\frac{1}{2}+2n+m}}\, dt\right|$$

is finite. Then, [20, p. 30, Thm. 2.1] permits us to interchange the order of summation and integration in (5.21) so that

$$A_n(x,w) = \frac{4x^{2n}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+2n\right)\int_0^\infty \frac{t^{2n}\cos(2t)}{(x^2+t^2)^{\frac{1}{2}+2n}}\sum_{m=0}^\infty \frac{(-w^2x^2/2)^m}{m!(x^2+t^2)^m}\,dt$$
$$= \frac{4x^{2n}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+2n\right)\int_0^\infty \frac{t^{2n}\cos(2t)}{(x^2+t^2)^{\frac{1}{2}+2n}}\exp\left(-\frac{w^2x^2}{2(x^2+t^2)}\right)dt. \quad (5.22)$$

Now (5.12) and (5.22) imply

$$I_2(x,w) = 4\sum_{n=1}^\infty \frac{\left(\frac{-w^4x^2}{16}\right)^n}{n!\left(\frac{1}{2}\right)_n}\int_0^\infty \frac{t^{2n}\cos(2t)}{(x^2+t^2)^{\frac{1}{2}+2n}}\exp\left(-\frac{w^2x^2}{2(x^2+t^2)}\right)dt.$$

Note that $\sum_{n=1}^\infty \frac{\left(-w^4x^2/16\right)^n}{n!\left(\frac{1}{2}\right)_n(x^2+t^2)^{2n}}$ converges uniformly to $\cos\left(\frac{w^2xt}{2(x^2+t^2)}\right)-1$ on compact intervals of $(0,\infty)$ and

$$\sum_{n=1}^\infty \int_0^\infty \left|\frac{(-w^4x^2/16)^n}{n!\left(\frac{1}{2}\right)_n}\frac{t^{2n}\cos(2t)}{(x^2+t^2)^{\frac{1}{2}+2n}}\exp\left(-\frac{w^2x^2}{2(x^2+t^2)}\right)dt\right|$$

is finite. Hence another appeal to [20, p. 30, Thm. 2.1] allows us to interchange the order of summation and integration so that

$$I_2(x,w) = 4\int_0^\infty \exp\left(-\frac{w^2x^2}{2(x^2+t^2)}\right)\frac{\cos(2t)}{\sqrt{x^2+t^2}}\left(\cos\left(\frac{w^2xt}{2(x^2+t^2)}-1\right)\right)dt, \quad (5.23)$$

so that from (5.15) and (5.23), we finally arrive at

$$I(x,w) = 4\int_0^\infty e^{-t^2-\frac{x^2}{t^2}}\cos(\sqrt{2}wt)\frac{dt}{t}$$
$$+4\int_0^\infty \exp\left(-\frac{w^2x^2}{2(x^2+t^2)}\right)\frac{\cos(2t)}{\sqrt{x^2+t^2}}\left(\cos\left(\frac{w^2xt}{2(x^2+t^2)}-1\right)\right)dt$$
$$= 4\int_0^\infty \exp\left(-\frac{w^2x^2}{2(x^2+t^2)}\right)\frac{\cos(2t)}{\sqrt{x^2+t^2}}\cos\left(\frac{w^2xt}{2(x^2+t^2)}\right)dt,$$

as can be seen from Lemma 5.1.2. From (5.10), we now obtain (5.8) upon change of variable. This completes the proof of Theorem 5.1.3 for $x > 0$. Since both sides of

(5.8) are analytic when $|\arg x| < \pi/4$, the identity holds for $|\arg x| < \pi/4$ by analytic continuation. □

5.2 Some series representations for the function $K_{z,w}(x)$

The first series representation for the generalized modified Bessel function $K_{z,w}(2x)$ is given in terms of the modified Bessel function $K_z(2x)$ as follows.

Theorem 5.2.1. *For $z, w \in \mathbb{C}$ and $|\arg x| < \frac{\pi}{4}$, we have*

$$K_{z,w}(2x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-w^2x)^{n+m}}{(2n)!(2m)!} K_{n-m+z}(2x). \tag{5.24}$$

Proof. Using Theorem 5.1.1, expanding each of the cosines into its Taylor series and then interchanging the order of summation and integration each time, we arrive at

$$\begin{aligned}
K_{z,w}(2x) &= x^{-z}\int_0^{\infty} e^{-t^2-\frac{x^2}{t^2}} \cos(wt)\cos\left(\frac{wx}{t}\right) t^{2z-1}\, dt \\
&= x^{-z}\int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-w^2)^n}{(2n)!} e^{-t^2-\frac{x^2}{t^2}} \cos\left(\frac{wx}{t}\right) t^{2n+2z-1}\, dt \\
&= x^{-z}\sum_{n=0}^{\infty} \frac{(-w^2)^n}{(2n)!} \int_0^{\infty} e^{-t^2-\frac{x^2}{t^2}} \cos\left(\frac{wx}{t}\right) t^{2n+2z-1}\, dt \\
&= x^{-z}\sum_{n=0}^{\infty} \frac{(-w^2)^n}{(2n)!} \int_0^{\infty} \sum_{m=0}^{\infty} \frac{(-w^2x^2)^m}{(2m)!} e^{-t^2-\frac{x^2}{t^2}} t^{2n-2m+2z-1}\, dt \\
&= x^{-z}\sum_{n=0}^{\infty} \frac{(-w^2)^n}{(2n)!} \sum_{m=0}^{\infty} \frac{(-w^2x^2)^m}{(2m)!} \int_0^{\infty} e^{-t^2-\frac{x^2}{t^2}} t^{2(n-m+z)-1}\, dt \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-w^2x)^{n+m}}{(2n)!(2m)!} K_{n-m+z}(2x),
\end{aligned}$$

where in the last step we used (5.4). □

From the above series representation (5.24), we get the following triple series representation for $K_{z,w}(2x)$.

Theorem 5.2.2. *For* $w \in \mathbb{C}$, $z \notin \mathbb{Z}$ *and* $|\arg x| < \frac{\pi}{4}$,

$$K_{z,w}(2x) = \frac{\pi}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{w^{2(n+m)} x^{2k}}{(2n)!\ (2m)!\ k! \sin(\pi z)} \times \left(\frac{x^{2m-z}}{\Gamma(k+m-n-z+1)} - \frac{x^{2n+z}}{\Gamma(k+n-m+z+1)} \right) \quad (5.25)$$

Proof. The following series representation for the modified Bessel function $K_z(x)$ is derived from the definition of $K_z(x)$ (1.1) using the series representation for the Bessel function $I_z(x)$ (1.10). For $z \notin \mathbb{Z}$,

$$K_z(2x) = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{x^{2k}}{k! \sin(\pi z)} \left(\frac{x^{-z}}{\Gamma(k-z+1)} - \frac{x^{z}}{\Gamma(k+z+1)} \right). \quad (5.26)$$

Using this series representation for $K_z(x)$ in the double series representation (5.24), the desired triple series representation for $K_{z,w}(x)$ is obtained. □

Another series representation for the function $K_{z,w}(x)$ in terms of Bessel functions is given as follows.

Theorem 5.2.3. *For* $-\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}$, $w \in \mathbb{C}$, *and* $|\arg x| < \frac{\pi}{4}$,

$$K_{z,w}(2x) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n K_{z+n}(2x) \left(I_{2n}(2w\sqrt{x}) + J_{2n}(2w\sqrt{x}) \right). \quad (5.27)$$

Proof. Let $x \in \mathbb{C}$ such that $|\arg x| < \pi/4$ and $w \in \mathbb{C}$. By Basset's formula (5.7) and the fact that $K_\nu(x)$ is an even function of its order, we have

$$K_{n-m+z}(2x) = \begin{cases} \dfrac{\Gamma(n-m+z+\frac{1}{2})(2x)^{n-m+z}}{2^{n-m+z}\Gamma(\frac{1}{2})} \displaystyle\int_0^\infty \frac{\cos(2u)\,du}{(u^2+x^2)^{n-m+z+\frac{1}{2}}}, \\ \qquad \text{if } \operatorname{Re}(n-m+z) \geq -\frac{1}{2}, \\ \dfrac{\Gamma(m-n-z+\frac{1}{2})(2x)^{m-n-z}}{2^{m-n-z}\Gamma(\frac{1}{2})} \displaystyle\int_0^\infty \frac{\cos(2u)\,du}{(u^2+x^2)^{m-n-z+\frac{1}{2}}}, \\ \qquad \text{if } \operatorname{Re}(m-n-z) \geq -\frac{1}{2}. \end{cases} \tag{5.28}$$

By an application of Lemma 5.2.1,

$$K_{z,w}(2x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-w^2x)^{n+m}}{(2n)!(2m)!} K_{n-m+z}(2x) + \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} \frac{(-w^2x)^{n+m}}{(2n)!(2m)!} K_{m-n-z}(2x).$$

By the hypothesis, $-\frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2}$. Now $\operatorname{Re}(z) > -1/2$ implies $\operatorname{Re}(z) + n + \frac{1}{2} > n$. So if $m \leq n$, then $m < \operatorname{Re}(z) + n + \frac{1}{2}$, that, is, $\operatorname{Re}(n-m+z) > -\frac{1}{2}$. Also, $\operatorname{Re}(z) < \frac{1}{2}$ implies $\operatorname{Re}(z) + n - \frac{1}{2} < n$. Hence if $m \geq n+1$, then $\operatorname{Re}(z) + n - \frac{1}{2} < m$, that is, $\operatorname{Re}(m-n-z) > -\frac{1}{2}$. Hence along with (5.28), we find that

$$K_{z,w}(2x) = S_1(z,w,x) + S_2(z,w,x), \tag{5.29}$$

where

$$S_1(z,w,x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-w^2x)^{n+m}}{(2n)!(2m)!} \frac{\Gamma(n-m+z+\frac{1}{2})x^{n-m+z}}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos(2u)\,du}{(u^2+x^2)^{n-m+z+\frac{1}{2}}} \tag{5.30}$$

$$S_2(z,w,x) = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} \frac{(-w^2x)^{n+m}}{(2n)!(2m)!} \frac{\Gamma(m-n-z+\frac{1}{2})x^{m-n-z}}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos(2u)\,du}{(u^2+x^2)^{m-n-z+\frac{1}{2}}}. \tag{5.31}$$

We first simplify $S_1(z,w,x)$. Writing $S_1(z,w,x)$ as a doubly infinite series, we see that

$$S_1(z,w,x) = \sum_{m=0}^{\infty}\sum_{d=0}^{\infty} \frac{(-w^2)^{2m+d} x^{2m+2d+z}}{(2m)!(2m+2d)!} \frac{\Gamma(d+z+\frac{1}{2})}{\Gamma(\frac{1}{2})} \int_0^{\infty} \frac{\cos(2u)\,du}{(u^2+x^2)^{d+z+\frac{1}{2}}}$$
$$=: T_1(z,w,x) + T_2(z,w,x), \tag{5.32}$$

where

$$T_1(z,w,x) = \frac{\Gamma(z+\frac{1}{2})}{\Gamma(\frac{1}{2})} \sum_{m=0}^{\infty} \frac{w^{4m} x^{2m+z}}{((2m)!)^2} \int_0^{\infty} \frac{\cos(2u)\,du}{(u^2+x^2)^{z+\frac{1}{2}}}$$
$$T_2(z,w,x) = \sum_{m=0}^{\infty} \frac{w^{4m} x^{2m+z}}{(2m)!} \int_0^{\infty} \frac{\cos(2u)\,du}{(u^2+x^2)^{z+\frac{1}{2}}} \sum_{d=1}^{\infty} \frac{\left(-\frac{w^2x^2}{x^2+u^2}\right)^d}{(2m+2d)!} \frac{\Gamma(d+z+\frac{1}{2})}{\Gamma(\frac{1}{2})}.$$

Employing (5.7), we have

$$\begin{aligned} T_1(z,w,x) &= \frac{\Gamma(z+\frac{1}{2})}{\Gamma(\frac{1}{2})} \sum_{m=0}^{\infty} \frac{w^{4m} x^{2m+z}}{((2m)!)^2} \frac{\sqrt{\pi} x^{-z} K_z(2x)}{\Gamma(z+\frac{1}{2})} \\ &= K_z(2x) \sum_{m=0}^{\infty} \frac{w^{4m} x^{2m}}{((2m)!)^2} \\ &= \frac{1}{2} K_z(2x) \left(I_0(2w\sqrt{x}) + J_0(2w\sqrt{x}) \right), \end{aligned} \tag{5.33}$$

where the last step follows from the definitions (1.10) and (1.11) of the two Bessel functions. Now it is easy to see that

$$\sum_{d=1}^{\infty} \frac{\left(-\frac{w^2x^2}{x^2+u^2}\right)^d}{(2m+2d)!} \frac{\Gamma(d+z+\frac{1}{2})}{\Gamma(\frac{1}{2})} = \frac{-w^2x^2}{(u^2+x^2)} \frac{\Gamma(z+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(2m+3)} \times$$
$${}_2F_2\left(1, z+\frac{3}{2}; m+\frac{3}{2}, m+2; \frac{-w^2x^2}{4(u^2+x^2)}\right).$$

To see this, we use the series representation of the right side and apply twice the duplication formula (3.21) for the gamma function to arrive at the left side. Thus,

$$T_2(z,w,x) = -w^2x^{z+2}\frac{\Gamma(z+\frac{3}{2})}{\Gamma(\frac{1}{2})}\sum_{m=0}^{\infty}\frac{w^{4m}x^{2m}}{(2m)!(2m+2)!}$$
$$\times\int_0^{\infty}\frac{\cos(2u)}{(u^2+x^2)^{z+\frac{3}{2}}}{}_2F_2\left(1, z+\frac{3}{2}; m+\frac{3}{2}, m+2; \frac{-w^2x^2}{4(u^2+x^2)}\right)du.$$

The case $d = 0$ in (5.32) is singled out to guarantee absolute convergence. This then allow us to interchange the order of summation and integration as well as the order of two summations. (Note that we could not have done the interchange had we kept the $d = 0$ term in the infinite sum over d in (5.32). Thus writing the ${}_2F_2$ in the form of a series and making the interchanges, we arrive at

$$T_2(z,w,x) = -w^2x^{z+2}\frac{\Gamma\left(z+\frac{3}{2}\right)}{\Gamma(\frac{1}{2})}\sum_{k=0}^{\infty}\left(z+\frac{3}{2}\right)_k\left(\frac{-w^2x^2}{4}\right)^k$$
$$\times\int_0^{\infty}\frac{\cos(2u)\,du}{(u^2+x^2)^{z+k+\frac{3}{2}}}\sum_{m=0}^{\infty}\frac{w^{4m}x^{2m}}{(2m)!(2m+2)!(m+\frac{3}{2})_k(m+2)_k}$$

After representing the rising factorials in the inner series over m in terms of gamma functions and applying the duplication formula (3.21) for the gamma function, we are led upon simplification to

$$\sum_{m=0}^{\infty}\frac{w^{4m}x^{2m}}{(2m)!(2m+2)!(m+\frac{3}{2})_k(m+2)_k} = 2^{2k}\sum_{m=0}^{\infty}\frac{w^{4m}x^{2m}\Gamma(2m+3)}{(2m)!(2m+2)!\Gamma(2m+2k+3)}$$
$$= 2^{2k}\sum_{m=0}^{\infty}\frac{w^{4m}x^{2m}}{(2m)!\Gamma(2m+2k+3)}$$
$$= \frac{2^{2k-1}}{w^{2k+2}x^{k+1}}\left(I_{2k+2}(2w\sqrt{x}) + J_{2k+2}(2w\sqrt{x})\right),$$

where the last step follows, as in (5.33), by employing the series definitions of the two

Bessel functions and simplifying. Thus,

$$T_2(z,w,x) = -\frac{x^{z+1}}{2\Gamma(\frac{1}{2})}\sum_{k=0}^{\infty}\Gamma\left(z+k+\frac{3}{2}\right)(-x)^k\left(I_{2k+2}(2w\sqrt{x})+J_{2k+2}(2w\sqrt{x})\right)$$
$$\times\int_0^{\infty}\frac{\cos(2u)\,du}{(u^2+x^2)^{z+k+\frac{3}{2}}}$$
$$= \frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}K_{k+1+z}(2x)\left(I_{2k+2}(2w\sqrt{x})+J_{2k+2}(2w\sqrt{x})\right), \tag{5.34}$$

where in the last step, we applied (5.7) again. Therefore from (5.32), (5.33) and (5.34), we see that

$$S_1(z,w,x) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k K_{k+z}(2x)\left(I_{2k}(2w\sqrt{x})+J_{2k}(2w\sqrt{x})\right). \tag{5.35}$$

We still need to evaluate $S_2(z,w,x)$. To that end, let $\ell = m-n$ in (5.31) so that

$$S_2(z,w,x) = \sum_{n=0}^{\infty}\sum_{\ell=1}^{\infty}\frac{(-w^2)^{2n+\ell}x^{2n+2\ell-z}\Gamma(\ell+\frac{1}{2}-z)}{(2n)!(2n+2\ell)!\Gamma(\frac{1}{2})}\int_0^{\infty}\frac{\cos(2u)\,du}{(u^2+x^2)^{\ell-z+\frac{1}{2}}}$$
$$= S_1(-z,w,x) - \sum_{n=0}^{\infty}\frac{(-w^2)^{2n}x^{2n-z}\Gamma(\frac{1}{2}-z)}{((2n)!)^2\Gamma(\frac{1}{2})}\int_0^{\infty}\frac{\cos(2u)\,du}{(u^2+x^2)^{-z+\frac{1}{2}}}$$

as can be seen from (5.32). Since $\mathrm{Re}(z) < 1/2$, we can employ (5.7) in the single series over n in the above equation. Then simplifying as in (5.33) and making use of the fact that $K_\nu(\lambda)$ is an even function of ν, we see from the above equation that

$$S_2(z,w,x) = S_1(-z,w,x) - \frac{1}{2}K_z(2x)\left(I_0(2w\sqrt{x})+J_0(2w\sqrt{x})\right)$$
$$= \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k K_{k-z}(2x)\left(I_{2k}(2w\sqrt{x})+J_{2k}(2w\sqrt{x})\right), \tag{5.36}$$

where the last step follows from (5.35). Now replace k by $-k$ in (5.36), again make use of the fact that $K_\nu(\lambda)$ is an even function of ν along with the identities $J_{-n}(\lambda) =$

$(-1)^n J_n(\lambda)$ and $I_{-n}(\lambda) = I_n(\lambda)$ so as to obtain

$$S_2(z, w, x) = \frac{1}{2} \sum_{k=-\infty}^{-1} (-1)^k K_{k+z}(2x) \left(I_{2k}(2w\sqrt{x}) + J_{2k}(2w\sqrt{x}) \right). \tag{5.37}$$

Finally, (5.29), (5.35) and (5.37) imply (5.27). □

5.3 More representations for the function $K_{z,w}(x)$

The following representation for the function $K_{z,w}(x)$ is an infinite series of Laplace transformation of a special function. The integral in the below representation is indeed a Laplace transform as [21, p. 934, formula **8.468**]

$$K_{n+\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2y}} e^{-y} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2y)^k}. \tag{5.38}$$

Theorem 5.3.1. *Let $w \in \mathbb{C}$, $\mathrm{Re}(z) > -\frac{1}{2}$ and $|\arg x| < \frac{\pi}{4}$. Then*

$$K_{z,w}(x) = \frac{(2x)^{z+\frac{1}{2}}}{\Gamma\left(z+\frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{\left(-\frac{w^2 x}{2}\right)^n}{(2n)!} \int_0^\infty t^{z-\frac{1}{2}} (t+1)^{z-\frac{1}{2}} (2t+1)^{-n+\frac{1}{2}} K_{n+\frac{1}{2}}(x(2t+1)) \times {}_0F_2\left(-; \frac{1}{2}, \frac{1}{2}+z; -\frac{w^2 x^2 t(t+1)}{4}\right) dt. \tag{5.39}$$

Proof. Replace x by $x/2$ in (5.1) and then let $t = \sqrt{\frac{xu}{2}}$ in the resulting equation to arrive at

$$K_{z,w}(x) = \frac{1}{2} \int_0^\infty \exp\left(-\frac{x}{2}\left(u + \frac{1}{u}\right)\right) \cos\left(\frac{w\sqrt{xu}}{\sqrt{2}}\right) \cos\left(\frac{w\sqrt{x}}{\sqrt{2u}}\right) u^{-z-1}\, du, \tag{5.40}$$

where the last step follows from the fact that $K_{z,w}(x) = K_{-z,w}(x)$. Now using [19,

p. 186, Equation (4.25)] [1], it can be seen that from $\mathrm{Re}(z) > -1/2$,

$$u^{-z-\frac{1}{2}} \cos\left(\frac{w\sqrt{x}}{\sqrt{2u}}\right) = \frac{1}{\Gamma\left(z+\frac{1}{2}\right)} \int_0^\infty e^{-yu} y^{z-\frac{1}{2}} {}_0F_2\left(-;\frac{1}{2},\frac{1}{2}+z;-\frac{w^2xy}{8}\right) dy. \quad (5.41)$$

This result can be easily obtained by writing the ${}_0F_2$ as a series and then integrating term by term. Now substitute (5.41) in (5.40) and interchange the order of integration, which is permissible due to absolute convergence, to arrive at

$$K_{z,w}(x) = \frac{1}{2\Gamma\left(z+\frac{1}{2}\right)} \int_0^\infty y^{z-\frac{1}{2}} {}_0F_2\left(-;\frac{1}{2},\frac{1}{2}+z;-\frac{w^2xy}{8}\right) dy \\ \times \int_0^\infty \exp\left(-u\left(y+\frac{x}{2}\right)-\frac{x}{2u}\right) \cos\left(\frac{w\sqrt{xu}}{\sqrt{2}}\right) \frac{du}{\sqrt{u}}. \quad (5.42)$$

Arguing as in the first part of Lemma 5.1.2, we find that

$$\int_0^\infty e^{-v^2-\frac{x^2}{v^2}} \cos(wv) v^{2z-1}\, dv = x^z \sum_{n=0}^\infty \frac{(-w^2x)^n}{(2n)!} K_{n+z}(2x).$$

In the above equation let $z = 1/2$, $v = \sqrt{y+\frac{x}{2}}\sqrt{u}$ and replace w by $\frac{w\sqrt{x/2}}{\sqrt{y+x/2}}$ and x by $\sqrt{\frac{x}{2}\left(y+\frac{x}{2}\right)}$ so that

$$\int_0^\infty \exp\left(-u\left(y+\frac{x}{2}\right)-\frac{x}{2u}\right) \cos\left(\frac{w\sqrt{xu}}{\sqrt{2}}\right) \frac{du}{\sqrt{u}} \\ = \frac{2(\frac{x}{2})^{1/4}}{(y+\frac{x}{2})^{1/4}} \sum_{n=0}^\infty \frac{1}{(2n)!} \left(\frac{-w^2x^{3/2}}{2^{3/2}\sqrt{y+x/2}}\right)^n K_{n+\frac{1}{2}}\left(2\sqrt{\frac{x}{2}\left(y+\frac{x}{2}\right)}\right). \quad (5.43)$$

Substituting (5.43) in (5.42) and interchanging the order of summation and integra-

[1] There is a typo in the argument of ${}_0F_2$ in the version given there in that the $-\frac{a^2y}{2}$ should be $-\frac{a^2y}{4}$.

tion due to absolute convergence, we see that

$$K_{z,w}(x) = \frac{\left(\frac{x}{2}\right)^{1/4}}{\Gamma\left(z+\frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{-w^2 x^{3/2}}{2^{3/2}}\right)^n}{(2n)!} \times \int_0^{\infty} \frac{y^{z-\frac{1}{2}}}{\left(y+\frac{x}{2}\right)^{\frac{n}{2}+\frac{1}{4}}} K_{n+\frac{1}{2}}\left(2\sqrt{\frac{x}{2}\left(y+\frac{x}{2}\right)}\right) {}_0F_2\left(-;\frac{1}{2},\frac{1}{2}+z;-\frac{w^2xy}{8}\right) dy$$

Next, make a change of variable $l = \sqrt{y+\frac{x}{2}}/\sqrt{\frac{x}{2}}$ so as to obtain

$$K_{z,w}(x) = \frac{x^{z+\frac{1}{2}} 2^{-z+\frac{1}{2}}}{\Gamma\left(z+\frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{-w^2 x}{2}\right)^n}{(2n)!} \times \int_1^{\infty} (l^2-1)^{z-\frac{1}{2}} l^{-n+\frac{1}{2}} K_{n+\frac{1}{2}}(xl) {}_0F_2\left(-;\frac{1}{2},\frac{1}{2}+z;-\frac{w^2x^2(l^2-1)}{16}\right) dl. \tag{5.44}$$

Finally let $l = 2t+1$ to arrive at (5.39). □

The following is a double integral representation for the function $K_{z,w}(x)$ involving a product of two hypergeometric functions.

Theorem 5.3.2. *Let $w \in \mathbb{C}$. For $\mathrm{Re}(z) > -1$ and $|\arg x| < \frac{\pi}{4}$,*

$$K_{z,w}(x) = \frac{1}{2\Gamma(1+z)} \int_0^{\infty}\int_0^{\infty} \frac{y^z t^{-1/2}}{\sqrt{y+\frac{x}{2}}} \exp\left(-2\sqrt{\left(t+\frac{x}{2}\right)\left(y+\frac{x}{2}\right)}\right) \times {}_0F_2\left(-;\frac{1}{2},1+z;-\frac{w^2xy}{8}\right) {}_0F_2\left(-;\frac{1}{2},\frac{1}{2};-\frac{w^2xt}{8}\right) dt\, dy. \tag{5.45}$$

Proof. Replace z by $z+\frac{1}{2}$ in (5.41) so as to have for $\mathrm{Re}(z) > -1$,

$$u^{-z-1}\cos\left(\frac{w\sqrt{x}}{\sqrt{2u}}\right) = \frac{1}{\Gamma(z+1)} \int_0^{\infty} e^{-yu} y^z {}_0F_2\left(-;\frac{1}{2},1+z;-\frac{w^2xy}{8}\right) dy. \tag{5.46}$$

Substitute the above equation in (5.40) and interchange the order of integration, which is valid by absolute convergence, so that

$$K_{z,w}(x) = \frac{1}{2\Gamma(z+1)} \int_0^\infty y^z {}_0F_2\left(-; \frac{1}{2}, 1+z; -\frac{w^2xy}{8}\right) dy \times \int_0^\infty \exp\left(-u\left(y+\frac{x}{2}\right) - \frac{x}{2u}\right) \cos\left(\frac{w\sqrt{xu}}{\sqrt{2}}\right) du. \tag{5.47}$$

Next, replace u by $1/u$ in (5.46) and then let $z = -1/2$ so that

$$\cos\left(\frac{w\sqrt{xu}}{\sqrt{2}}\right) = \frac{1}{\sqrt{\pi u}} \int_0^\infty e^{-t/u} t^{-1/2} {}_0F_2\left(-; \frac{1}{2}, \frac{1}{2}; -\frac{w^2xt}{8}\right) dt. \tag{5.48}$$

Now substitute (5.48) in (5.47) and again interchange the order of integration. This gives

$$K_{z,w}(x) = \frac{1}{2\sqrt{\pi}\Gamma(z+1)} \int_0^\infty \int_0^\infty y^z t^{-1/2} {}_0F_2\left(-; \frac{1}{2}, 1+z; -\frac{w^2xy}{8}\right) \times {}_0F_2\left(-; \frac{1}{2}, \frac{1}{2}; -\frac{w^2xt}{8}\right) \int_0^\infty \exp\left(-u\left(y+\frac{x}{2}\right) - \left(t+\frac{x}{2}\right)\frac{1}{u}\right) \frac{du}{\sqrt{u}}\, dt\, dy. \tag{5.49}$$

Using (5.4), the innermost integral is now evaluated to

$$2\left(\frac{t+\frac{x}{2}}{y+\frac{x}{2}}\right)^{1/4} K_{\frac{1}{2}}\left(2\sqrt{\left(t+\frac{x}{2}\right)\left(y+\frac{x}{2}\right)}\right) = \frac{\sqrt{\pi}}{\sqrt{y+\frac{x}{2}}} \exp\left(-2\sqrt{\left(t+\frac{x}{2}\right)\left(y+\frac{x}{2}\right)}\right), \tag{5.50}$$

since we have $K_{\frac{1}{2}}(x) = \sqrt{\dfrac{\pi}{2x}} e^{-x}$ (see [21, p. 934, formula **8.468**]). The representation in (5.45) now follows from (5.49) and (5.50). □

5.4 Asymptotic exapansion of the function $K_{z,w}(x)$

The asymptotic expansion of $K_{z,w}(x)$ for large values of $|x|$ stated below is obtained by Nico M. Temme, and its proof is given in the Appendix of the paper [22].

Theorem 5.4.1. *Let the complex variables w and z belong to compact domains, then for large values of $|x|$, $|\arg x| < \frac{1}{4}\pi$, we have the representation*

$$K_{z,w}(2x) = \frac{1}{4}\sqrt{\frac{\pi}{x}}\, e^{-2x}\big(\cos(2w\sqrt{x})P - \sin(2w\sqrt{x})Q + e^{-\frac{1}{4}w^2}R\big),$$

where P, Q and R have the asymptotic expansions

$$P = 1 + \frac{32z^2 - 3w^2 - 8}{128x} + O\left(x^{-2}\right),$$

$$Q = \frac{w}{8\sqrt{x}} + O\left(x^{-\frac{3}{2}}\right),$$

$$R = 1 + \frac{(4z^2-1)(2-w^2)}{32x} + O\left(x^{-2}\right).$$

For small values of x, the following result is obtained, of which the first part is proved using (5.1) and the second using (5.27).

Theorem 5.4.2. (i) *Let $w \in \mathbb{C}$ be fixed. Consider a fixed z such that* $\mathrm{Re}(z) > 0$. *Let $\mathfrak{D} = \{x \in \mathbb{C} : |\arg x| < \frac{\pi}{4}\}$. Then as $x \to 0$ along any path in $\mathfrak{D}$, we have*

$$K_{z,w}(x) \sim \frac{1}{2}\Gamma(z)\left(\frac{x}{2}\right)^{-z} {}_1F_1\left(z; \frac{1}{2}; \frac{-w^2}{4}\right). \tag{5.51}$$

(ii) *Let $w \in \mathbb{C}$ be fixed. Let $|\arg x| < \frac{\pi}{4}$. As $x \to 0$,*

$$K_{0,w}(x) \sim -\log x - \frac{w^2}{2}\,{}_2F_2\left(1, 1; \frac{3}{2}, 2; -\frac{w^2}{4}\right). \tag{5.52}$$

Proof. Replacing x by $x/2$ in (5.1), we get

$$K_{z,w}(x) = \left(\frac{x}{2}\right)^{-z} \int_0^\infty e^{-t^2 - \frac{x^2}{4t^2}} \cos(wt) \cos\left(\frac{wx}{2t}\right) t^{2z-1}\, dt. \tag{5.53}$$

Next, for $\mathrm{Re}(z) > 0$,

$$\begin{aligned}\lim_{x\to 0}\int_0^\infty e^{-t^2-\frac{x^2}{4t^2}}\cos(wt)\cos\left(\frac{wx}{2t}\right)t^{2z-1}\,dt &= \int_0^\infty \lim_{x\to 0} e^{-t^2-\frac{x^2}{4t^2}}\cos(wt)\cos\left(\frac{wx}{2t}\right)t^{2z-1}\,dt\\ &= \int_0^\infty e^{-t^2}\cos(wt)t^{2z-1}\,dt\\ &= \frac{1}{2}\Gamma(z)_1F_1\left(z;\frac{1}{2};-\frac{w^2}{4}\right),\end{aligned}$$

as can be seen from [19, p. 47, Eqn. 5.30]. The above two equations lead us to (5.51) for x lying in the region $\mathfrak{D}$ and tending to 0.

To prove (ii) of Theorem (5.4.2), we note the following asymptotic formulas for the modified Bessel functions $I_z(x)$ and $K_z(x)$ as $x \to 0$ [32, p. 375, equations (9.7.1), (9.7.2)]:

$$I_z(x) \sim \frac{(x/2)^z}{\Gamma(z+1)}, z \neq -1, -2, -3, \cdots . \tag{5.54}$$

and

$$K_z(x) \sim \begin{cases} \frac{1}{2}\Gamma(z)\left(\frac{x}{2}\right)^{-z}, \text{if Re } z > 0,\\ -\log x, \qquad \text{if } z = 0.\end{cases} \tag{5.55}$$

From (5.27), for $|\arg x| < \frac{\pi}{4}$,

$$K_{0,w}(x) = \frac{1}{2}K_0(x)\left(I_0(w\sqrt{2x}) + J_0(w\sqrt{2x})\right) + \sum_{n=1}^{\infty}(-1)^n K_n(x)\left(I_{2n}(w\sqrt{2x}) + J_{2n}(w\sqrt{2x})\right).$$

Consider the first term on the above right-hand side. Note that as $x \to 0$, $I_0(w\sqrt{2x}) \to 0$ and $J_0(w\sqrt{2x}) \to 0$, so along with the second part of (5.55), this implies that

$$\frac{1}{2}K_0(x)\left(I_0(w\sqrt{2x}) + J_0(w\sqrt{2x})\right) \to -\log x. \tag{5.56}$$

Now from (5.54), as $x \to 0$,

$$I_{2n}(w\sqrt{2x}) \sim \frac{\left(w\sqrt{\frac{x}{2}}\right)^{2n}}{\Gamma(2n+1)}.$$

Also, from (1.11) and (5.54), we find that as $x \to 0$,

$$J_{2n}(w\sqrt{2x}) = (-1)^n I_{2n}(-iw\sqrt{2x}) \sim (-1)^n \frac{\left(-iw\sqrt{\frac{x}{2}}\right)^{2n}}{\Gamma(2n+1)}.$$

Interchanging the order of limit and summation using [33, p. 149, Theorem 7.11] and combining the above two equations with the first part of (5.55), we find that

$$\begin{aligned}
&\lim_{x\to 0}\sum_{n=1}^{\infty}(-1)^n K_n(x)\left(I_{2n}(w\sqrt{2x}) + J_{2n}(w\sqrt{2x})\right)\\
&= \sum_{n=1}^{\infty}\lim_{x\to 0}(-1)^n K_n(x)\left(I_{2n}(w\sqrt{2x}) + J_{2n}(w\sqrt{2x})\right)\\
&= \sum_{n=1}^{\infty}\lim_{x\to 0}\frac{(-1)^n}{2}\Gamma(n)\left(\frac{x}{2}\right)^{-n}\left(\frac{\left(w\sqrt{\frac{x}{2}}\right)^{2n}}{\Gamma(2n+1)} + (-1)^n\frac{\left(-iw\sqrt{\frac{x}{2}}\right)^{2n}}{\Gamma(2n+1)}\right)\\
&= \sum_{n=1}^{\infty}\frac{\Gamma(n)}{\Gamma(2n+1)}(-w^2)^n\\
&= -\frac{w^2}{2}{}_2F_2\left(1,1;\frac{3}{2},2;-\frac{w^2}{4}\right), \qquad (5.57)
\end{aligned}$$

where in the last step we used (3.21). The required asymptotic formula is obtained from (5.56) and (5.57). □

5.5 A differential-difference equation for $K_{z,w}(x)$

The differential-difference equation for $K_{z,w}(x)$ is proved below for which we state and prove two simple lemmas.

Lemma 5.5.1. *For* $z, w \in \mathbb{C}$ *and* $|\arg x| < \frac{\pi}{4}$,

$$x^z K_{z,w}(2x) = \frac{e^{2x}}{2} \int_0^\infty e^{-(t+\frac{x}{t})^2} \cos\left(w\left(t+\frac{x}{t}\right)\right) t^{2z-1}\, dt + \frac{e^{-2x}}{2} \int_0^\infty e^{-(t-\frac{x}{t})^2} \cos\left(w\left(t-\frac{x}{t}\right)\right) t^{2z-1}\, dt.$$

Proof. The proof readily follows from (5.1) and the elementary trigonometric identity $2\cos A \cos B = \cos(A+B) + \cos(A-B)$. □

Lemma 5.5.2. *Let* $z, w \in \mathbb{C}$ *and* $|\arg x| < \frac{\pi}{4}$. *Let*

$$I(z,w,x) := \frac{e^{2x}}{2} \int_0^\infty e^{-(t+\frac{x}{t})^2} \cos\left(w\left(t+\frac{x}{t}\right)\right) t^{2z-1}\, dt.$$

Then

$$-\frac{d^2 I(z,w,x)}{dw^2} = I(z+1,w,x) + 2xI(z,w,x) + x^2 I(z-1,w,x).$$

Proof.

$$\frac{d}{dw}\left(\cos\left(w\left(t+\frac{x}{t}\right)\right)\right) = -\left(t+\frac{x}{t}\right)\sin\left(w\left(t+\frac{x}{t}\right)\right)$$

$$-\frac{d^2}{dw^2}\left(\cos\left(w\left(t+\frac{x}{t}\right)\right)\right) = \left(t+\frac{x}{t}\right)^2 \cos\left(w\left(t+\frac{x}{t}\right)\right) = t^2\cos\left(w\left(t+\frac{x}{t}\right)\right) + 2x\cos\left(w\left(t+\frac{x}{t}\right)\right) + \frac{x^2}{t^2}\cos\left(w\left(t+\frac{x}{t}\right)\right) \quad (5.58)$$

The identity now follows by differentiating under the integral sign. □

Theorem 5.5.3. *Let* $z, w \in \mathbb{C}$ *and* $|\arg x| < \frac{\pi}{4}$. *Then*

$$\frac{d^4}{dw^4} K_{z,w}(2x) + 2x\left(\frac{d^2}{dw^2} K_{z+1,w}(2x) + \frac{d^2}{dw^2} K_{z-1,w}(2x)\right) + x^2\left(K_{z+2,w}(2x) - 2K_{z,w}(2x) + K_{z-2,w}(2x)\right) = 0.$$

Proof. Let $K(z,w,x) = x^z K_{z,w}(2x)$. From Lemma 5.5.1,

$$K(z,w,x) = I(z,w,x) + I(z,w,-x), \tag{5.59}$$

where as Lemma 5.5.2 gives

$$\frac{d^2 I(z,w,x)}{dw^2} = -I(z+1,w,x) - 2xI(z,w,x) - x^2 I(z-1,w,x), \tag{5.60}$$

$$\frac{d^2 I(z,w,-x)}{dw^2} = -I(z+1,w,-x) + 2xI(z,w,-x) - x^2 I(z-1,w,-x). \tag{5.61}$$

From (5.59), (5.60) and (5.61), we obtain

$$\frac{d^2 K(z,w,x)}{dw^2} = -K(z+1,w,x) - x^2 K(z-1,w,x) - 2x(I(z,w,x) - I(z,w,-x)). \tag{5.62}$$

Taking the second derivative with respect to w on both sides of the above equation leads to

$$\begin{aligned} \frac{d^4 K(z,w,x)}{dw^4} = -\frac{d^2 K(z+1,w,x)}{dw^2} - x^2 \frac{d^2 K(z-1,w,x)}{dw^2} \\ - 2x\left(\frac{d^2 I(z,w,x)}{dw^2} - \frac{d^2 I(z,w,-x)}{dw^2}\right). \end{aligned} \tag{5.63}$$

Using (5.60), (5.61) and (5.62) in (5.63), we arrive at

$$\begin{aligned} \frac{d^4 K(z,w,x)}{dw^4} &= K(z+2,w,x) + 6x^2 K(z,w,x) + x^4 K(z-2,w,x) \\ &+ 4x(I(z+1,w,x) - I(z+1,w,-x)) + 4x^3(I(z-1,w,x) - I(z-1,w,-x)). \end{aligned} \tag{5.64}$$

Employing (5.62) in (5.64) twice, we get

$$\frac{d^4}{dw^4}K(z,w,x) + 2\frac{d^2}{dw^2}K(z+1,w,x) + 2x^2\frac{d^2}{dw^2}K(z-1,w,x)$$
$$= -K(z+2,w,x) + 2x^2K(z,w,x) - x^4K(z-2,w,x).$$

The desired differential-difference equation follows readily by substituting back $K(z,w,x) = x^z K_{z,w}(2x)$. □

Chapter 6

2-adic valuations of sums of four integer squares

6.1 Introduction

Lagrange's four-square theorem says every natural number can be written as a sum of four integer squares. Let $S = \{a^2 + b^2 + c^2 + d^2 : a, b, c, d \in \mathbb{Z}\}$, then set S is effectively the set of natural numbers with each number repeating a certain number of times. Given any power of 2, no matter how large say v, there are infinitely many natural numbers that are divisible by 2^v periodically spaced in the number line at intervals of length 2^v. Consequently, the set S also contains infinitely many numbers divisible by any power of 2, though not necessarily in the same frequency as above.

Let the set S be restricted by fixing one of the four squares, say a^2, while letting the other three squares take all the possible values, and the subset so obtained be denoted as $S(a) = \{a^2 + b^2 + c^2 + d^2 : b, c, d \in \mathbb{Z}\}$. Counterintuitively, the set $S(a)$ no longer contains numbers divisible by all powers of 2. For any number a, there is a large enough power of 2, say v, such that no number in the set $S(a)$ is divisible by 2^v (see Theorem 6.2.1). For example, for $a = 1$, the set $S(1) = \{1 + b^2 + c^2 + d^2 : b, c, d \in \mathbb{Z}\}$

has no numbers divisible by 8. On the other hand, given any power of 2, no matter how large, there exists infinitely many natural numbers a such that the set $S(a)$ has infinitely many numbers divisible by that power (see Theorem 6.2.2). The existence of such an a such that $S(a)$ has numbers divisible by any given power of 2 is not surprising since the union of the sets $S(a)$ as a vary over natural numbers gives back the set S that has numbers divisible by all powers of 2. However, the existence of infinitely many such a's placed in a characteristic pattern in the number line (see Remark 6.4.6) such that in each set $S(a)$ there are infinitely many numbers divisible by that power of 2 presents an interesting pattern of divisibility by 2. The next question analysed is for a fixed v, how often the sets of the type $S(a)$ contains no numbers divisible by 2^v (see Theorem 6.2.3). For example, as we look at sets $S(a)$ for different values of the natural number a, the proportion of the sets that contains no numbers divisible by 16 is exactly half.

Apart from the sets $S(a)$, two more kinds of sets obtained by fixing two or three sqaures respectively in the sums of four squares are studied. The sets $S(a, b) = \{a^2 + b^2 + c^2 + d^2 : c, d \in \mathbb{Z}\}$ and $S(a, b, c) = \{a^2 + b^2 + c^2 + d^2 : d \in \mathbb{Z}\}$ exhibits the properties very similar in nature to $S(a)$.

It is noteworthy to mention that this property of infinite sets having no numbers divisible by higher powers of a prime p no longer holds true if either the number of squares or the prime is changed. For the sums of five or more squares, even the sets of the smallest cardinality among them viz the ones obatined by fixing all but one squares, often contains numbers divisible by all powers of 2, for example $\{1 + 1 + 4 + 9 + n^2 : n \in \mathbb{Z}\}$. Similarly for any odd prime and the sets obtained by summing any number of squares, some of which fixed, a number of such sets contains numbers divisible by all powers of p. For example, the set $\{1 + 1 + 0 + n^2 : n \in \mathbb{Z}\}$ contains numbers divisible by all powers of 3.

The combination of $p = 2$ and sums of four squares is special in view of the following facts:

1. Hensel's lemma does not hold true for certain cases and the quadratic $x^2 + n$ and $p = 2$ precisely fits the criteria.

2. For a fixed integer n, the set of integer polynomials $\{x^2+n : x \in N\}$ has numbers divisible by any high power of 2 if and only if n is of the form $4^m(8l + 7)$.

3. Legendre's three-square theorem: An integer n can be written as sum of three or less squares if and only if n is not of the form $4^m(8l + 7)$.

It is also noteworthy to mention the inherent connection of this analysis with the widely known function $r_k(n)$ that counts the number of ways n can be represented as sum of k integer squares, allowing zeros and distinguishing signs and order in counting the number of ways. The set S itself is natural numbers repeated $r_4(n)$ number of times. If two numbers are considered different even if they are equal in value but comes from two different represenation as sums of squares, then the collection of subsets $S(a)$ as a varies over integers forms a disjoint partition of the set S. From this perspective, the chapter considers the three different ways of partitioning the set S viz using subsets of the types $S(a)$, $S(a, b)$ and $S(a, b, c)$ respectively and studies the divisibility by powers of 2 of the subsets taken as an whole as well as the numbers the subsets contains. The most direct connection with the function $r_k(n)$ in this analysis is evident in Theorem 6.2.9 that says the proportion of subsets of the type $S(a, b, c)$ that has numbers divisible by 2^{2v} is $1/2^{3v}$. On the other hand, the function $r_3(n)$ satisfies the identity asymptotically [23]:

$$\frac{1}{2^{3v}} \sum_{n=1}^{2^{2v}k} r_3(n) \sim \sum_{n=1}^{k} r_3(n)$$

Theorems 6.2.3, 6.2.6 and 6.2.9 indicates that in contrast with the highly regular spacing of numbers divisible by powers of 2 in the number line, the numbers divisible by powers of 2 are distributed in an highly uneven pattern in all the three partitions of S. For example, 31 out of 32 subsets of the type $S(a, b, c)$ picked at random will have no numbers divisible by 2^4. The partitioning inherently involves $r_3(n)$ and $r_2(n)$, so it is likely these functions play a role in the skewed proportions. However, the rest of the analysis concerning the divisibility by powers of 2 especially Theorems 6.2.1, 6.2.4 and 6.2.7 does not seem to show a direct connection with the functions $r_k(n)$. Incidentally, the triangular numbers kept recurring in the proofs of these theorems in interesting ways.

The proofs presented here are straighforward, elementary and self-sufficient to appreciate the results. The second section of the chapter lists all the results. The third section covers the preliminary lemmas that can be skipped altogether. The fourth, fifth and sixth sections comprises of Propositions 6.4.1, 6.5.1 and 6.6.1 that gives insight into the divisibility of sums of four squares by powers of 2 while fixing one, two and three of the squares respectively and prove all the theorems from the second section. The concluding section discusses the failing of the phenomena for any other combination of primes and the sums of squares.

6.2 Main Theorems

Definition. For $n \in \mathbb{Z}$, the 2-adic valuation of n denoted as $\nu_2(n)$ is defined as the exponent of the highest power of 2 that divides n.

Theorems 6.2.1, 6.2.2 and 6.2.3 concerns the set obtained by fixing one out of the four squares:

Theorem 6.2.1. *Given $a \neq 0$, there exists v such that no number in the set $S(a) := \{a^2 + x^2 + y^2 + z^2 : x, y, z \in \mathbb{Z}\}$ is divisible by 2^v. Moreover, the highest power of 2 that divides any number in the set $S(a)$ is directly proportional to the highest power of 2 that divides a, i.e. it is equal to $2\nu_2(a) + 2$.*

Theorem 6.2.2. *Given v, there exists infinitely many non-zero numbers a's such that the sets $S(a) = \{a^2 + x^2 + y^2 + z^2 : x, y, z \in \mathbb{Z}\}$ has infinitely many numbers divisible by 2^v.*

Theorem 6.2.3. *For arbitrary but fixed $a \neq 0$, the probability that the set $S(a) = \{a^2 + x^2 + y^2 + z^2 : x, y, z \in \mathbb{Z}\}$ has a number divisible by 2^{2v} is equal to $1/2^{v-1}$.*

Theorems 6.2.4, 6.2.5 and 6.2.6 concerns the set obtained by fixing two out of the four squares:

Theorem 6.2.4. *Given a couple $(a, b) \neq (0, 0)$, there exists v such that no term in the set $S(a, b) := \{a^2 + b^2 + x^2 + y^2 : x, y \in \mathbb{Z}\}$ is divisible by 2^v. Moreover, the highest power of 2 that divides any number in the set $S(a, b)$ is directly proportional to the highest power of 2 that divides $a^2 + b^2$, i.e. it is equal to $\nu_2(a^2 + b^2) + 1$.*

Theorem 6.2.5. *Given v, there exists infinitely many couples $(a, b) \neq (0, 0)$ such that the sets $S(a, b) = \{a^2 + b^2 + x^2 + y^2 : x, y \in \mathbb{Z}\}$ has infinitely many numbers divisible by 2^v.*

Theorem 6.2.6. *For arbitrary but fixed couples $(a, b) \neq (0, 0)$, the probability that the set $S(a, b) = \{a^2 + b^2 + x^2 + y^2 + z^2 : x, y \in \mathbb{Z}\}$ has a number divisible by 2^v is equal to $1/2^{v-1}$.*

Theorems 6.2.7, 6.2.8 and 6.2.9 concerns the set obtained by fixing three out of the four squares:

Theorem 6.2.7. *Given a triplet $(a, b, c) \neq (0, 0, 0)$, there exists v such that no number in the set $S(a, b, c) := \{a^2 + b^2 + c^2 + x^2 : x \in \mathbb{Z}\}$ is divisible by 2^v. Moreover, the highest power of 2 that divides any number in the set $S(a, b, c)$ is directly proportional to the highest power of 2 that divides $a^2+b^2+c^2$ i.e. it is proportional to $\nu_2(a^2+b^2+c^2)$.*

Theorem 6.2.8. *Given v, there exists infinitely many triplets $(b, c, d) \neq (0, 0, 0)$ such that the sets $S(a, b, c) = \{a^2+b^2+c^2+x^2 : x \in \mathbb{Z}\}$ has infintely many numbers divisible by 2^v periodically spaced if the set is considered a sequence.*

Theorem 6.2.9. *For arbitrary but fixed triplets* $(a, b, c) \neq (0, 0, 0)$*, the probability that the sets* $S(a, b, c) = \{a^2 + b^2 + c^2 + x^2 : x \in \mathbb{Z}\}$ *has a number divisible by* 2^v *is equal to* $1/2^{f(v)}$ *where* $f(v)$ *equals* $\left\lfloor \frac{3v}{2} \right\rfloor - 1$.

Remark 6.2.10. The function $f(v)$ takes the values $0, 2, 3, 5, 6, 8, 9, 11, 12 \ldots$ for $v = 1, 2, 3, \ldots$.

6.3 Preliminaries

Definition. The 2-adic valuation of a number, denoted as $\nu_2(n)$, is defined as the exponent of the highest power of 2 that divides n.

Notation. For a set A, we define $\{\nu_2(A)\}$ as the set of the 2-adic valuations of each number in the set A. Moreover, $\nu_2(A) \to \infty$ means the sequence $\{\nu_2(A)\}$ diverges to infinity. In other words, $\nu_2(A) \to \infty$ implies any given power of 2 divides some number in the set A.

The following result is proved in [34] using modular trees.

Theorem 6.3.1. $\nu_2(x^2 + k) \to \infty \iff$ *k is of the form* $4^m(8l + 7)$.

Lemma 6.3.2. *For any integer x:*

1. $x^2 \equiv 0 \pmod{2^{2r}} \iff x \equiv 0 \pmod{2^r}$
2. $x^2 \equiv 2^{2r} \pmod{2^{2r+2}} \iff x \equiv 2^r \pmod{2^{r+1}}$
3. $x^2 \equiv 0 \pmod{2^{2r+1}} \iff x \equiv 0 \pmod{2^{r+1}}$
4. $x^2 \equiv 2^{2r} \pmod{2^{2r+1}} \iff x \equiv 2^r \pmod{2^{r+1}}$

Proof. By Unique Factorization Theorem: $x^2 = 2^{2r}(2k + 1) \iff x = 2^r(2l + 1)$. It follows. □

Lemma 6.3.3. *For* $x, y \in \mathbb{Z}$ *and* $r > 0$*:*

1. $x^2 + y^2 \equiv 0 \pmod{2^{2r}} \iff x \equiv y \equiv 0 \pmod{2^r}$.

2. $x^2 + y^2 \equiv 0 \pmod{2^{2r+1}} \iff x \equiv y \equiv 0$ *or* $2^r \pmod{2^{r+1}}$.

Proof. 1. Let $x^2 + y^2 \equiv 0 \pmod{4^r}$ which implies $x^2 + y^2 \equiv 0 \pmod 4$. Since x^2 can only be 0 or 1 $\pmod 4$, $x^2 \equiv y^2 \equiv 0 \pmod 4$ which means $x = 2x_1,\ y = 2y_1$ with $x_1^2 + y_1^2 \equiv 0 \pmod{4^{r-1}}$. Proceeding this way we get $x = 2^r x_r,\ y = 2^r y_r$ and thus, $x \equiv y \equiv 0 \pmod{2^r}$.

Conversely, using Lemma 6.3.2 it is obvious $x \equiv y \equiv 0 \pmod{2^r} \implies x^2 + y^2 \equiv 0 \pmod{4^r}$

2. Let $x^2 + y^2 \equiv 0 \pmod{2^{2r+1}}$ which implies $x^2 + y^2 \equiv 0 \pmod 4$ for $r \geq 1$. Similar arguments as above gives $x = 2^{r-1}x_{r-1},\ y = 2^{r-1}y_{r-1}$ with $x_{r-1}^2 + y_{r-1}^2 \equiv 0 \pmod 2$, the solution for which is $x_{r-1}^2 \equiv y_{r-1}^2 \equiv 0$ or $1 \pmod 2$ and thus, $x \equiv y \equiv 0$ or $2^r \pmod{2^{r+1}}$.

Conversely, using Lemma 6.3.2 it is obvious $x \equiv y \equiv 0$ or $2^r \pmod{2^{r+1}} \implies x^2 + y^2 \equiv 0 \pmod{2^{2r+1}}$

□

Lemma 6.3.4. *For* $x, y, z \in \mathbb{Z}$*:*

1. $x^2 + y^2 + z^2 \equiv 0 \pmod{2^{2r}} \iff x \equiv y \equiv z \equiv 0 \pmod{2^r}$.

2. $x^2 + y^2 + z^2 \equiv 3 \cdot 4^{r-1} \pmod{2^{2r}} \iff x \equiv y \equiv z \equiv 2^{r-1} \pmod{2^r}$.

Proof. 1. Let $x^2 + y^2 + z^2 \equiv 0 \pmod{4^r}$ which implies $x^2 + y^2 + z^2 \equiv 0 \pmod 4$. Since x^2 can only be 0 or 1 $\pmod 4$, $x^2 \equiv y^2 \equiv z^2 \equiv 0 \pmod 4$ which means $x = 2x_1,\ y = 2y_1,\ z = 2z_1$ with $x_1^2 + y_1^2 + z_1^2 \equiv 0 \pmod{4^{r-1}}$. Proceeding this way we get $x = 2^r x_r,\ y = 2^r y_r,\ z = 2^r z_r$ and thus, $x \equiv y \equiv z \equiv 0 \pmod{2^r}$.

Conversely, using Lemma 6.3.3 it is obvious $x \equiv y \equiv z \equiv 0 \pmod{2^r} \implies x^2 + y^2 + z^2 \equiv 0 \pmod{4^r}$

2. Using similar arguments we get, $x = 2^{r-1}x_{r-1},\ y = 2^{r-1}y_{r-1},\ z = 2^{r-1}z_{r-1}$ and $x_{r-1}^2 + y_{r-1}^2 + z_{r-1}^2 \equiv 3 \pmod 4$, the solution for which is $x_{r-1}^2 \equiv y_{r-1}^2 \equiv z_{r-1}^2 \equiv 1 \pmod 4$. This means $x_{r-1} \equiv y_{r-1} \equiv z_{r-1} \equiv 1 \pmod 2)$ by Lemma 6.3.3. Hence, $x \equiv y \equiv z \equiv 2^{r-1} \pmod{2^r}$.

   Conversely, using Lemma 6.3.3 it is obvious $x \equiv y \equiv z \equiv 2^{r-1} \pmod{2^r} \implies x^2 + y^2 + z^2 \equiv 3 \cdot 4^{r-1} \pmod{2^{2r}}$

□

6.4 Fixing one of the four squares in the sums of squares

Definition. Let $\lambda(n) := \sup\limits_{x,y,z\in\mathbb{Z}} \{\nu_2(x^2 + y^2 + z^2 + n)\}$ i.e. $\lambda(n)$ is the supremum of the powers of 2 that divides any number in the set $\{x^2 + y^2 + z^2 + n : x, y, z \in \mathbb{Z}\}$.

Proposition 6.4.1. *1. For any $r, k \in \mathbb{N}$, let $n = 4^r(8k + 1)$, then $\lambda(n) = 2r + 2$. Moreover, there are infinitely many numbers in the set $S(n) = \{x^2 + y^2 + z^2 + n : x, y, z \in \mathbb{Z}\}$ divisible by 2^{2r+2}.*

2. For n not of the form $4^r(8k + 1)$, $\lambda(n) \to \infty$.

Remark 6.4.2 (Legendre's Three square theorem). A non-zero number n is sum of three squares $\iff n \neq 4^r(8k + 7)$. In other words, any number n not of the form $4^r(8k + 7)$ can be written as sum of three squares.

Proof of Proposition 6.4.1. 1. Let $n = 4^r(8k+1)$. By the remark 6.4.2, $x^2+y^2+z^2$ takes all values except the numbers of the form $4^s(8l + 7)$. The possible cases for $x^2 + y^2 + z^2$:

(a) Let $x^2 + y^2 + z^2 = 2^{2s+1}(2l + 1)$, then

$$\nu_2(x^2 + y^2 + z^2 + n) = \nu_2(2^{2s+1}(2l + 1) + 4^r(8k + 1)) = \min(2r, 2s + 1) \leq 2r$$

.

(b) Let $x^2+y^2+z^2=4^s(4l+1)$, then

$$\nu_2(x^2+y^2+z^2+n)=\begin{cases}\min(2r,2s)\leq 2r & \text{if } r\neq s\\ 2r+\nu_2(8k+1+4l+1)=2r+1 & \text{if } r=s\end{cases}$$

(c) Let $x^2+y^2+z^2=4^s(8l+3)$, then

$$\nu_2(x^2+y^2+z^2+n)=\begin{cases}\min(2r,2s)\leq 2r & \text{if } r\neq s\\ 2r+\nu_2(8k+1+8l+3)=2r+2 & \text{if } r=s\end{cases}$$

From the remark 6.4.2, for every natural number l, there exists $x,y,z\in\mathbb{Z}$ such that $x^2+y^2+z^2=4^r(8l+3)$ for which $\nu_2(x^2+y^2+z^2+n)=2r+2$. Hence there are infinitely many numbers in the set $S(n)=\{x^2+y^2+z^2+n : x,y,z\in\mathbb{Z}\}$ divisible by 2^{2r+2}.

2. Let n be not of the form $4^r(8k+1)$. Then n must be of one of the form: $2^{2r+1}(2k+1)$, $4^r(4k+3)$ and $4^r(8k+5)$. In each of these forms of n, a suitable form for $x^2+y^2+z^2$, that is not equal to $4^s(8l+7)$ in view of remark 6.4.2, can be chosen as follows:

(a) For $n=2^{2r+1}(2k+1)$, let $x^2+y^2+z^2=2^{2r+1}(2l+1)$. Then,

$$\nu_2(x^2+y^2+z^2+n)=\nu_2(2^{2r+1}(2l+1)+2^{2r+1}(2k+1))=2r+2+\nu_2(k+l+1)$$

(b) For $n=4^r(4k+3)$, let $x^2+y^2+z^2=4^r(4l+1)$. Then,

$$\nu_2(x^2+y^2+z^2+n)=\nu_2(4^r(4l+1)+4^r(4k+3))=2r+2+\nu_2(k+l+1)$$

(c) For $n = 4^r(8k+5)$, let $x^2+y^2+z^2 = 4^r(8l+3)$ Then,

$$\nu_2(x^2+y^2+z^2+n) = \nu_2(4^r(8l+3)+4^r(8k+5)) = 2r+3+\nu_2(k+l+1)$$

In all the above cases n is given, and the sum $x^2+y^2+z^2$, whose form is chosen based on the form of n, can take different values. Thus k is fixed once n is given, but l can take different values. Moreover, the remark 6.4.2 implies for any number of the form $2^{2r+1}(2l+1), 4^r(4l+1)$ or $4^r(8l+3)$, it is equal to $x^2+y^2+z^2$ for some $x, y, z \in \mathbb{Z}$. Thus l ranges over all natural numbers which implies $\nu_2(k+l+1) \to \infty$. Hence, $\nu_2(x^2+y^2+z^2+n) \to \infty$ for all n not of the form $4^r(8k+1)$.

□

Lemma 6.4.3. *A non-zero number n is a square implies $n = 4^r(8k+1)$ for some $r, k \in \mathbb{N}$.*

Proof. If $n = (2m+1)$, then $n^2 = 4m^2+4m+1 = 4m(m+1)+1 = 8k+1$.
For $n = 2^r(2m+1)$, it follows that $n^2 = 4^r(8k+1)$. □

The converse is not true, i.e. a number of the form $n = 4^r(8k+1)$ need not be a square. In fact, it is a square if and only if k is a triangular number as defined below.

Definition 6.4.4. Triangular numbers are numbers of the form $\frac{n(n+1)}{2}$ for $n \in \mathbb{N}$.

Lemma 6.4.5. *A number of the form $8t+1$ is a square $\iff$ t is a triangular number.*

Proof. $t = \frac{n(n+1)}{2} \iff 8t+1 = 4n(n+1)+1 = (2n+1)^2$ □

Now, we are ready to give proofs of the first three main theorems given in Section 6.2.

Proof of Theorem 6.2.1. Given $a \neq 0$, Lemma 6.4.3 implies $a^2 = 4^r(8k+1)$ where $r = \nu_2(a)$. Proposition 6.4.1 implies $\lambda(a^2) = 2r+2$ which means the highest power of 2 that divides any number in the set $S(a)$ is $2\nu_2(a)+2$. Moreover, for any $v > 2r+2$, no number in the set $S(a) = \{a^2 + x^2 + y^2 + z^2 : x, y, z \in \mathbb{Z}\}$ is divisible by 2^v. □

Proof of Theorem 6.2.2. Given $v > 0$, let r be defined in the folowing manner: if v is even, then let $r = v/2$, else let $r = (v-1)/2$ so that in either case $2r \geq v$. From Proposition 6.4.1, for any k, $\nu(4^r(8k+1)) = 2r \geq v$ and there are infinitely many $x, y, z \in \mathbb{Z}$ such that $x^2 + y^2 + z^2 + 4^r(8k+1)$ is divisible by 2^v. On the other hand, from Lemma 6.4.5, $8k+1$ is a square whenever k is a triangular number. Since there are infinitely many triangular numbers, it implies there are infinitely many a's of the form $a^2 = 4^{r-1}(8k+1)$. Hence there exists infinitely many non-zero numbers a's of the form $a^2 = 4^{r-1}(8k+1)$ such that the sets $S(a) = \{a^2 + x^2 + y^2 + z^2 : x, y, z \in \mathbb{Z}\}$ has infinitely many numbers divisible by 2^v. □

Remark 6.4.6. In Theorem 6.2.2, the even exponent of all such a's is solely determined by the power v whereas for every traingular number $k = n(n+1)/2$, there is a corresponding a whose odd part equals $2n+1$.

Proof of Theorem 6.2.3. For arbitrary but fixed $a \neq 0$, the probability that the set $S(a) = \{a^2 + x^2 + y^2 + z^2 : x, y, z \in \mathbb{Z}\}$ has a number divisible by 2^{2v} is given by the probability that $\lambda(a^2) \geq 2v$. From Proposition 6.4.1, $\lambda(a^2) = 2v \iff a^2 = 4^{v-1}(8k+1) \iff \nu_2(a) = v-1$. Therefore $\lambda(a^2) \geq 2v \iff \nu_2(a) \geq v-1 \iff a \equiv 0 \pmod{2^{v-1}}$.

Hence, $\text{Prob}(\lambda(a^2) \geq 2v) = \text{Prob}(a \equiv 0 \pmod{2^{v-1}}) = \frac{1}{2^{v-1}}$. □

6.5 Fixing two of the four squares in the sums of squares

Definition. Let $\mu(n) := \sup_{x,y\in\mathbb{Z}} \{\nu_2(x^2+y^2+n)\}$ i.e. $\mu(n)$ is the supremum of the powers of 2 that divides any number in the set $\{x^2+y^2+n : x,y\in\mathbb{Z}\}$.

Proposition 6.5.1. *1. For any $r,k\in\mathbb{N}$, let $n=2^r(4k+1)$, then $\mu(n)=r+1$. Moreover, there are infinitely many numbers in the set $\{x^2+y^2+n : x,y\in\mathbb{Z}\}$ divisible by 2^{r+1}.*

2. $\mu(2^r(4k+3))\to\infty$.

Before we proceed to the proof of Proposition 6.5.1, we need a few lemmas.

Lemma 6.5.2. *A non-zero number n is sum of two squares $\iff$ n has no prime $p\equiv 3 \pmod 4$ with an odd exponent in its factorization. The latter implies $n=2^r(4k+1)$ for some $r,k\in\mathbb{N}$*

Lemma 6.5.2 is well-known in number theory and its proof is readily available in textbooks.

Lemma 6.5.3. *A number of the form $4l+1$ is a sum of two squares $\iff$ l is either itself a triangular number or a sum of two triangular numbers.*

Proof. Let $l=\frac{n(n+1)}{2}$. Then, $4l+1=n^2+(n+1)^2$.
Let $l=\frac{n(n+1)}{2}+\frac{m(m+1)}{2}$. Then, $4l+1=(m-n)^2+(m+n+1)^2$.
Conversely, let $4l+1=c^2+d^2$. Then, setting $a=m-n$ and $b=m+n+1$, we get $m=\frac{a+b-1}{2}$ and $n=\frac{b-a-1}{2}$. let $l_1=\frac{m(m+1)}{2}=\frac{(a+b)^2-1}{8}$ and $l_2=\frac{n(n+1)}{2}=\frac{(b-a)^2-1}{8}$. Then, $4(l_1+l_2)+1=a^2+b^2=4l+1$ which implies $l=l_1+l_2$. □

Lemma 6.5.4. *Let $\{l\}=\{1,3,6,10,15,21,\dots\}$ be the sequence of all triangular numbers. Then for arbitray but fixed integer m, the sequence $\{\nu_2(l+m)\}\to\infty$.*

Proof. From the definition 6.4.4 of triangular numbers, $l = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$. Let $m = 2n+1$, then $l = \frac{m^2-1}{8}$ where m is odd. For any fixed $r \in \mathbb{Z}$, $\nu_2(l+r) = \nu_2(\frac{m^2-1}{8} + r) = \nu_2(m^2 + 8r - 1) - 3$. From Proposition 6.6.1, since $8r-1$ is of the form $8k+7$, $\nu_2(m^2+8r-1) \to \infty$ when m ranges over odd numbers and hence $\{\nu_2(l+m)\} \to \infty$. □

Proof of Proposition 6.5.1.

1. Let $n = 2^r(4k+1)$. From Lemma 6.5.2, $x^2+y^2 = 2^s(4l+1)$ for some $s, l \in \mathbb{N}$.

$$\nu_2(x^2+y^2+n) = \begin{cases} \min(r,s) \le r & \text{if } r \neq s \\ r + \nu_2(4k+1+4l+1) = r+1 & \text{if } r = s \end{cases}$$

   Thus $\nu_2(x^2+y^2+n) \le r+1$ for all $x, y \in \mathbb{N}$. Lemma 6.5.2 implies given any r and $l \in \mathbb{N}$, there exists $x, y \in \mathbb{Z}$ such that $x^2+y^2 = 2^r(4l+1)$ for which $\nu_2(x^2+y^2+n) = r+1$. Here r is fixed once n is given, however l ranges over all natural numbers giving infinitely many numbers in the set $\{x^2+y^2+n : x, y \in \mathbb{Z}\}$ divisible by 2^{r+1}.

2. Let $n = 2^r(4k+3)$. As explained above, there exists x, y such that $x^2+y^2 = 2^r(4l+1)$.

$$\nu_2(x^2+y^2+n) = \nu_2(2^r(4l+1) + 2^r(4k+3)) = r + 2 + \nu_2(k+l+1)$$

   Here n is given, and the sum x^2+y^2 can take different values. Thus, k is fixed while l can only take values for which $4l+1$ is a sum of two squares x^2+y^2. Lemma 6.5.3 implies $4l+1$ is a sum of two squares whenever l is a triangular number and Lemma 6.5.4 implies $\{\nu_2(k+l+1)\} \to \infty$ for the set of triangular numbers l. Hence, $\{\nu_2(x^2+y^2+n)\} \to \infty$.

□

Now, we are ready to give proofs of the three main theorems given in Section 6.2.

Proof of Theorem 6.2.4. Given a couple $(a, b) \neq (0, 0)$, Lemma 6.5.2 implies $a^2+b^2 = 2^r(4k+1)$ where $r = \nu_2(a^2+b^2)$. Proposition 6.5.1 implies $\mu(a^2+b^2) = r+1$ which means the highest power of 2 that divides any number in the set $S(a, b)$ is $\nu_2(a^2+b^2)+1$. Moreover, for any $v > r+1$, no number in the set $S(a, b) = \{a^2+b^2+x^2+y^2 : x, y \in \mathbb{Z}\}$ is divisible by 2^v. □

Proof of Theorem 6.2.5. Given $v > 0$, from Proposition 6.5.2, $\mu(2^{v-1}(4k+1)) = v$ for any k and there are infinitely many $x, y \in \mathbb{Z}$ such that $x^2+y^2+2^{v-1}(4k+1)$ is divisible by 2^v. On the other hand, from Lemma 6.5.3, $4k+1$ is a sum of two squares whenever k is itself a triangular number or a sum of two triangular numbers. Since there are infinitely many triangular numbers, it implies there are infinitely many couples (a, b) such that their sum a^2+b^2 equals $2^{v-1}(4k+1)$. Hence there exists infinitely many couples (a, b) with $a^2+b^2 = 2^{v-1}(4k+1)$ such that the sets $S(a, b) = \{a^2+b^2+x^2+y^2 : x, y \in \mathbb{Z}\}$ has infinitely many numbers divisible by 2^v. □

Proof of Theorem 6.2.6. For arbitrary but fixed couples $(a, b) \neq (0, 0)$, the probability that the set $S(a, b) = \{a^2+b^2+x^2+y^2+z^2 : x, y \in \mathbb{Z}\}$ has a number divisible by 2^v is given by the probability that $\mu(a^2+b^2) \geq v$.

1. For v odd, say $2r+1$:

$$\begin{aligned}
\text{Prob}\left(\mu(a^2+b^2) \geq 2r+1\right) &= \text{Prob}\left(a^2+b^2 \equiv 0 \pmod{2^{2r}}\right) \\
&= \text{Prob}\left(a \equiv 0 \;\&\; b \equiv 0 \pmod{2^r}\right) \\
&= \frac{1}{2^r} \cdot \frac{1}{2^r} = \frac{1}{2^{2r}} = \frac{1}{2^{v-1}}
\end{aligned}$$

2. For v even, say $2r+2$:

$$\begin{aligned}
\text{Prob}\ \big(\mu(a^2+b^2) \geq 2r+2\big) &= \text{Prob}\ \big(a^2+b^2 \equiv 0 \pmod{2^{2r+1}}\big) \\
&= \text{Prob}\ \big(a \equiv 0\ \&\ b \equiv 0 \pmod{2^{r+1}}\big) + \\
&\quad\ \text{Prob}\ \big(a \equiv 2^r\ \&\ b \equiv 2^r \pmod{2^{r+1}}\big) \\
&= \left(\frac{1}{2^{r+1}}\right)^2 + \left(\frac{1}{2^{r+1}}\right)^2 = \frac{1}{2^{2r+1}} = \frac{1}{2^{v-1}}
\end{aligned}$$

In both the cases, Lemma 6.5.1 implies the first step and Lemma 6.3.3 implies the second step. □

6.6 Fixing three of the four squares in the sums of squares

Definition. Let $\nu(n) := \sup_{x\geq 0} \{\nu_2(x^2+n)\}$ i.e. $\nu(n)$ is the supremum of the powers of 2 that divides any number in the set $\{x^2+n : x \in \mathbb{Z}\}$.

Proposition 6.6.1. *For any number k:*

1. $\nu(4k) = 2 + \nu(k)$
2. $\nu(4k+1) = 1$. *Moreover, x^2+4k+1 is divisible by 2 iff x is an odd integer.*
3. $\nu(4k+2) = 1$. *Moreover, x^2+4k+2 is divisible by 2 iff x is an even integer.*
4. $\nu(8k+3) = 2$. *Moreover, x^2+8k+3 is divisible by 4 iff x is an odd integer.*
5. $\nu(8k+7) \to \infty$. *Moreover, x^2+8k+7 is divisible by powers of 2 only when x is an odd integer.*

Remark 6.6.2. To rephrase Proposition 6.6.1, $\nu(n) \to \infty$ if and only if n is of the form $4^m(8l+7)$. For any number n not of this form, $\nu(n)$ takes either of the three values $\nu_2(n)$, $\nu_2(n)+1$ or $\nu_2(n)+2$.

Proof. 1. If $x = 2m$, then $x^2 + 4k = 4(m^2 + k)$ else let $x = 2m + 1$, then $x^2 + 4k = (2m + 1)^2 + 4k = 4m^2 + 4m + 4k + 1$. Hence, $\nu(4k) = 2 + \nu(k)$.

2. If $x = 2m$, then $x^2 + 4k + 1 = 4m^2 + 4k + 1$ else let $x = 2m + 1$, then $x^2 + 4k + 1 = (2m + 1)^2 + 4k + 1 = 2(2m^2 + 2m + 2k + 1)$. Hence, $\nu(4k + 1) = 1$.

3. If $x = 2m$, then $x^2 + 4k + 2 = 4m^2 + 4k + 2 = 2(2m^2 + 2k + 1)$ else let $x = 2m + 1$, then $x^2 + 4k + 2 = (2m + 1)^2 + 4k + 2 = 4m^2 + 4m + 4k + 3$. Hence, $\nu(4k + 2) = 1$.

4. If $x = 2m$, then $x^2 + 8k + 3 = 4m^2 + 8k + 3$ else let $x = 2m + 1$, then $x^2 + 8k + 3 = (2m + 1)^2 + 8k + 3 = 4(m^2 + m + 2k + 1)$. Hence, $\nu(8k + 3) = 2$.

5. Follows from Theorem 6.3.1.

□

Remark. $\nu(2k) \neq 1 + \nu(k)$. For e.g. $\nu(2) = \nu(1) = 1$.

Corollary 6.6.3. $\nu(n) \to \infty \iff$ *n is of the form* $4^r(8k+7)$. *Moreover, the highest power of* 2 *that divides any number in the set* $\{x^2 + n : x \in \mathbb{Z}\}$, *say* v, *equals either* $\nu_2(n)$, $\nu_2(n) + 1$ *or* $\nu_2(n) + 2$ *depending on* n. *The* x*'s corresponding to the numbers* x^2+n *that are divisible by the above mentioned* v *are spaced periodically in the number line with period lengths that are powers of* 2.

Now, we give the proof for Theorem 6.2.7 stated in Section 6.2.

Proof of Theorem 6.2.7. Given a triple $(a, b, c) \neq (0, 0, 0)$, the remark 6.4.2 implies $a^2+b^2+c^2$ takes one of the three forms $4^r(4k+1)$, $4^r(4k+2)$ or $4^r(8k+3)$. Proposition 6.6.1 implies $\nu(a^2 + b^2 + c^2) \leq \nu_2(a^2 + b^2 + c^2) + 2$ as explained in Corollary 6.6.3. This means the highest power of 2 that divides any number in the set $S(a, b, c)$ is directly proportional to $\nu_2(a^2 + b^2 + c^2)$. Moreover, for any $v > \nu_2(a^2 + b^2 + c^2) + 2$, no number in the set $S(a, b, c) = \{a^2 + b^2 + c^2 + x^2 : x \in \mathbb{Z}\}$ is divisible by 2^v. □

Lemma 6.6.4. *A number of the form $8n+3$ is a square of three squares $\iff$ n is a sum of three triangular number.*

Proof. From Lemma 6.4.3, a square can only be $0, 1$ or $4 \pmod 8$. Hence, $8n+3$ is a sum of three squares if and only if all the three are odd squares.
$8k+3 = (2k+1)^2 + (2l+1)^2 + (2m+1)^2 \iff 8n+3 = 4k(k+1)+1+4l(l+1)+1+4m(m+1)+1 \iff n = \frac{k(k+1)}{2} + \frac{l(l+1)}{2} + \frac{m(m+1)}{2}$. □

Remark 6.6.5. Guass's Eureka theorem that every number n is the sum of 3 triangular numbers and hence every $8n+3$ is a sum of 3 squares.

Proof of Theorem 6.2.8. Given $v > 0$, let r be defined in the folowing manner: if v is even, then let $r = v/2$, else let $r = (v-1)/2$ so that in either case $2r \geq v$. From Proposition 6.6.6, for any k, $\lambda(4^{r-1}(8k+3)) = 2r \geq v$ and there are infinitely many x periodically spaced in the number line such that $x^2 + 4^{r-1}(8k+3)$ is divisible by 2^v. On the other hand, from Lemma 6.6.4, $8k+3$ is a sum of two squares whenever k is a sum of three triangular numbers. Since there are infinitely many triangular numbers, it implies there are infinitely many triplets (a, b, c) such that their sum $a^2 + b^2 + c^2$ equals $4^{r-1}(8k+3)$. Hence there exists infinitely many triplets (a, b, c) such that the sequences $S(a, b, c) = \{a^2 + b^2 + c^2 + x^2 : x \in \mathbb{Z}\}$ has infinitely many periodically spaced numbers divisible by 2^v. □

Lemma 6.6.6. *For any $r \in \mathbb{N}$ and n not of the form $4^r(8k+7)$,*

(i) $\nu(n) \geq 2r+1 \iff n$ is of the form $4^r k$ (k need not be odd).

(ii) $\nu(n) \geq 2r \iff n$ is either of the two forms $4^r k$ or $n = 4^{r-1}(8k+3)$.

Proof. (i) Proof by induction on r: The case $r = 0$ and $r = 1$ trivially holds true from Proposition 6.6.1 and let $\nu(n) \geq 2(r-1)+1 \iff n = 4^{r-1}k$ holds true. Let $\nu(n) \geq 2r+1$, then $\nu(n) \geq 3$ since $r \geq 1$. From Proposition 6.6.1, this is possible only if n is divisible by 4, so let $n = 4l$. Then again using Proposition

6.6.1, $\nu(n) = 2 + \nu(l)$. Thus, $\nu(n) \geq 2r + 1$ implies $\nu(l) \geq 2(r-1) + 1$. By induction hypothesis, $l = 4^{r-1}k$ and hence, $n = 4l = 4^r k$. Conversely, let $n = 4^r k$, then $\nu(n) = \nu(4^r k) = 2 + \nu(4^{r-1}k) \geq 2 + 2(r-1) + 1$. Thus, $\nu(n) \geq 2r + 1$.

(ii) It is only left to be proved that $\nu(n) = 2r \iff n = 4^{r-1}(8k+3)$ for some k. Proof by induction on r: The case $r = 1$ trivially holds true from Proposition 6.6.1 and let $\nu(n) = 2(r-1) \iff n = 4^{r-2}(8k+3)$ holds true.

Let $\nu(n) = 2r$, then $\nu(n) \geq 4$ since $2r \geq 4$. From Proposition 6.6.1, the same argument as above gives $n = 4l$ and $\nu(n) = 2 + \nu(l)$, which in turn implies $\nu(l) = 2(r-1)$. By induction hypothesis, $l = 4^{r-2}(8k+3)$ and hence, $n = 4^{r-1}(8k+3)$.

□

Proof of Theorem 6.2.9. For arbitrary but fixed triplets $(a, b, c) \neq (0, 0, 0)$, the probability that the sets $S(a, b, c) = \{a^2 + b^2 + c^2 + x^2 : x \in \mathbb{Z}\}$ has a number divisible by 2^v is given by the probability that $\nu(a^2 + b^2 + c^2) > v$.

(i) Let v be odd, say $2r + 1$:

$$\begin{aligned}
\text{Prob}\left(\nu(a^2 + b^2 + c^2) \geq 2r + 1\right) &= \text{Prob}\left(a^2 + b^2 + c^2 \equiv 0 \bmod 4^r\right) \\
&= \text{Prob}\left(a \equiv 0 \ \&\ b \equiv 0 \ \&\ c \equiv 0 \bmod 2^r\right) \\
&= \frac{1}{2^r} \cdot \frac{1}{2^r} \cdot \frac{1}{2^r} = \frac{1}{2^{3r}}.
\end{aligned}$$

(ii) Let v be even, say $2r$:

$$\begin{aligned}
\text{Prob}\left(\nu(a^2+b^2+c^2)\geq 2r\right) &= \text{Prob}\left(a^2+b^2+c^2\equiv 0 \bmod 4^r\right)+\\
&\quad \text{Prob}\left(a^2+b^2+c^2\equiv 3\cdot 4^{r-1} \bmod 4^r\right)\\
&= \text{Prob}\left(a\equiv 0\ \&\ b\equiv 0\ \&\ c\equiv 0 \bmod 2^r\right)+\\
&\quad \text{Prob}\left(a\equiv 2^{r-1}\ \&\ b\equiv 2^{r-1}\ \&\ c\equiv 2^{r-1} \bmod 2^r\right)\\
&= \left(\frac{1}{2^r}\right)^3+\left(\frac{1}{2^r}\right)^3=\frac{1}{2^{3r-1}}
\end{aligned}$$

In both the cases, Lemma 6.6.6 implies the first step and Lemma 6.3.4 implies the second step. □

6.7 Combination of prime $p = 2$ and sums of four squares

From Proposition 6.6.1 and the remark 6.4.2, the set $\{x^2+n : x \in \mathbb{Z}\}$ has no numbers divisible by large enough powers of 2 if and only if $n = a^2 + b^2 + c^2$. Moreover, from Proposition 6.5.1, if $n = a^2 + b^2$ then the set $\{x^2 + y^2 + n : x, y \in \mathbb{Z}\}$ has no numbers divisible by large enough powers of 2. Similarly, from Proposition 6.4.1, if then $n = a^2$ the set $\{x^2+y^2+z^2+n : x, y, z \in \mathbb{Z}\}$ has no numbers divisible by large enough powers of 2. However, the converse fails to hold true in the second and third cases unlike the first one. This means that there are sets of the kinds $\{x^2 + y^2 + n : x, y \in \mathbb{Z}\}$ and $\{x^2 + y^2 + z^2 + n : x, y, z \in \mathbb{Z}\}$ that do not come from sums of four squares but still display the phenomena occuring with the sums of squares. Nevertheless, it is still striking that the sets coming from sums of four squares fixing one or more squares always follows the property - that it has no numbers divisible by large enough powers of 2.

For the sums of five or more squares, even if all except one of the squares are fixed, it is evident that the sums of fixed squares in certain cases would add upto numbers of the form $4^r(8k+7)$ in light of remark 6.4.2. From Lemma 6.6.1, it follows that in such cases, the set so obtained will contain numbers divisible by all powers of 2. However, depending on the constant obtained by the sums of fixed squares, there will also be sets that have no number divisible by high enough powers of 2. For example, the set $\{1+1+1+4+n^2 : n \in \mathbb{Z}\}$ contains numbers divisible by all powers of 2 whereas no numbers in the set $\{1+0+0+n^2 : n \in \mathbb{Z}\}$ is divisible by 4. Similarly, for the odd primes and the sums of two or more squares some of which are fixed, the sets obtained may or may not have numbers divisible by all powers of prime p. For example, the set $\{1+1+0+n^2 : n \in \mathbb{Z}\}$ contains numbers divisible by all powers of 3 whereas all numbers in the set $\{1+0+0+n^2 : n \in \mathbb{Z}\}$ are indivisible by 3.

Both squares and triangular numbers are polygonal numbers given by quadratic formula. The equivalent of Lagrange's four square theorem is Guass's Eureka Theorem 6.6.5 that says every number is the sum of 3 triangular numbers. Unlike the sums of four squares, the subsets obtained by taking sums of three triangular numbers with one or more of the triangular number fixed always contains numbers divisible by all powers of 2 as evident from Lemma 6.5.4. In general, any number can be written as sum of s s-gonal numbers and only the sum of four squares is unique among the polygonal numbers having this property.

Chapter 7

2-adic valuations of the translations of sequences of polygonal numbers

7.1 Introduction

The numbers that are squares considered as a sequence $s_n = \{n^2 : n \in \mathbb{Z}\}$ are spread throughout the number line getting sparser as $n \to \infty$. On the other hand, for any fixed power of 2, say 4, the multiples of 4 are distributed evenly in the number line generating a sequence given by $4n$. The two sequences intersect at regular intervals even as the gap in the sequence of square grows wider as $n \to \infty$. On the contrary the similarly spaced sequence given by $n^2 + 1$ contains no numbers divisible by 4. In fact, the sequence $n^2 + 1$ is not the only example, the sequence $n^2 + 2$ do not contain any number divisible by 4 either. The result 7.2.2 below shows that much more often than not, when the sequence of squares is shifted uniformly by adding a fixed but arbitrary integer to each term, the resulting sequence contains no numbers divisible by high enough power of 2.

On the other hand the sequence of triangular numbers, that are spread in the number line in a manner similar to squares, considered as a sequence $t_n = \{n(n+$

$1)/2 : n \in \mathbb{Z}\}$ contains numbers divisible by all powers of 2. However unlike squares, when shifted uniformly by adding an integer k the resulting sequence always contains numbers divisible by all powers of 2 regardless of the integer k. For both the sequences of the squares and the triangular numbers, the gap between consecutive terms, that is $2n+1$ and $n+1$ respectively, increases linearly with n. Unlike the widening gap between the terms in these two sequences, the sequence of multiples of a fixed power of 2 have a constant difference between consecutive terms, though this difference is larger for higher powers of 2. The results stated below presents an interesting insight in the way the squares and triangular numbers are distributed in the number line and how their distribution intersect with that of fixed but arbitrary powers of 2.

Both squares and triangular numbers are examples of figurate numbers, also known as polygonal or s-gonal numbers, given by a quadratic formula $P_s(n) := \left(n^2(s-2) - n(s-4)\right)/2$. By the very definition it follows that the sequence of s-gonal numbers contains numbers divisible by all powers of 2 for any s, as shown in Theorem 7.2.1. The sequences obtained by shifting the s-gonal numbers by the addition of a number k may or may not have numbers divisible by all powers of 2 depending the combination of s and k. Except for the truly unique case of the squares, that is $s = 4$, all other s-gonal numbers can be categorized into two classes depending on whether $s \equiv 0 \pmod 4$ or not. For the later case that includes triangular numbers, the sequences resulting from adding integer k always have numbers divisible by all powers of 2 regardless of the value of k. For the former case, the sequence obtained by uniformly adding integer k contains terms divisible by all powers of 2 if and only if k is of the form $4^m(8l+7)$. For the squares, similar to the case of $s \equiv 0 \pmod 4$ the behavior of the resulting sequence depends on k. As seen previously, the sequence $n^2 + k$ contains terms divisible by all powers of 2 if and only if $k \equiv 0 \pmod{2^{2u-1}}$.

7.2 Main results

Definition. The s-gonal number is defined as:

$$P_s(n) := \frac{n^2(s-2) - n(s-4)}{2}.$$

Theorem 7.2.1. *The sequence of s-gonal numbers always contains numbers divisible by all powers of 2, that is $\nu_2(P_s(n)) \to \infty$, for any s.*

Proof.

$$\begin{aligned}\nu_2(P_s(n)) &= \nu_2\left(\frac{n^2(s-2) - n(s-4)}{2}\right) \\ &= \nu_2(n) + \nu_2\left(n(s-2) - (s-4)\right) - 1\end{aligned}$$

Since $\nu_2(n) \to \infty$ as $n \to \infty$, we have $\nu_2(P_s(n)) \to \infty$. □

Theorem 7.2.2. *Let k be a non-zero integer.*

1. *For $s = 4$, $\nu_2(P_s(n) + k) \to \infty \iff k$ is of the form $4^m(8l + 7)$. In other words, the sequence obtained by adding an integer k to s-gonal numbers (that is squares) have numbers divisible by all powers of 2 if and only if k is of the form $4^m(8l + 7)$.*

2. *For $s \not\equiv 0 \pmod 4$, $\nu_2(P_s(n) + k) \to \infty$ for any k. In other words, the sequence obtained by adding an integer k to s-gonal numbers have numbers divisible by all powers of 2 for any k.*

3. *For $s \neq 4$ and $s \equiv 0 \pmod 4$, let $\nu_2(s-4) = u$, then $\nu_2(P_s(n) + k) \to \infty \iff k \equiv 0 \pmod{2^{2u-1}}$. In other words, the sequence obtained by adding an integer k to s-gonal numbers have numbers divisible by all powers of 2 if and only if $k \equiv 0 \pmod{2^{2u-1}}$.*

The first case of Theorem 7.2.2 follows directly from Theorem 6.3.1. To prove the other two cases, we need the following lemmas:

Lemma 7.2.3. *For fixed but arbitrary $s \geq 2$ and $k \in \mathbb{Z}$, let $a = 8(s-2)k-(s-4)^2$, then $\nu_2(P_s(n)+k) \to \infty$ if and only if $\nu_2(t^2+a) \to \infty$ where t is restricted to take only the values that satisfy $t \equiv s \pmod{2(s-2)}$.*

Proof of Lemma 7.2.3. For $s = 2$, we have $P_2(n) = n$ and $a = -4$. Clearly $\nu_2(n+k) \to \infty$ for every k as well as $\nu_2(t^2-4) = \nu(t-2)+\nu(t+2) \to \infty$. Now for $s > 2$,

$$\begin{aligned}
P_s(n) &= \frac{n^2(s-2)-n(s-4)}{2}+k \\
&= \frac{n^2(s-2)-n(s-4)+2k}{2} \\
&= \frac{4n^2(s-2)^2-4n(s-2)(s-4)+8k(s-2)}{8(s-2)} \\
&= \frac{(2n(s-2)-(s-4))^2+8k(s-2)-(s-4)^2}{8(s-2)} \\
&= \frac{t^2+8(s-2)k-(s-4)^2}{8(s-2)} \text{ where } t = 2n(s-2)-(s-4)
\end{aligned}$$

Let $a = 8(s-2)k-(s-4)^2$, then

$$\nu_2(P_s(n)) = \nu_2\left(t^2+a\right) - \nu_2(s-2) - 3$$

The condition $t = 2n(s-2)-(s-4)$ as n varies over $\mathbb{N}$ can be given by $t \equiv s \pmod{2(s-2)}$. Hence $\nu_2(P_s(n)+k) \to \infty$ if and only if $\nu_2(t^2+a) \to \infty$. □

From Theorem 6.3.1 we know that $\nu_2(t^2+a) \to \infty$ if and only if a is of the form $4^m(8l+7)$ when t takes all values in $\mathbb{Z}$. Hence even if a is in the desired form $4^m(8l+7)$, the restriction on t given in Lemma 7.2.3 needs to be checked to conclude $\nu_2(t^2+a) \to \infty$. For example when $s = 5$, then $a = 24k-1 = 8(3k-1)+7$ and the restriction on t becomes $t \equiv 5 \pmod 6$. So to confirm whether $\nu_2(t^2 + a) \to \infty$ or not, the congruence $t \equiv 5 \pmod 6$ needs to be taken into account. For some cases of s, it suffices to use Chinese remainder theorem whereas for the cases that needs more consideration, the following lemmas are used.

Lemma 7.2.4. *For an arbitrary but fixed integer l, let $u = 3 + \nu_2(l + 1)$. When t is restricted to $t \equiv 2^{u-1} + 1 \pmod{2^u}$, it follows that $\nu_2(t^2 + (8l + 7)) \to \infty$, that is, the sequence $\{t^2 + (8l + 7)|\ t \equiv 2^{u-1} + 1 \pmod{2^u}\}$ have numbers divisible by any power of 2.*

Proof. From Theorem 6.3.1, we know that $\nu_2(t^2 + (8l + 7)) \to \infty$ when t takes all values in $\mathbb{Z}$. The condition $\nu_2(t^2 + (8l + 7)) \to \infty$ is equivalent to the existence of a root of the quadratic $t^2 + (8l + 7)$ in the 2-adic field $\mathbb{F}_2$. There are atmost two roots of the quadratic $t^2 + (8l + 7)$ in $\mathbb{F}_2$ and the modular class given by $t \equiv d \pmod{2^u}$ corresponds to a root of $t^2 + (8l + 7)$ in $\mathbb{F}_2$ if and only if $t^2 + (8l + 7) \equiv 0 \pmod{2^{u+1}}$. Hence, to prove that $\nu_2(t^2 + (8l + 7)) \to \infty$ when t is restricted to $t \equiv 2^{u-1} + 1 \pmod{2^u}$ it suffices to show that $t^2 + (8l + 7) \equiv 0 \pmod{2^{u+1}}$ for the modular class given by $t \equiv 2^{u-1} + 1 \pmod{2^u}$. Now

$$
\begin{aligned}
t \equiv 2^{u-1} + 1 \pmod{2^u} &\implies t = 2^u n + 2^{u-1} + 1 \text{ for some integer } n \\
&\implies t = 2^{u-1}(2n + 1) + 1 \\
&\implies t^2 = 2^{2u-2}(2n + 1)^2 + 2^u(2n + 1) + 1 \\
&\implies t^2 + (8l + 7) \equiv 2^u + 8(l + 1) \pmod{2^{u+1}} \text{ since } u \geq 3 \\
&\implies t^2 + (8l + 7) \equiv 0 \pmod{2^{u+1}} \text{ since } \nu_2(l + 1) = u - 3
\end{aligned}
$$

Hence proved. □

Lemma 7.2.5. *For an arbitrary but fixed integer l. When t is restricted to $t \equiv 3 \pmod{2^2}$, it follows that $\nu_2(t^2 + (8l + 7)) \to \infty$.*

Lemma 7.2.6. *The set of congruences*

$$t \equiv a \pmod{2^c}$$
$$t \equiv a \pmod{2b+1}$$

have a solution if and only if the congruence

$$t \equiv a \pmod{2^c(2b+1)}$$

has a solution.

Proof. Since $gcd(2^u, 2b+1) = 1$, it follows from the Chinese remainder theorem given below. Let $\gcd(m, n) = 1$, then

$$x \equiv a \pmod{m}$$
$$x \equiv b \pmod{n}$$

have a unique solution $\pmod{m\,n}$ □

Lemma 7.2.7. *If $\nu_2(t^2 + a) \to \infty$ when t is restricted to $t \equiv c \pmod{2^u}$, then $\nu_2(t^2 + a) \to \infty$ when t is restricted to $t \equiv c \pmod{2^u}$ and $t \equiv d \pmod{2b+1}$ for any numbers b and d.*

Proof. If $\nu_2(t^2+a) \to \infty$ for the modular class $t \equiv c \pmod{2^u}$, it means that for any k, there exists a c_k satisfyting $c_k \equiv c \pmod{2^u}$ such that $t \equiv c_k \pmod{2^{u+k}}$ and for this modular class, we have $\nu_2(t^2+a) \geq u+k+1$. From Lemma (7.2.6) (the Chinese remainder theorem), it follows that infinitely many of such t's also satisfy $t \equiv d \pmod{2b+1}$ for any numbers b and d. Thus for any k, we have $\nu_2(t^2+a) \geq u+k+1$ when t is restricted to $t \equiv c \pmod{2^u}$ and $t \equiv d \pmod{2b+1}$. Hence proved. □

Proof of Theorem 7.2.2. For $k \neq 0$.

1. For $s = 4$, $P_4(n) = n^2$. Hence it follows directly from Theorem 6.3.1 that $\nu_2(P_s(n) + k) \to \infty \iff k$ is of the form $4^m(8l + 7)$.

2. For $s \not\equiv 0 \pmod 4$, there are two cases:

   (a) To use Lemma 7.2.3 with $s = 2b+1$ and hence $a = 8k(2b-1)-(2b-3)^2 \equiv -1 \equiv 7 \pmod 8$, we need to first prove that $\nu_2(t^2 + a) \to \infty$ when t is restricted to $t \equiv s \pmod{2(s-2)}$, which is equivalent to the two modular congruences, viz. $t \equiv 1 \pmod 2$ and $t \equiv 2 \pmod{2b-1}$. Since a is of the form $8l + 7$, clearly $\nu_2(t^2 + a) \to \infty$ even with the restriction $t \equiv 1 \pmod 2$. From Lemma 7.2.7, it follows that $\nu_2(t^2 + a) \to \infty$ when t is restricted to $t \equiv 1 \pmod 2$ and $t \equiv 2 \pmod{2b-1}$. From Lemma 7.2.3, it follows that $\nu_2(P_s(n) + k) \to \infty$ for any k.

   (b) Similar as above for $s = 2^u(2b+1)+2$ with $u \geq 2$ so that $s-2 = 2^u(2b+1)$ and hence $a = 4[2^{u+1}(2b + 1)k - (2^{u-1}(2b + 1) - 1)^2]$, we first prove that $\nu_2(t^2 + a) \to \infty$ when t is restricted to $t \equiv s \pmod{2(s-2)}$. Since $(2^{u-1}(2b + 1) - 1)^2 \equiv 1 \pmod 8$ and $u \geq 2$, we have $a = 4(8l + 7)$ where l is given as below for two cases $u = 2$ and $u \geq 3$. For $u \geq 3$, we have $l = 2^{u-2}(2b+1)k-2^{2u-1}b^2(b+1)^2-2^{2u-1}(b(b+1)+1)+2^{u-2}b(b+1)+2^{u-3}-1$ and hence $\nu_2(l + 1) = u - 3$ for $u \geq 3$. On the other hand, for $u = 2$, we have $l = (2b + 1)k - b(2b + 1) - 1$.

   Now, let $t = 2t_1$, then $t^2+a = 4(t_1^2+(8l+7))$. The restriction on t viz. $t \equiv s \pmod{2(s-2)}$ is equivalent to $2t_1 \equiv s \pmod{2^{u+1}(2b+1)}$ which in turn is equivalent to the two modular congruences, viz. $2t_1 \equiv 2^u + 2 \pmod{2^{u+1}}$ and $2t_1 \equiv 2 \pmod{2b+1}$ that reduces to $t_1 \equiv 2^{u-1} + 1 \pmod{2^u}$ and $t_1 \equiv 1 \pmod{2b+1}$ respectively. We have $\nu_2(t_1^2 + (8l + 7)) \to \infty$ when t_1 is restricted to $t_1 \equiv 2^{u-1} + 1 \pmod{2^u}$ from Lemma 7.2.4 in case of $u \geq 3$ and from Lemma 7.2.5 for $u = 3$. Moreover, from Lemma 7.2.7, it follows that $\nu_2(t_1^2+(8l+7)) \to \infty$ when t_1 is restricted to $t_1 \equiv 2^{u-1}+1 \pmod{2^u}$ and $t_1 \equiv 1 \pmod{2b+1}$. This in turn implies that $\nu_2(t^2+a) \to \infty$ when t is restricted to $t \equiv s \pmod{2(s-2)}$. From Lemma 7.2.3, it follows that $\nu_2(P_s(n)+k) \to \infty$ for any k.

3. For $s \equiv 0 \pmod 4$, let $s = 2^u(2b+1)+4$, where $u \geq 2$ so that $s-2 = 2(2c+1)$ where $c = 2^{u-1}(2b+1)$. Then $a = 8(2(2c+1)k - 2^{2u}(2b+1)^2) = 16((2c+1)k - 2^{2u-4}(8m+1))$. For a to be of the form $4^m(8l+7)$, we must have $((2c+1)k - 2^{2u-4}(8m+1))$ in the form $4^m(8l+7)$ that is possible if and only if k has a factor of 2^{2u-1}, that $k \equiv 0 \pmod{2^{2u-1}}$. For $k = 2^{2u-1}k'$, we have $a = 4^u(8l+7)$ where $l = (2c+1)k' - (m+1)$. Let $t = 2^u t_1$, then $t^2+a = 4^u(t_1^2+(8l+7))$ and hence $\nu_2(t^2+a) \to \infty \iff \nu_2(t_1^2+(8l+7)) \to \infty$. The restriction on t, viz. $t \equiv s \pmod{2(s-2)}$ is equivalent to $2^u t_1 \equiv s \pmod{4(2c+1)}$ that reduces to $2^u t_1 \equiv 2 \pmod{2c+1}$ since $u \geq 2$ and further simplifies to $t_1 \equiv d \pmod{2c+1}$ where d is such that $2^u d \equiv 1 \pmod{2c+1}$. From Theorem 6.3.1 and Lemma 7.2.7, it follows that $\nu_2(t_1^2+(8l+7)) \to \infty$ when t_1 is restricted to $t_1 \equiv d \pmod{2c+1}$. This in turn implies that $\nu_2(t^2+a) \to \infty$ when t is restricted to $t \equiv s \pmod{2(s-2)}$. From Lemma 7.2.3, it follows that $\nu_2(P_s(n)+k) \to \infty$ for $k \equiv 0 \pmod{2^{2u-1}}$. From Theorem **??**, it is clear that $\nu_2(P_s(n)+k)$ is finite for $k \not\equiv 0 \pmod{2^{2u-1}}$. Hence proved.

□

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