ALGEBRAIC PROPERTIES OF SQUAREFREE MONOMIAL IDEALS

AN ABSTRACT
Submitted On The Fifteenth Day Of April, 2016
To The Department Of Mathematics
Of The Graduate School Of
Tulane University
In Partial Fulfillment Of The Requirements
For The Degree Of
Doctor Of Philosophy
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Abstract

The class of squarefree monomial ideals is a classical object in commutative algebra, which has a strong connection to combinatorics. Our main goal throughout this dissertation is to study the algebraic properties of squarefree monomial ideals using combinatorial structures and invariants of hypergraphs. We focus on the following algebraic properties and invariants: the persistence property, non-increasing depth property, Castelnuovo-Mumford regularity and projective dimension.

It has been believed for a long time that squarefree monomial ideals satisfy the persistence property and non-increasing depth property. In a recent work, Kaiser, Stehlik and Skrekovski provided a family of graphs and showed that the cover ideal of the smallest member of this family gives a counterexample to the persistence and non-increasing depth properties. We show that the cover ideals of all members of their family of graphs indeed fail to have the persistence and non-increasing depth properties.

Castelnuovo-Mumford regularity and projective dimension are both important invariants in commutative algebra and algebraic geometry that govern the computational complexity of ideals and modules. Our focus is on finding bounds for the regularity in terms of combinatorial data from associated hypergraphs. We provide two upper bounds for the edge ideal of any vertex decomposable graph in terms of induced matching number and the number of cycles. We then give an upper bound for the edge ideal of a special class of vertex decomposable hypergraphs. Moreover, we generalize a domination parameter from graphs to hypergraphs and use it to give an upper bound for the projective dimension of the edge ideal of any hypergraph.
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Acknowledgments

First and foremost I would like to express my sincere gratitude to my advisor Dr. Tài Huy Hà. I would not have finished this thesis without his guidance. He has taught me everything I know about scientific research, and his immense knowledge, patience and enthusiasm have always inspired me.

My sincere thanks also goes to my thesis committee member: Dr. Mahir Can, Dr. Slawomir Kwasik, Dr. Victor H. Moll and Dr. Albert Vitter, for their valuable time and suggestions.

I am grateful to my twin sister, Mengxiao Sun, for always being there for me as a best friend. Thank you for listening to my complaints and frustrations, giving me more self-confidence and believing me.

I am forever indebted to my parents for giving me the opportunities and experiences that have made me who I am. Thank you for your unconditional love that has been my greatest strength. This journey would not have been possible if not for them.

Finally, I would like to dedicate this thesis to my grandmother. I have been extremely fortunate in my life to have her who have shown me unconditional love and support.
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Chapter 1

Introduction

The class of squarefree monomial ideals is a classical object in commutative algebra, which has a strong connection to combinatorics. In the 1970s, the Stanley-Reisner ring construction provides an essential link between these two areas. The construction arises by identifying minimal generators of a squarefree monomial ideal with the minimal nonfaces of a simplicial complex. Stanley’s proof [1] of the Upper Bound Conjecture for simplicial spheres has been seen as one of the important results of exploiting the connection between two fields. Other celebrated results are Reisner’s criterion for Cohen-Macaulayness [2] and Hochster’s formula [3].

The edge ideal construction is another well-studied correspondence between commutative algebra and combinatorics, which identifies minimal generators of a squarefree monomial ideal with the edges of a simple hypergraph. The edge ideal construction was first introduced by Villarreal in [4] for graphs and later generalized to hypergraphs in [5]. The notion of edge ideals enables one to study the properties of a squarefree monomial ideal using the properties of the associated graph, and vice versa. Fröberg [6], Villarreal [4], and Simis, Vasconcelos, and Villarreal [7] were among the early pioneers in this field.

Moreover, any monomial ideal can be reduced to a squarefree monomial ideal using the technique “polarization”. This allows us to translate a problem about

Our main goal is to study the algebraic properties of squarefree monomial ideals using combinatorial structures and invariants of hypergraphs. Much recent work has been devoted to studying the algebraic and combinatorial properties of squarefree monomial ideals and hypergraphs. We refer the reader to [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 24]. In this thesis, we focus on the following algebraic properties and invariants: the persistence property, non-increasing depth property, Castelnuovo-Mumford regularity and projective dimension.

Associated prime ideals have long been known in commutative algebra for their key role in the theory of primary decomposition (see for details [22, Chapter 3]). Given an ideal $I$ in a commutative Noetherian ring $R$, the theory of primary decomposition enables one to write $I$ as an intersection of primary ideals that are closely connected with its associated prime ideals. While the existence of primary decompositions is assured by working with finitely generated modules over a noetherian ring, the uniqueness of such decompositions does not always exist. One of many important properties of the associated primes comes from the fact that the set of associated primes of a module is always unique. Moreover, if a primary decomposition of $I$ is irredundant, then the radicals of those primary ideals coincide with the associated prime ideals of $I$.

In algebraic geometry, an algebraic set can be decomposed as a union of irreducible algebraic sets. And algebraic sets correspond to radical ideals and irreducible algebraic sets correspond to prime ideals. According to these correspondences, the decomposition of algebraic sets into irreducible components is equivalent to writing a radical ideal as an intersection of prime ideals. Thus the primary decomposition
expresses the algebraic set as a union of the irreducible algebraic sets.

Depth, which is an algebraic notion parallel to the geometric notion of codimension, can be defined to be the maximal length of all \( M \)-regular sequences. As a geometric interpretation, the depth can be thought of as a measure of how much we can cut down things using hypersurfaces, where each intersection has to drop the dimension by 1. In [22, Chapter 17], the notion of depth has been studied with a homological tool called Koszul complex, and Theorem 17.4 in [22] characterizes the depth in terms of the vanishing of the homology of the Koszul complex.

Depth is also used to define special classes of rings and modules with nice properties, such as Cohen-Macaulay rings and modules. For a local noetherian ring, the Krull-dimension and depth of a module do not coincide in general. A module or a ring is Cohen-Macaulay if its Krull-dimension equals its depth. Macaulay’s Unmixedness Theorem[23] states that if a ring is Cohen-Macaulay then the hyperplane sections of the associated variety do not have embedded components.

A large body of research has also been concerned with certain sequences of associated prime ideals and depth functions. Let \( k \) be a field and let \( R = k[x_1, \ldots, x_n] \) be a polynomial ring over \( k \). Let \( I \subseteq R \) be a homogeneous ideal. The ideal \( I \) is said to have the persistence property if

\[
\text{Ass}(R/I^s) \subseteq \text{Ass}(R/I^{s+1}), \quad \forall s \geq 1.
\]

It is known by Brodmann [25] that the set of associated primes of \( I^s \) stabilizes for large \( s \). But the behavior of these sets can be very strange for small values of \( s \). The ideal \( I \) is said to have non-increasing depth if

\[
\text{depth}(R/I^s) \geq \text{depth}(R/I^{s+1}), \quad \forall s \geq 1.
\]
And it is also known by Brodmann [26] that depth($R/I^s$) takes a constant value for large $s$. The behavior of depth($R/I^s$), for small values of $s$, can also be very complicated.

Associated primes and depth of powers of ideals have been extensively investigated in the literature (cf. [15, 16, 18, 27, 28, 29, 30, 31, 33, 34, 35, 36]). In general, it is difficult to classify which ideals possess the persistence property or non-increasing depth, even for monomial ideals. Squarefree monomial ideals behave considerably better than monomial ideals in general, and many classes of squarefree monomial ideals were shown to have the persistence property.

In an attempt to tackle the persistence property, at least in identifying a large class of squarefree monomial ideals having the persistence property, Francisco, Hà and Van Tuyl in [28], made a graph-theoretic conjecture about expansion of critically $s$-chromatic graphs and proved that this conjecture implies the persistence property for the cover ideals of graphs. The conjecture is stated as follows.

**Conjecture 1.1.** ([28, Conjecture 1.1]) Let $G$ be a critically $s$-chromatic graph. Then there exists a subset $W$ of the vertices such that the expansion of $G$ at $W$ is critically $(s + 1)$-chromatic.

Recently, Kaiser, Stehlík and Škrekovski [45] gave a family of counterexamples to this graph-theoretic conjecture. Computational experiment showed that the first member of their family of graphs also gives a counterexample to the persistence property and non-increasing depth for squarefree monomial ideals.

In Chapter 3, we prove that all members of this family indeed give counterexamples to the persistence property. As a consequence, they also provide counterexamples to non-increasing depth property.
Theorem 1.2. Let $H_q$ be the graph constructed in Figure 1.1. Let $J = J(H_q)$ be the cover ideal of $H_q$ and let $m$ be the maximal homogeneous ideal in the polynomial ring $R = k[x_{i,j} | i = 1, \ldots, q, j = 0, 1, 2]$. Then $m \in \text{Ass}(R/J^3)$ and $m \notin \text{Ass}(R/J^4)$. As a consequence, $J$ fails to have non-increasing depth.

In Chapter 4, we study the Castelnuovo-Mumford regularity of squarefree monomial ideals. Castelnuovo Mumford regularity is an important invariant in commutative algebra and algebraic geometry that governs the computational complexity of ideals, modules and sheaves. It can be viewed as the width of the minimal resolution. In general, computing or finding bounds for the regularity is a difficult problem. Our focus will be on studies that find bounds and/or compute the regularity of squarefree monomial ideals in terms of combinatorial data from associated simplicial complexes and hypergraphs.

Let $H = (V, E)$ be a simple hypergraph with vertices set $V = \{x_1, \ldots, x_n\}$ and edge set $E$. The edge ideal of $H$ is defined to be

$$I(H) = (x^e \mid e \in E \text{ is an edge in } H).$$
For any simple hypergraph $H$, if $x$ is a vertex in $H$, then it was shown in [37, Lemma 2.10] that $\text{reg}(I(H))$ is always equal to either $\text{reg}(I : x) + 1$ or $\text{reg}(I, x)$. For a vertex decomposable hypergraph, it was proved in [38, Theorem 1.5] that $\text{reg}(I(H))$ is always equal to the larger one of $\text{reg}(I : x) + 1$ and $\text{reg}(I, x)$. With this inductive equality, it is natural to seek for combinatorial invariants that measure the regularity of these hypergraphs.

In [39], Khosh-Ahang and Moradi considered the class of vertex decomposable graphs which contain no cycles of length 5 and showed that $\text{reg}(I(G)) = \nu(G) + 1$, where $\nu(G)$ is the \textit{induced matching number}. The same result was proved by Biyikoglu and Civan in [40] for vertex decomposable graphs containing no induced 5-cycles and 4-cycles. Our goal is to find bounds for the regularity of any vertex decomposable graph in general. We provide the following upper bound in terms of induced matching number and the number of cycles.

**Theorem 1.3.** (Theorem 4.2.2) \textit{If $G$ is a vertex decomposable graph, then}

$$\text{reg}(I(G)) \leq \nu(G) + 1 + \min\{W_5(G), C_4(G) + C_5(G)\}$$

where $W_5(G)$ denotes the number of cycles of length 5 in $G$, $C_4(G)$ and $C_5(G)$ denote the number of induced 4-cycles and induced 5-cycles in $G$.

In [35, Corollary 3.9], it was proved that for any simple hypergraph $H$, the regularity of edge ideal is always bounded below by $\nu(H) + 1$. As a corollary, we are able to recover the results in [39] and [40].

In [24], Woodrooffe showed that a graph with no induced cycles of length other than 3 or 5 is vertex decomposable. We improve our bound for this class of graphs.

**Theorem 1.4.** (Theorem 4.2.17) \textit{If $G$ is a graph containing only $C_3$ or $C_5$, then}

$$\nu + 1 + c \leq \text{reg}(G) \leq \nu + 1 + C_5(G) - b,$$

where $c$ is the number of isolated $C_5$ and $b$
is the number of induced 5-cycles whose neighbors contains exactly one leaf.

We then consider vertex decomposable hypergraphs and give an upper bound on regularity of a special class of vertex decomposable hypergraphs.

**Definition 1.5.** Let $H$ be a simple hypergraph. An $n$-cycle in $H$ is defined to be $x_1E_1 \cdots x_nE_n$, where $x_i$s are distinct vertices, $E_i$s are distinct edges, and $x_1, x_n \in E_n$, $x_i, x_{i+1} \in E_i$ for $1 \leq n - 1$. We consider $x'_1E_1 \cdots x'_nE_n$ and $x_1E_1 \cdots x_nE_n$ as the same $n$-cycle.

Here we denote an $n$-cycle of a hypergraph by $W_n$.

**Definition 1.6.** Let $H$ be a simple hypergraph. We define $b(H)$ by

$$
b(H) = \max \{|\bigcup_{i=1}^k E_i| - k\mid \text{the induced subhypergraph of } H \text{ on } \bigcup_{i=1}^k E_i \text{ only contains } E_1, \ldots, E_k\}.
$$

**Theorem 1.7.** (Theorem 4.3.4) Suppose that $H$ is a vertex decomposable hypergraph and for any two edges $E_i$ and $E_j$, $|E_i \cap E_j| \leq 1$. Then

$$
\text{reg}(I(H)) \leq W_5(H) + b(H) + 1.
$$

Projective dimension is also an important invariant which measures the computational complexity of an ideal. It can be viewed as the measure of the length of a free resolution. It is a celebrated result of Terai [41] that the regularity of a squarefree monomial ideal can be related to the projective dimension of its Alexander dual. The main result in Chapter 5 provides an upper bound for projective dimension of any simple hypergraph.

In [42], Dao and Schweig provided two upper bounds on the projective dimensions of edge ideals of graphs in terms of domination parameters $\epsilon$ and $\tau$. 
Theorem 1.8. ([42, Corollary 4.2]) \(\text{Let } G \text{ be a graph on } n \text{ vertices. Then}
\[
    \text{pd}(G) \leq n - \max\{\epsilon(G), \tau(G)\}.
\]

In a later paper [43], they generalized one of the domination parameters \(\epsilon\) from graphs to hypergraphs. They showed that if \(\epsilon(H)\) is this new domination parameter of a simple hypergraph \(H\) and \(n\) is the number of vertices, then the following holds.

Theorem 1.9. ([43, Theorem 3.2]) \(\text{For any simple hypergraph } H \text{ on } n \text{ vertices,}
\[
    \text{pd}(H) \leq n - \epsilon(H).
\]

In Chapter 5, we generalized the other domination parameter \(\tau\) from graphs to hypergraphs. We show that if \(\alpha(H)\) is this new domination parameter of a hypergraph \(H\), then the following holds.

Theorem 1.10. (Theorem 5.2.5) \(\text{If } H \text{ is a simple hypergraph, then}
\[
    \text{pd}(H) \leq n - \alpha(H),
\]

where \(n\) is the number of vertices of \(H\).
Chapter 2

Preliminaries

2.1 Graphs and Hypergraphs

In this section, we recall basic definitions from graph theory.

Definition 2.1.1. A finite graph $G$ is a pair $G = (V(G), E(G))$ where $V(G) = \{x_1, \ldots, x_n\}$ is the set of vertices of $G$, and $E(G)$ is the set of edges of $G$, which is a collection of 2-element subsets of $V(G)$. A finite graph is simple if we do not allow multiple edges between vertices and we do not allow loops at vertices.

Example 2.1.2. Let $G = (V(G), E(G))$, where $V(G) = \{x_1, \ldots, x_6\}$ and $E(G) = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_6\}, \{x_6, x_1\}\}$. Then $G$ is a finite simple graph.

Figure 2.1. A finite simple graph.
Example 2.1.3. The figure below is not a simple graph. It has a loop at the vertex $x_1$ and it has two edges from $x_5$ to $x_6$.

![Figure 2.2](image)

Figure 2.2. A graph that is not simple.

Definition 2.1.4. Let $G$ be a graph.

(1) A cycle is a closed walk with all vertices (and hence all edges) distinct (except the first and last vertices). A cycle consisting of $n$ distinct vertices is called an $n$-cycle.

(2) A chordless cycle of length $n$ is an induced $n$-cycle.

Throughout the thesis, we denote an $n$-cycle by $W_n$ and an induced $n$-cycle by $C_n$.

Example 2.1.5. The graph shown in Figure 2.3 is a 6-cycle, but not an induced 6-cycle.

![Figure 2.3](image)

Figure 2.3. A 6-cycle.
Example 2.1.6. The graph in Figure 2.1 is an induced 6-cycle.

Definition 2.1.7. Let $G = (V, E)$ be a graph, and let $S \subseteq V$ be a subset of vertices of $G$. Then the induced subgraph of $G$ on $S$ is the graph whose vertex set is $S$ and whose edge set consists of all of the edges in $E$ that have both endpoints in $S$.

Example 2.1.8. Consider the graph $G$ in Figure 2.3. The induced subgraph of $G$ on $\{x_1, x_2, x_3, x_4\}$ is the graph in Figure 2.4.

![Graph](image)

Figure 2.4. The induced subgraph of $G$ on $\{x_1, x_2, x_3, x_4\}$.

The construction of hypergraphs is a generalization of graphs. We can think of graphs as a special class of hypergraphs. In fact, a graph is a hypergraph for which every edge has cardinality two.

Definition 2.1.9. Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let $E = \{E_1, \ldots, E_m\}$ be a collection of distinct subsets of $X$. The pair $H = (X, E)$ is a hypergraph if $E_i \neq \emptyset$ for each $i$. The elements of $X$ are called the vertices, and the elements of $E$ are called the edges of $H$. A hypergraph $H$ is simple if $H$ has no loops, i.e., $|e| \geq 2$ for all $e \in E$, and none of its edges is contained within another.

Example 2.1.10. Let $H = \{\{x_1, x_2, x_3\}, \{x_2, x_4, x_5\}, \{x_3, x_5, x_6\}\}$. Then $H$ is a simple hypergraph.
Throughout the thesis, we only consider finite simple graphs and hypergraphs, so we will simply call $G$ a graph and $H$ a hypergraph.

### 2.2 Edge Ideals and Cover Ideals

In this section, we introduce edge and cover ideals of hypergraphs. We illustrate how a one-to-one correspondence can be built between squarefree monomial ideals and hypergraphs.

We use the graph $G$ to construct two squarefree monomial ideals.

**Definition 2.2.1.** Let $G$ be a simple graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and the edge set $E(G)$. The edge ideal of $G$ is defined by

\[ I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E \rangle \subseteq R = k[x_1, \ldots, x_n]. \]

The cover ideal of $G$ is defined by

\[ J(G) = \bigcap_{\{x_i, x_j\} \in E} \langle x_i, x_j \rangle \subseteq R = k[x_1, \ldots, x_n]. \]
\textbf{Example 2.2.2.} Let $G$ be the graph in Example 2.1.1. Then the edge ideal and cover ideal of $G$ are

\[ I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1 \rangle \]
\[ J(G) = \langle x_1, x_2 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_3, x_4 \rangle \cap \langle x_4, x_5 \rangle \cap \langle x_5, x_6 \rangle \cap \langle x_6, x_1 \rangle \]

Observe that we can reverse the construction of Definition 2.1.3 in the following sense. If $I$ is any quadratic squarefree monomial ideal in $R = k[x_1, \ldots, x_n]$ of the form $I = \langle x_{1,1}x_{1,2}, \ldots, x_{s,1}x_{s,2} \rangle$ we can construct a simple graph $G$ whose edge ideal is $I$.

\[ I \rightarrow G = (\{x_1, \ldots, x_n\}, \{\{x_{1,1}, x_{1,2}\}, \ldots, \{x_{s,1}, x_{s,2}\}\}). \]

If $J$ is a squarefree unmixed height two monomial ideal, say $J = \cap_{i=1}^s \langle x_{i,1}, x_{i,2} \rangle$, we can construct a graph $G$ whose cover ideal is $J$.

\[ J \rightarrow G = (\{x_1, \ldots, x_n\}, \{\{x_{i,1}, x_{i,2}\} | i = 1, \ldots, s\}). \]

The edge ideal construction gives a one-to-one correspondence between simple graphs and quadratic squarefree monomial ideals. And the cover ideal construction gives a one-to-one correspondence between simple graphs and unmixed, codimension two squarefree monomial ideals.

The ideal $J(G)$ is called the cover ideal because its minimal generators correspond to the minimal vertex covers of the graph $G$. To see this, let’s first recall the definition of a \textit{vertex cover}.

\textbf{Definition 2.2.3.} Let $G = (V(G), E(G))$. A subset $S \subseteq V(G)$ is a \textit{vertex cover} if $S \cap e \neq \emptyset$ for all $e \in E(G)$. A vertex cover $W$ is a \textit{minimal vertex cover} if no proper subset of $S$ is a vertex cover.
Example 2.2.4. Consider the graph $G$ in Example 2.1.2. $S_1 = \{x_1, x_2, x_3, x_4, x_5\}$ is a vertex cover of $G$ but not a minimal vertex cover. $S_2 = \{x_1, x_3, x_5\}$ is a minimal vertex cover.

![Graph](image)

Figure 2.6. A finite simple graph with a minimal vertex cover $\{x_1, x_3, x_5\}$.

We can prove the following lemma which justifies the name cover ideal directly from the definitions.

Lemma 2.2.5. Let $G$ be a finite simple graph with cover ideal $J(G)$. Then

$$J(G) = \langle x_S \mid S \subseteq V(G) \text{ is a minimal vertex cover of } G \rangle.$$ 

Example 2.2.6. We still consider the graph $G$ in Example 2.1.2. The minimal vertex covers of $G$ are $\{\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}\}$. The cover ideal of $G$ is $J(G) = \langle x_1, x_2 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_3, x_4 \rangle \cap \langle x_4, x_5 \rangle \cap \langle x_5, x_6 \rangle \cap \langle x_6, x_1 \rangle = \langle x_1 x_3 x_5, x_2 x_4 x_6 \rangle$.

The definitions of edge and cover ideals of a graph can be extended to a hypergraph.

Definition 2.2.7. Let $H$ be a hypergraph with the vertex set $V = \{x_1, \ldots, x_n\}$ and
edge set. The edge ideal of $H$ is defined by

$$I(H) = \left\langle \prod_{x \in e} x \mid e \in E \right\rangle \subseteq R = k[x_1, \ldots, x_n].$$

The cover ideal of $H$ is defined by

$$J(H) = \bigcap_{e \in E} \langle x \mid x \in e \rangle \subseteq R = k[x_1, \ldots, x_n].$$

**Example 2.2.8.** Let $H$ be the hypergraph in Example 2.1.4. Then the edge ideal and cover ideal of $H$ are

$$I(G) = \langle x_1x_2x_3, x_2x_4x_5, x_3x_5x_6 \rangle$$

$$J(G) = \langle x_1, x_2, x_3 \rangle \cap \langle x_2, x_4, x_5 \rangle \cap \langle x_3, x_5, x_6 \rangle.$$

This construction gives a one-to-one correspondence between squarefree monomial ideals and hypergraphs, which allows us to study algebraic properties of squarefree monomial ideals in terms of combinatorial data from associated hypergraphs.

### 2.3 Vertex Decomposable Hypergraphs

In this section, we introduce a special class of hypergraphs, which are called vertex decomposable hypergraphs. We will study the regularity of this class of hypergraphs in Chapter 4.

**Definition 2.3.1.** A simplicial complex $\Delta$ is called vertex decomposable if $\Delta$ is a simplex, or $\Delta$ contains a vertex $x$ such that

(1) both $\text{del}_\Delta(x)$ and $\text{link}_\Delta(x)$ are vertex decomposable, and

(2) any facet of $\text{del}_\Delta(x)$ is a facet of $\Delta$.

A vertex $x$ that satisfies condition (2) is called a shedding vertex of $\Delta$. 
Example 2.3.2. The simplicial complex as shown below is *vertex decomposable* with \( \{x_5, x_4\} \) as a shedding order.

![Figure 2.7. A vertex decomposable simplicial complex.](image)

Definition 2.3.3. Let \( H \) be a simple hypergraph. A collection of vertices \( X \) of \( V(H) \) is called an *independent set* if there is no edge \( e \in E(H) \) such that \( e \subseteq X \). The *independence complex* of \( H \) is the simplicial complex whose faces are independent sets in \( G \).

Definition 2.3.4. A hypergraph \( H \) is called *vertex decomposable* if its independence complex is vertex decomposable.

Example 2.3.5. The simplicial complex in Figure 2.4 is the independence complex of the graph in Figure 2.5. So the graph in Figure 2.5 is *vertex decomposable*.

![Figure 2.8. A graph whose independence complex is Figure 2.4.](image)

Also, the notion of independence complex relates the edge ideal and Stanley-Reisner ideal. Let’s recall the definition of Stanley-Reisner ideal first.
Definition 2.3.6. Let $\Delta$ be a simplicial complex on $X$. The Stanley-Reisner ideal of $\Delta$ is defined to be

$$I_\Delta = \langle x_{i_1} \cdots x_{i_t} \mid \{x_{i_1}, \ldots, x_{i_t}\} \subseteq X \text{ is not a face of } \Delta \rangle.$$

The notion of Stanley-Reisner ideal also gives a one-to-one correspondence between squarefree monomial ideals and simplicial complexes, which enables us to move back and forth from squarefree monomial ideals to simplicial complexes and simple hypergraphs.

The following lemma can be proved directly from definitions.

Lemma 2.3.7. Let $H$ be a simple hypergraph and let $\Delta = \Delta(H)$ be its independence complex. Then

$$I_\Delta = I(H).$$

Example 2.3.8. The edge ideal of the graph in Figure 2.5 is

$$I(G) = \langle x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_5, x_4x_5 \rangle.$$

The Stanley-Reisner ideal of the simplicial complex $\Delta$ in Figure 2.4 is

$$I_\Delta = \langle x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_5, x_4x_5 \rangle.$$

Let $\overline{F}$ denote the simplex over the vertices of $F$. Recall that a simplicial complex $\Delta$ is said to be shellable if there exists a linear order of its facets $F_1, \ldots, F_t$ such that for all $k = 2, \ldots, t$, the subcomplex $(\bigcup_{i=1}^{k-1} \overline{F_i}) \cap \overline{F_k}$ is pure and of dimension $(\dim F_k - 1)$. A ring or module is said to be sequentially Cohen-Macaulay if it has a filtration in which the factors are Cohen-Macaulay and their dimensions are increasing.
We know that shellability of a simplicial complex implies sequentially Cohen-Macaulayness and if a simplicial complex is vertex decomposable then it is shellable. So vertex decomposability can be thought of as a combinatorial criterion for shellability and sequentially Cohen-Macaulayness.

2.4 Associated Primes, Depth and Persistence Property

Definition 2.4.1. Let $I \subseteq R = k[x_1, \ldots, x_n]$ be an ideal. A prime ideal $P$ is said to be an associated prime of $I$ if there exists an element $x \in R$ such that $P = I : x$. The set of all associated primes of $I$ is denoted by Ass$(R/I)$.

In the case that $I$ is a monomial ideal, all the associated prime ideals must also be monomial. Furthermore, the only prime monomial ideals are those of the form $(x_{i_1}, \ldots, x_{i_r})$.

Example 2.4.2. Let $G$ be a graph with the edge ideal $I(G)$ and cover ideal $J(G)$.

\[
\text{Ass}(R/I(G)) = \{ (x_{i_1}, \ldots, x_{i_r}) \mid \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a minimal vertex cover} \} \\
\text{Ass}(R/J(G)) = \{ (x_i, x_j) \mid \{x_i, x_j\} \in E(G) \}.
\]

Definition 2.4.3. Let $M$ be a finitely generated $R$-module.

1. A sequence of elements $x_1, \ldots, x_t \in R$ is called an $M$-regular sequence if $M \neq (x_1, \ldots, x_t)M$ and $x_i$ is not a zero-divisor of $M/(x_1, \ldots, x_{i-1})M$ for all $i = 1, \ldots, t$.

2. The depth of $M$, denoted by depth$(M)$, is the largest length of an $M$-regular sequence in $R$.

Remark 2.4.4. It is easy to see that for an ideal $I \subseteq R$, depth$(R/I) > 0$ if and only if $m \notin \text{Ass}(R/I)$.
We recall the definitions of persistence property and non-increasing depth.

**Definition 2.4.5.** Let $I \subseteq R = k[x_1, \ldots, x_n]$ be an ideal.

(1) The ideal $I$ is said to have the *persistence property* if

$$\text{Ass}(R/I^s) \subset \text{Ass}(R/I^{s+1}) \quad \forall s \geq 1.$$ 

(2) The ideal $I$ is said to have *non-increasing depth* if

$$\text{depth}(R/I^s) \geq \text{depth}(R/I^{s+1}) \quad \forall s \geq 1.$$ 

**Remark 2.4.6.** If $I$ is a monomial ideal, then the two properties are related by the fact that $I$ possesses the persistence property if all monomial localizations of $I$ have non-increasing depth.

## 2.5 Castelnuovo-Mumford Regularity

Castelnuovo-Mumford regularity is an important invariant in commutative algebra and algebraic geometry that measures the computational complexity of ideals, modules and sheaves. The regularity of graded modules over the polynomial ring $R$ can be defined in various ways. Let $\mathfrak{m}$ denote the maximal homogeneous ideal in $R$.

**Definition 2.5.1.** Let $M$ be a finitely generated $R$-module. For $i \geq 0$, let

$$a^i(M) = \begin{cases} 
\max \{ l \in \mathbb{Z} \mid [H^i_{\mathfrak{m}}(M)]_l \neq 0 \} & \text{if } H^i_{\mathfrak{m}}(M) \neq 0 \\
-\infty & \text{otherwise.}
\end{cases}$$

The *regularity* of $M$ is defined to be

$$\text{reg}(M) = \max_{i \geq 0} \{ a^i(M) + i \}.$$
Castelnuovo-Mumford regularity can also be computed via the minimal free resolution.

**Definition 2.5.2.** Let \( M \) be a graded \( R \)-module and let

\[
0 \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{ij}(M)} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{ij}(M)} \to M \to 0
\]

be its minimal free resolution. Then the *regularity* of \( M \) is given by

\[
\text{reg}(M) = \max\{ j - i \mid \beta_{ij}(M) \neq 0 \}.
\]

**Remark 2.5.3.** From Definition 2.4.2, it is easy to see that \( \text{reg}(R/I) = \text{reg}(I) - 1 \), so we can work with \( \text{reg}(I) \) and \( \text{reg}(R/I) \) interchangeably.

**Remark 2.5.4.** For convenience, if \( I = I(H) \) is the edge ideal of \( H \), then we sometimes write \( \text{reg}(H) \) for \( \text{reg}(I) \).

### 2.6 Projective Dimension

Projective dimension is another important invariant that measures the complexity of a module because it measures the size of the minimal free resolution of the module.

**Definition 2.6.1.** Let \( M \) be a finitely generated \( R \)-module. The *projective dimension* of \( M \), denoted by \( \text{pd}(M) \), is the length of the minimal free resolution associated to \( M \), that is,

\[
\text{pd}(M) = \max\{ i \mid \beta_{ij}(M) \neq 0 \text{ for some } j \}.
\]

The regularity of a squarefree monomial ideal can be related to the projective dimension via its *Alexander dual*. Let’s first recall the definition of Alexander duality for a simplicial complex.
**Definition 2.6.2.** Let $\Delta$ be a simplicial complex with the vertex set $X$. The *Alexander dual* of $\Delta$, denoted by $\Delta^\vee$, is the simplicial complex over $X$ with faces

$$\{X \setminus F \mid F \notin \Delta\}.$$

If $I$ is the Stanley-Reisner ideal of a simplicial complex $\Delta$, then we denote by $I^\vee$ the Stanley-Reisner ideal of $\Delta^\vee$. If $I(H)$ is the edge ideal of a hypergraph $H$, then we denote by $H^\vee$ the hypergraph associated to $I^\vee$.

The following result due to Terai states that the regularity of a squarefree monomial ideal can be related to the projective dimension of its *Alexander dual*.

**Theorem 2.6.3.** ([41, Corollary 0.3]) Let $I \subseteq R$ be a squarefree monomial ideal. Then

$$\text{reg}(I) = \text{pd}(R/I^\vee).$$

This theorem tells us that studying the regularity of squarefree monomial ideals is equivalent to studying the projective dimension of squarefree monomial ideals.
Chapter 3

Associated Primes and Depth

Let $k$ be a field and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over $k$. Let $I \subseteq R$ be a homogeneous ideal. It is known by Brodmann([25]) that the set of associated primes of $I^s$ stabilizes for large $s$, that is,

$$\text{Ass}(R/I^s) = \text{Ass}(R/I^{s+1}) \quad \text{for all } s \gg 0.$$ 

However, the behavior of these sets can be very strange for small values of $s$. It is also known by Brodmann([26]) that depth($R/I^s$) takes a constant value for large $s$. The behavior of depth($R/I^s$), for small values of $s$, can also be very complicated. In general, it is difficult to classify which ideals possess the persistence property or non-increasing depth.

**Example 3.1.** Let $G$ be a graph in Figure 3.1. Let $R = k[x_1, \ldots, x_5]$ and

$$I = I(G) = (x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5).$$

Then

$$\text{Ass} \left( R/I^s \right) \subseteq \text{Ass} \left( R/I^{s+1} \right) \quad \text{for all } s \geq 1.$$
Figure 3.1. A graph $G$ whose edge ideal has the persistence property.

The following monomial ideal constructed by Bandari, Herzog and Hibi in [27] fails to have the persistence property and non-increasing depth function.

**Example 3.2.** ([27, Theorem 0.1])

Let $n \geq 0$ be any integer and $R = k[a, b, c, d, x_1, y_1, \ldots, x_n, y_n]$. Let

\[ I = (a^6, a^5b, ab^5, b^6, a^4b^4c, a^4b^4d, a^4x_1y_1^2, b^4x_1y_1, \ldots, a^4x_ny_n^2, b^4x_ny_n) \]

Then

\[ P = (a, b, c, d, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n) \in \text{Ass} \left( R/I^k \right) \]

for $k$ even with $k \leq 2n$, but

\[ P = (a, b, c, d, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n) \notin \text{Ass} \left( R/I^k \right) \]

for $k$ odd with $k \leq 2n + 1$.

Squarefree monomial ideals behave considerably better than monomial ideals in general, and many classes of squarefree monomial ideals were shown to have the persistence property. For instance, edge ideals of graphs ([33]), cover ideals of perfect graphs, cover ideals of cliques, odd holes and odd antiholes ([28]), and polymatroidal
ideals ([31]). It is natural to ask whether all squarefree monomial ideals have the persistence property (see [28] and [44, Question 3.28]).

We have seen in Remark 2.4.6, a monomial ideal $I$ satisfies the persistence property if all monomial localizations of $I$ have a non-increasing depth function. Herzog and Hibi in [29] noted that the depth function of most monomial ideals is non-increasing and in [27] they asked whether all squarefree monomial ideals have a non-increasing depth function. A large class of squarefree monomial ideals with constant depth was constructed in [32]. And according to [27], a positive answer was expected.

### 3.1 A Graph-theoretic Conjecture

In order to identify a large class of squarefree monomial ideals having the persistence property, Francisco, Hà, and Van Tuyl in [28] made a graph-theoretic conjecture about the coloring of critical graphs and proved that this conjecture implies the persistence property for the cover ideals of graphs. Before stating our conjecture, we recall the following definitions.

**Definition 3.1.1.** Let $G$ be a graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E$. The graph $G$ has a $s$-coloring if there exists a partition $V = C_1 \cup \cdots \cup C_s$ such that for every $e \in E$, $e \notin C_i$ for $i = 1, \ldots, s$.

The minimal integer $s$ such that $G$ has a $s$-coloring is called the chromatic number of $G$ and is denoted by $\chi(G)$.

**Example 3.1.2.** Let $G$ be the graph in Figure 3.2. Then $\chi(G) = 2$. 
Definition 3.1.3. A graph $G$ is said to be critically $s$-chromatic if $\chi(G) = s$ but $\chi(G \setminus x) = s - 1$ for every $x \in V$, where $G \setminus x$ is the graph obtained from $G$ by removing the vertex $x$ and all edges incident to $x$. A graph that is critically $s$-chromatic for some $s$ is called critical.

Example 3.1.4. The graph $G$ in Figure 3.1 is not critical. $\chi(G) = 2 = \chi(G \setminus x_1)$.

Definition 3.1.5. For any vertex $x_i \in V$, the expansion of $G$ at the vertex $x_i$ is the graph $G' = G[[x_i]]$ whose vertex set is $V' = (V \setminus \{x_i\}) \cup \{x_{i1}, x_{i2}\}$ and whose edge set is $E' = \{\{u, v\} \in E \mid u \neq x_i \text{ and } v \neq x_i\} \cup \{\{u, x_{i1}\}, \{u, x_{i2}\} \mid \{u, x_i\} \in E\} \cup \{\{x_{i1}, x_{i2}\}\}$. For any $W \in V$, the expansion of $G$ at $W$, denoted by $G[W]$, is obtained by successively expanding all the vertices of $W$ (in any order).

Example 3.1.6. The expansion of a graph $G$ at $W = \{x_1, x_2\}$. 
We now state the conjecture about constructing critically $(s+1)$-chromatic graphs from critically $s$-chromatic graphs. This conjecture implies that any unmixed height two squarefree monomial ideal $I$ in a polynomial ring $R$, i.e., the cover ideal of a finite simple graph, has the persistence property.

**Conjecture 3.1.7.** ([28, Conjecture 1.1]) Let $s$ be a positive integer, and let $G$ be a finite simple graph that is critically $s$-chromatic. Then there exists a subset $W \subseteq V(G)$ such that $G[W]$ is a critically $(s+1)$-chromatic graph.

Conjecture 3.1.7 arose out of investigations of the associated primes of powers of the cover ideal of a graph. It was proved that Conjecture 3.1.7 is true if $\chi_f(G)$, the fractional chromatic number of $G$, is close to $\chi(G)$.

**Theorem 3.1.8.** ([28, Theorem 1.3]) Conjecture 3.1.7 holds if we also assume $\chi(G) - 1 < \chi_f(G) \leq \chi(G)$.

As a corollary, all odd holes and odd antiholes satisfy Conjecture 3.1.7.

**Corollary 3.1.9.** ([28, Corollary 3.11]) Conjecture 3.1.7 holds for the following critical graphs: cliques, odd holes, and odd antiholes.

If Conjecture 3.1.7 is true, then unmixed height two squarefree monomial ideals have the persistence property.
**Theorem 3.1.10.** ([28, Theorem 1.2]) Suppose that $I$ is any unmixed squarefree monomial ideal of height 2. If Conjecture 3.1.7 holds for $(s + 1)$, then

$$\text{Ass}(R/I^s) \subseteq \text{Ass}(R/I^{s+1}).$$

In particular, if Conjecture 3.1.7 holds for all $s$ then $I$ has the persistence property.

**Remark 3.1.11.** We make the following two remarks.

(1) The converse of the theorem is not necessarily true.

(2) A hypergraph version of Conjecture 3.1.7 can also be formulated. And by adapting the proof of Theorem 3.1.10, we can remove the unmixed and height hypothesis from Theorem 3.1.10. Because of the one-to-one correspondence between hypergraphs and squarefree monomial ideals, the hypergraph version of Conjecture 3.1.7 would imply the persistence property for all squarefree monomial ideals.

### 3.2 Construction of A Counterexample

We have seen in last section, motivated by questions about properties of squarefree monomial ideals, a conjecture about expansion in color-critical graphs was provided. In a recent work [45], Kaiser, Stehlík and Škrekovski provide a family of critically 3-chromatic graphs whose expansions do not result in critically 4-chromatic graphs, and thus give counterexamples to Conjecture 3.1.7. The cover ideal of the smallest member of this family also gives a counterexample to the persistence and non-increasing depth properties.

This infinite family of graphs was originally constructed in [46] by Gallai and it provided the first example of a $k$-edge-critical graph without vertices of degree $k - 1$. We start with describing in detail the construction of the family of graphs that serves as a counterexample.
Let $K_3$ denote the complete graph on 3 vertices, and let $P_q$, for $q \geq 4$ denote a path of length $q-1$. The graph $H_q = K_3 \boxtimes P_q$ is formed by taking $q$ copies of $K_3$ with vertices $\{x_{i,0}, x_{i,1}, x_{i,2}\}$, $i = 1, \ldots, q$, connecting $x_{i,j}$ and $x_{i+1,j}$ for $i = 1, \ldots, q-1$ to get 3 paths of length $q-1$, and finally connecting $x_{1,j}$ and $x_{q,2-j}$ for $j = 0, 1, 2$.(see Figure 3.5).

![Figure 3.5. The graph $H_q = K_3 \boxtimes P_q$.](image)

One of the interesting properties of $H_q$s is that they embed in the Klein bottle as quadrangulations.

**Example 3.2.1.** This example show that how $H_4$ can be embedded as a quadrangulation in the Klein bottle.

![Figure 3.6. The graph $H_q = K_3 \boxtimes P_q$.](image)
Remark 3.2.2. The construction of the graph $H_q$ can be generalized to a pair consisting of a path and a complete graph of any size. Indeed, let $P_q$ be a path of length $q - 1$ and let $K_p$ be the complete graph of size $p$. We can then construction the graph $H_{p,q} = K_p \boxtimes P_q$ by taking $q$ copies of $K_p$ with vertices $\{x_{i,0}, \ldots, x_{i,p-1}\}$, $i = 1, \ldots, q$, connecting $x_{i,j}$ to $x_{i+1,j}$ for $i = 1, \ldots, q - 1$ to get $p$ paths of length $q - 1$, and finally connecting $x_{1,j}$ to $x_{q,p-1-j}$ for $j = 0, \ldots, p - 1$. In this construction, $H_q = H_{3,q}$.

Kaiser, Stehlík and Škrekovski showed that Gallai’s graphs are counterexamples to Cojecture 3.1.7. First, they show that for certain sets $W$, the chromatic number of $H_n[W]$ is at least 5, but $H_n[W]$ is not 5-critical. They then prove that for any other set $W$, $H_n[W]$ is 4-chromatic.

Theorem 3.2.3. ([45, Theorem 2]) For any $n \geq 4$ and any $W \subseteq V(H_n)$, the graph $H_n[W]$ is not 5-critical.

While this counterexample to the conjecture does not necessarily imply a negative answer to the question about the persistence property, it was pointed out in [45], the cover ideal of $H_4$ does in fact show that the answer is negative.

Let $R = k[x_1, \ldots, x_{12}]$. Let $J = J(H_4)$ denote the cover ideal of $H_4$. Using
Macaulay2, we can compute the set of associated primes of $J^3$ and $J^4$. By comparing
the output, we can find that

$$\text{Ass}(R/J^4) = \text{Ass}(J^3) - \{m\},$$

where $m$ is the maximal ideal of $R$. In particular,

**Theorem 3.2.4.** ([45, Theorem 11]) The cover ideal $J(H_4)$ does not have the persistence property.

Moreover, the cover ideal of $H_4$ does not have a non-increasing depth function. Using Macaulay2, we find that

$$\text{depth}(R/J^3) = 0 < 4 = \text{depth}(R/J^4),$$

so we have the following

**Theorem 3.2.5.** ([45, Theorem 13]) The depth function of the cover ideal $J(H_4)$ is not non-increasing.

Theorem 3.2.4 and Theorem 3.2.5 says that the cover ideal of the smallest member of this family also gives a counterexample to the persistence and non-increasing depth properties. In fact, the cover ideals of all members of this family of graphs fail to have the persistence and non-increasing depth properties.

**Theorem 3.2.6.** Let $H_q$ be the graph constructed as before. Let $J = J(H_q)$ and let $m$
be the maximal homogeneous ideal in the polynomial ring $R = k[x_{i,j} | i = 1, \ldots, q, j = 0, 1, 2]$. Then

$$m \in \text{Ass}(R/J^3) \text{ and } m \notin \text{Ass}(R/J^4).$$

As a consequence, $J$ fails to have non-increasing depth.
3.3 Proof of Main Result

This section is devoted to the proof of our main result, Theorem 3.2.6. This theorem will be proved as a combination of Propositions 3.3.1 and 3.3.2 and Corollary 3.3.8. For simplicity of terminology, we call the complete graph $K_3$ on $\{x_{i,0}, x_{i,1}, x_{i,2}\}$ the $i$th triangle in $H_q$.

**Proposition 3.3.1.** Let $H_q$ be the graph constructed as in Section 3.2. Let $J = J(H_q)$ and let $m$ be the maximal homogeneous ideal in $R = k[x_{i,j}] \mid i = 1, \ldots, q, j = 0, 1, 2$. Then $m \in \text{Ass}(R/J^3)$.

**Proof.** It was shown in [16, Proposition 9] that $H_q$ is critically 4-chromatic. Thus, it follows from [6, Corollary 4.5] that $m \in \text{Ass}(R/J^3)$. □

**Proposition 3.3.2.** Let $H_q$ be the graph constructed as in Section 3.2. Let $J = J(H_q)$ and let $m$ be the maximal homogeneous ideal in $R = k[x_{i,j}] \mid i = 1, \ldots, q, j = 0, 1, 2$. Then $m \notin \text{Ass}(R/J^4)$.

**Proof.** Suppose, by contradiction, that $m \in \text{Ass}(R/J^4)$. That is, there exists a monomial $T$ in $R$ such that $T \notin J^4$ and $J^4 : T = m$. Since the generators of $J$ are squarefree, the powers of each variable in minimal generators of $J^4$ are at most 4. This implies that the power of each variable in $T$ is at most 3, i.e., $T$ divides $(\prod_{i,j} x_{i,j})^3$. We shall now make a number of observations to reduce the number of cases that we need to consider later. □

**Observation 3.3.3.** We have $M = (\prod_{i,j} x_{i,j})^3 \in J^4$. Indeed, we can write $M = M_1M_2M_3M_4N$ as follows.
(1) If \( q \) is odd then choose \( N = \prod_{i=1}^{q} x_{i,0} \) and

\[
M_1 = \left( \prod_{i \text{ odd}} x_{i,0}x_{i,1} \right) \left( \prod_{i \text{ even}} x_{i,1}x_{i,2} \right) \left( x_{q,0}x_{q,2} \right)
\]

\[
M_2 = \left( \prod_{i \text{ odd}} x_{i,0}x_{i,2} \right) \left( \prod_{i \text{ even}} x_{i,1}x_{i,2} \right) \left( x_{q,0}x_{q,1} \right)
\]

\[
M_3 = \left( \prod_{i \text{ odd}} x_{i,1}x_{i,2} \right) \left( \prod_{i \text{ even}} x_{i,0}x_{i,1} \right) \left( x_{q,1}x_{q,2} \right)
\]

\[
M_4 = \left( \prod_{i \text{ odd}} x_{i,1}x_{i,2} \right) \left( \prod_{i \text{ even}} x_{i,0}x_{i,2} \right) \left( x_{q,1}x_{q,2} \right)
\]

(2) If \( q \) is even then choose \( N = \prod_{i=1}^{q} x_{i,1} \) and

\[
M_1 = \left( \prod_{i \text{ odd}} x_{i,0}x_{i,2} \right) \left( \prod_{i \text{ even}} x_{i,1}x_{i,1} \right)
\]

\[
M_2 = \left( \prod_{i \text{ odd}} x_{i,0}x_{i,2} \right) \left( \prod_{i \text{ even}} x_{i,1}x_{i,2} \right)
\]

\[
M_3 = \left( \prod_{i \text{ odd}} x_{i,0}x_{i,1} \right) \left( \prod_{i \text{ even}} x_{i,0}x_{i,2} \right)
\]

\[
M_4 = \left( \prod_{i \text{ odd}} x_{i,1}x_{i,2} \right) \left( \prod_{i \text{ even}} x_{i,0}x_{i,2} \right)
\]

It is easy to verify that \( M_1, \ldots, M_4 \) are vertex covers of \( H_q \). Thus, \( M \in J^4 \). This observation allows us to assume that \( T \) strictly divides \( M \).

**Observation 3.3.4.** For each \( i = 1, \ldots, q \), the total power of \( x_{i,0}, x_{i,1} \) and \( x_{i,2} \) in \( T \) is at least 8. Indeed, take \( k \neq i \), then since \( J^4 : T = m \), we must have \( Tx_{k,0} \in J^4 \). That is, \( Tx_{k,0} \) can be written as the product of 4 vertex covers of \( H_q \). Notice also that to cover the triangle with vertices \( \{x_{i,0}, x_{i,1}, x_{i,2}\} \) each vertex cover needs at least two of those 3 vertices. Thus, 4 vertex covers contain in total at least 8 copies of those vertices. This observation and the fact that \( T \) divides \( M \) allow us to conclude that for each \( i = 1, \ldots, q \), either all three vertices \( \{x_{i,0}, x_{i,1}, x_{i,2}\} \) appear in \( T \) each with
power exactly 3, or two of them appear in \( T \) with power 3 and the third one appears in \( T \) with power exactly 2. For simplicity of language, we shall call the total powers of \( \{x_{i,0}, x_{i,1}, x_{i,2}\} \) in \( T \) the power of the \( i \)th triangle in \( T \).

**Observation 3.3.5.** Suppose that the power of the \( i \)th triangle in \( T \) is at least 8, and we already impose the conditions that 3 among the \( M_i \)s each has to contain a specific (but distinct) variable in the \( i \)th triangle. Then we can always distribute the remaining variables of the \( i \)th triangle from \( T \) into the \( M_i \)s so that each of them indeed covers the edges of the \( i \)th triangle. To see this, without loss of generality, we may assume that the 3 imposed conditions are \( x_{i,0} \mid M_1 \), \( x_{i,1} \mid M_2 \) and \( x_{i,2} \mid M_3 \), and assume that \( x_{i,1} \) and \( x_{i,2} \) appear in \( T \) with powers at least 3. This implies that \( x_{i,0} \) appears in \( T \) with power least 2, and we can distribute the variables of the \( i \)th triangle in \( T \) into the \( M_i \)s as follows:

\[
\begin{align*}
x_{i,0} & \mid M_1 \\
x_{i,1} & \mid M_2 \\
x_{i,1} & \mid M_3 \\
x_{i,0} & \mid M_4
\end{align*}
\]

**Observation 3.3.6.** Re-indexing the vertices of \( H_q \) as follows: label \( x_{q,0} \) by \( x_{1,2} \), label \( x_{q,1} \) by \( x_{1,1} \), label \( x_{q,2} \) by \( x_{1,0} \) (notice that we have switched the second indices 0 and 2 in the \( q \) triangle and bring it to be the first triangle), and then label \( x_{i,j} \) by \( x_{i+1,j} \) for all \( 1 \leq i \leq q - 1 \) and \( j = 0, 1, 2 \) (i.e., shifting the rest of the triangles one space to the right). We then obtain an isomorphic copy of \( H_q \) where the old \( q \)th triangle becomes the first one. This process can be repeated. Thus, coupled with Observation 3.3.3, we can assume that the power of the first triangle in \( T \) is exactly
8. Without loss of generality, we may further assume that \(x_{1,0}\) appears in \(T\) with power 2, while \(x_{1,1}\) and \(x_{1,2}\) appear in \(T\) with powers 3.

**Observation 3.3.7.** Fix an index \(i < q - 1\) where the power of the \(i\)th triangle in \(T\) is exactly 8, and assume that \(x_{i,0}\) appears in \(T\) with power 2 (and so, \(x_{i,1}\) and \(x_{i,2}\) appear in \(T\) both with power 3). Since \(J^1 : T = m\), in particular, we have \(T x_{q,0} \in J^1\). That is, we can write \(T x_{q,0} = M_1 M_2 M_3 M_4\) as the product of 4 vertex covers of \(H_q\). To distribute \(x_{i,0}^3 x_{i,1}^2 x_{i,2}^3\) into 4 vertex covers, there is only one possibility (up to permutation of the indices of the vertex covers), which is:

\[
\begin{align*}
x_{i,0} x_{i,1} & \mid M_1 \\
x_{i,0} x_{i,2} & \mid M_2 \\
x_{i,1} x_{i,2} & \mid M_3 \\
x_{i,0} x_{i,2} & \mid M_4 
\end{align*}
\]

This distribution of the vertices on the \(i\)th triangle will impose specific conditions on how the vertices of the \((i + 1)\)st triangle can be distributed into the 4 vertex covers. Particularly, we must have that \(x_{i+1,2} \mid M_1\), \(x_{i+1,1} \mid M_2\), and \(x_{i+1,0}\) divides both \(M_3\) and \(M_4\).

If the power of the \((i + 1)\)st triangle in \(T\) is 9 then we can distribute vertices in the \((i + 1)\)st triangle into the \(M_i\)s as follows:
where the extra copy of \( x_{i+1,0} \) could be assigned to either \( M_1 \) or \( M_2 \). Now, the only conditions imposed on the \((i + 2)\)nd triangle are \( x_{i+2,2} \bigg| M_3, x_{i+2,1} \bigg| M_4, \) and either \( x_{i+2,0} \bigg| M_2 \) or \( x_{i+1,0} \bigg| M_1 \). It follows from Observation 3.3.5 that the variables of the \((i + 2)\)nd triangle in \( T \) can be distributed into the \( M_i \)s, and we can think of the \((i + 2)\)nd triangle as our new starting point (if \( i + 2 < q \)).

If, on the other hand, the power of the \((i + 1)\)st triangle in \( T \) is 8, then we obtain the following possibilities depending on which variable in the \((i + 1)\)st triangle appears in \( T \) with power 2.

(1) If the power of \( x_{i+1,0} \) in \( T \) is 2 then (up to permuting \( M_3 \) and \( M_4 \)) we have:

\[
\begin{array}{ccc}
  x_{i,0} x_{i,1} & x_{i+1,1} x_{i+1,2} & M_1 \\
  x_{i,0} x_{i,2} & x_{i+1,1} x_{i+1,2} & M_2 \\
  x_{i,1} x_{i,2} & x_{i+1,0} x_{i+1,1} & M_3 \\
  x_{i,1} x_{i,2} & x_{i+1,0} x_{i+1,2} & M_4 \\
\end{array}
\]
(2) If the power of $x_{i+1,1}$ in $T$ is 2 then we must be in either of the following cases:

\[
\begin{array}{ccc}
  x_{i,0}x_{i,1} & x_{i+1,0}x_{i+1,2} & M_1 \\
  x_{i,0}x_{i,2} & x_{i+1,1}x_{i+1,2} & M_2 \\
  x_{i,1}x_{i,2} & x_{i+1,0}x_{i+1,1} & M_3 \\
  x_{i,1}x_{i,2} & x_{i+1,1}x_{i+1,2} & M_4 \\
\end{array}
\]

or

\[
\begin{array}{ccc}
  x_{i,0}x_{i,1} & x_{i+1,1}x_{i+1,2} & M_1 \\
  x_{i,0}x_{i,2} & x_{i+1,1}x_{i+1,1} & M_2 \\
  x_{i,1}x_{i,2} & x_{i+1,0}x_{i+1,2} & M_3 \\
  x_{i,1}x_{i,2} & x_{i+1,1}x_{i+1,2} & M_4 \\
\end{array}
\]

(3) If the power of $x_{i+1,2}$ in $T$ is 2 then we must be in either of the following cases:

\[
\begin{array}{ccc}
  x_{i,0}x_{i,1} & x_{i+1,0}x_{i+1,2} & M_1 \\
  x_{i,0}x_{i,2} & x_{i+1,1}x_{i+1,2} & M_2 \\
  x_{i,1}x_{i,2} & x_{i+1,0}x_{i+1,1} & M_3 \\
  x_{i,1}x_{i,2} & x_{i+1,0}x_{i+1,1} & M_4 \\
\end{array}
\]
or

\[
\begin{align*}
  x_{i,0}x_{i,1} & \quad x_{i+1,0}x_{i+1,1} \quad M_1 \\
  x_{i,0}x_{i,2} & \quad x_{i+1,0}x_{i+1,1} \quad M_2 \\
  x_{i,1}x_{i,2} & \quad x_{i+1,0}x_{i+1,1} \quad M_3 \\
  x_{i,1}x_{i,2} & \quad x_{i+1,0}x_{i+1,2} \quad M_4
\end{align*}
\]

The upshot of this observation is that we can successively distribute \( T \) and \( Tx_{q,0} \) (without the use of the extra variable \( x_{q,0} \)) into 4 vertex covers up to the \((q - 1)\)st triangle in the same way. At each step, moving from the \( i \)th triangle to the \((i + 1)\)st triangle, we might end up with a number of different choices. Moreover, if the power of the \((i + 1)\)st triangle in \( T \) is 9, then we can distribute the vertices in the \( i \)th and the \((i + 1)\)st triangles, and consider the \((i + 2)\)nd triangle as our new starting point to repeat the process. The difference, and what makes \( T \not\in J^4 \) but \( Tx_{q,0} \in J^4 \), occurs when we need to cover the \( q \)th triangle and edges connecting the \( q \)th and the 1st triangles (i.e., moving from the \((q - 1)\)st triangle to the last triangle).

By making use of Observation 3.3.7, we can successively distribute the variables appearing in \( T \) into the \( M_i \)s in the same way as that of \( Tx_{q,0} \) such that along the process, \( M_i \)s cover edges in the first \((q - 1)\) triangles. It remains to consider how the variables in the \( q \)th triangle are distributed. We shall show that a contradiction, either that \( T \in J^4 \) or that \( J^4 : T \neq m \), is always resulted in.

Notice that when the power of the \((q - 1)\)st triangle in \( T \) is 9, in our distribution process, a power 8 of this triangle is distributed to the \( M_i \)s, and there is possibly an extra copy of a variable left. This possible extra variable can then be assigned to one of the \( M_i \)s. Our argument will complete by exhausting cases depending on how the vertices in the \((q - 1)\)st triangle are distributed among the \( M_i \)s and which vertex is
possibly treated as the extra one. There are 3 choices for the possible extra vertex. For each choice of the possible extra vertex, the cases are considered depending on how the other two copies of this vertex are distributed among 4 vertex covers $M_i$. Observe that if the possible extra vertex is $x_{q-1, t}$ (where $t = 0, 1$ or 2, and we identify $x_{i,t}$ with $x_{i,t+3}$) then there are 6 cases to consider by assigning $x_{q-1, t}$ to 2 out of the 4 vertex covers $M_i$s. For example, if $x_{q-1, t}$ is assigned to $M_1$ and $M_2$, then there would be two possibilities depending on how $x_{q-1,t+1}$ and $x_{q-1,t+2}$ are distributed. These possibilities are described by conditions:

$$
\begin{align*}
&x_{1,0}x_{1,1} \cdots x_{q-1,t}x_{q-1,t+1} & M_1 \\
&x_{1,0}x_{1,1} \cdots x_{q-1,t}x_{q-1,t+2} & M_2 \\
&x_{1,1}x_{1,2} \cdots x_{q-1,t+1}x_{q-1,t+2} & M_3 \\
&x_{1,1}x_{1,2} \cdots x_{q-1,t+1}x_{q-1,t+2} & M_4
\end{align*}
$$

or

$$
\begin{align*}
&x_{1,0}x_{1,1} \cdots x_{q-1,t}x_{q-1,t+2} & M_1 \\
&x_{1,0}x_{1,2} \cdots x_{q-1,t}x_{q-1,t+1} & M_2 \\
&x_{1,1}x_{1,2} \cdots x_{q-1,t+1}x_{q-1,t+2} & M_3 \\
&x_{1,1}x_{1,2} \cdots x_{q-1,t+1}x_{q-1,t+2} & M_4
\end{align*}
$$

This case-by-case analysis is quite tedious, but the 18 cases are mostly similar. Thus, we will go through the argument carefully for one case and leave it to the interested reader to check the details of the remaining cases.
Consider the case where \( x_{q-1,0} \) is the possible extra vertex, and the other two copies of \( x_{q-1,0} \) are in \( M_1 \) and \( M_2 \). There are two possibilities depending on how \( x_{q-1,1} \) and \( x_{q-1,2} \) were distributed:

If it is the first possibility that occurs, and there is in fact no extra copy of \( x_{q-1,0} \) (i.e., the power of the \((q - 1)\)st triangle in \( T \) was exactly 8), then this impose the following conditions on the \( q \)th triangle: \( x_{q,0}x_{q,2} \big| M_1, M_3, M_4 \) and \( x_{q,1} \big| M_2 \). This implies that the product of the \( M_i \)s will use 4 copies of either \( x_{q,0} \) or \( x_{q,2} \). Thus, \( Tx_{q,1} \notin J^4 \). If it is the first possibility but there is an extra copy of \( x_{q,0} \) left, then we can distribute this extra copy of \( x_{q,0} \) to either \( M_3 \) or \( M_4 \), say \( M_4 \). In this case, the conditions imposed on the \( q \)th triangle are: \( x_{q,0}x_{q,2} \big| M_1 \) and \( M_3, x_{q,1} \big| M_2 \), and \( x_{q,2} \big| M_4 \). Thus, to cover the edges of the \( q \)th triangle, we must have
It follows that if \( T \) contains 3 copies of \( x_{q,0} \) and \( x_{q,2} \) then this distribution shows that \( T \in J^4 \). Otherwise, if \( T \) contains, for instance, only 2 copies of \( x_{q,0} \), then since the product of 4 vertex covers, as shown, must contain at least 3 copies of \( x_{q,0} \), we have that \( Tx_{q,1} \notin J^4 \).

If it is the second possibility and there is no extra copy of \( x_{q-1,0} \) then conditions imposed on the \( q \)th triangle are: \( x_{q,0}x_{q,1} | M_1 \), \( x_{q,1}x_{q,2} | M_2 \), \( x_{q,0}x_{q,2} | M_3 \) and \( M_4 \). Thus, the product of the 4 vertex covers contain at least 3 copies of \( x_{q,0} \) and \( x_{q,2} \). If \( T \) has at least 3 copies of \( x_{q,0} \) and \( x_{q,2} \) then \( T \in J^4 \). Otherwise, \( Tx_{q,1} \notin J^4 \). If it is the second possibility and there is an extra copy of \( x_{q-1,0} \) then we can distribute this extra copy of \( x_{q,0} \) to either \( M_3 \) and \( M_4 \), say \( M_4 \). In this case, the conditions imposed on the \( q \)th triangle are: \( x_{q,0}x_{q,1} | M_1 \), \( x_{q,1}x_{q,2} | M_2 \), \( x_{q,0}x_{q,2} | M_3 \) and \( x_{q,2} | M_4 \). Thus, if \( T \) contains at least 3 copies of \( x_{q,2} \) then by distributing either \( x_{q,0} \) or \( x_{q,1} \) to \( M_4 \), we get that \( T \in J^4 \). Otherwise, \( Tx_{q,0} \notin J^4 \).

For the remaining cases, it can be seen that covering the edges of the \( q \)th triangle and edges connecting to the first and the \((q-1)\)st triangles will impose a number of conditions on how vertices of the \( q \)th triangle in \( T \) can be distributed to the 4 vertex covers \( M_i \). These conditions will fall into one of the following situations.

(1) The conditions do not require \( \prod_{i=1}^4 M_i \) to contain any vertex of the \( q \)th triangle
with power more than 2. In this case, we can always distribute the vertices of the $q$th powers in $T$ into the 4 vertex covers $M_i$s in a way to satisfy these conditions. We thus have $T \in J^4$.

(2) The conditions require $\prod_{i=1}^4 M_i$ to contain one or two vertices of the $q$th triangle with powers at least 3. If $T$ does contain those vertices with powers at least 3, then we can also distribute the vertices of the $q$th triangle in $T$ into the 4 vertex covers $M_i$ to comply with those condition; we then have $T \in J^4$. If, otherwise, $T$ does not contain those one or two vertices with powers at least 3, then the product of $T$ with the third vertex will not be in $J^4$.

(3) The conditions require $\prod_{i=1}^4 M_i$ to contain a vertex of the $q$th triangle with power at least 4. In this case, the product of $T$ and another vertex of the $q$th triangle will not be in $J^4$.

**Corollary 3.3.8.** Let $H_q$ be the graph constructed as in Section 3.2. Let $J = J(H_q) \subseteq R = k[x_{i,j} | i = 1, \ldots, q, j = 0, 1, 2]$. Then $J$ fails to have non-increasing depth.

**Proof.** The conclusion is a direct consequence of Remark 2.3.4, Propositions 3.3.1 and 3.3.2.

### 3.4 Other Constructions

A natural generalization of the graphs $H_q$s are those of $H_{p,q}$s as constructed in the previous section. We end this chapter by showing that those graphs $H_{p,q}$ do not give counterexamples to Conjecture 3.1.7. In fact, we shall show that $H_{p,q}$, for $p > 3$ are not critical graphs.

**Theorem 3.4.1.** Let $p, q > 4$ and let $H_{p,q}$ be constructed as in Remark 3.2.2. Then, $\chi(H_{p,q}) = p$, but $H_{p,q}$ is not critical $p$-chromatic.
Proof. Clearly, any graph containing a complete subgraph of size $p$ has the chromatic number at least $p$. Thus, it suffices to show that $H_{p,q}$ can be colored using $p$ colors (and since $H_{p,q}$ contains more than one copies of $K_p$, this will also imply that $H_{p,q}$ is not critical $p$-chromatic). Indeed, we can assign $p$ colors to the vertices of $H_{p,q}$ as follows.

Case 1: $p$ is even and $q$ is odd. For $1 \leq i \leq q$ and $i$ is odd, assign to $x_{i,j}$ color $j$ for all $j = 0, \ldots, p-1$. For $1 \leq i \leq q$ and $i$ is even, assign to $x_{i,j}$ color $j + 1$, for $j = 0, \ldots, p-1$ (here, we identify colors congruent modulo $p$). It is easy to see that the vertices on each copy of $K_p$ get different colors. Also, on the $i$th and $(i+1)$st copies of $K_p$, since the parity of $i$ and $i+1$ are different, adjacent vertices $x_{i,j}$ and $x_{i+1,j}$ get different colors. Finally, on the first and the last copies of $K_p$, adjacent vertices are $x_{1,j}$ of color $j$ and $x_{q,p-1-j}$ of color $p-1-j$. Since $p$ is even $j \neq p-1-j$ for any $j$. Figure 3.8 gives the assigned 4-coloring for $H_{4,q}$ in this case.

![Figure 3.8. A 4-coloring for $H_{4,q}$ when $q$ is odd.](image)

Case 2: $p$ and $q$ are both even. For $1 \leq i \leq q$ and $i$ is odd, assign to $x_{i,j}$ color $j$ for all $j = 0, \ldots, p-1$. For $1 \leq i \leq q$ and $i$ is even, assign to $x_{i,j}$ the color $p+1-j$. Again, the vertices on each copy of $K_p$ get different colors. Also, since $p$ is even $j \neq p+1-j$, adjacent vertices on consecutive copies of $K_p$ also get different colors. On the first
and the last copies of $K_p$, adjacent vertices are $x_{1,j}$ of color $j$ and $x_{q,p-1-j}$ of color $j + 2$, and we have $j \not\equiv j + 2 (mod\ p)$. Figure 3.9 gives the assigned 4-coloring for $H_{4,q}$ in this case.

Case 3: $p$ is odd and $q$ is even. For $1 \leq i \leq q$ and $i$ is odd, assign to $x_{i,j}$ color $j$ for all $j = 0, \ldots, p - 1$. For $1 \leq i \leq q$ and $i$ is even, we assign the colors to $x_{i,j}$s as follows: first, we assign to $x_{i,j}$ color $p - j$, for $j = 0, \ldots, p - 1$, and then we switch the colors of $x_{i,0}$ and $x_{i, p + 1}$ (i.e., the vertex $x_{i,0}$ now has color $\frac{p - 1}{2}$ and the vertex $x_{i, p + 1}$ now has color 0). Again, the vertices on each copy of $K_p$ get different colors. On consecutive copies of $K_p$, since $j \not\equiv p - j (mod\ p)$ unless $j = 0$, together with the color switching between $x_{i,0}$ and $x_{i, p + 1}$, it can be seen that adjacent vertices get different colors. On the first and the last copies of $K_p$, adjacent vertices are $x_{1,j}$ of color $j$ and $x_{q,p-1-j}$ of colors $j + 1 \not\equiv j$, except when $j = p - 1$ or $j = \frac{p - 3}{2}$. Finally, $x_{1,p-1}$ and $x_{q,0}$ are adjacent and of colors $p - 1 \not\equiv \frac{p - 1}{2}$, while $x_{1, \frac{p-3}{2}}$ and $x_{q, \frac{p-3}{2}}$ are adjacent and of colors $\frac{p-3}{2} \not\equiv 0$ (this is where we make use of the hypothesis that $p \geq 4$). Figure 3.10 gives the assigned 5-coloring for $H_{5,q}$ in this case.
Figure 3.10. A 5-coloring for $H_{5,q}$ when $q$ is even.

Case 4: $p$ and $q$ are both odd. For $1 \leq i < q - 1$ and $i$ is odd, assign to $x_{i,j}$ color $j$ for all $j = 0, \ldots, p - 1$. For $1 \leq i \leq q - 1$ and $i$ is even, assign to $x_{i,j}$ color $j - 1$ for all $j = 0, \ldots, p - 1$. Finally, we assign the colors to $x_{q,j}$s as follows: first, we assign to $x_{q,j}$ color $p - j$, for $j = 0, \ldots, p - 1$, and then we switch the colors of $x_{q,0}$ and $x_{q,p+1}$ (i.e., the vertex $x_{q,0}$ now has color $\frac{p-1}{2}$ and the vertex $x_{q,p+1}$ now has color $0$). Clearly, vertices on each copy of $K_p$ get different colors, and adjacent vertices on consecutive copies of $K_p$ (except the last one) get different colors. On the $(q-1)$st and the $q$th copies of $K_p$, adjacent vertices are $x_{q-1,j}$ of color $j - 1$ and $x_{q,j}$ of color $p - j$, except exactly when $j = 0$ or $j = \frac{p+1}{2}$ due to the color switch. It can be seen that $j - 1 \neq p - j$ for all $j \neq \frac{p+1}{2}$. When $j = \frac{p+1}{2}$ the colors of $x_{q-1,\frac{p+1}{2}}$ and $x_{q,\frac{p+1}{2}}$ are $\frac{p-1}{2} \neq 0$. For adjacent vertices between the $q$th and the first copies of $K_p$, the argument follows from the last part of that of Case 3 (and again, we shall need the condition that $p \geq 4$). Figure 3.11 gives the assigned 5-coloring for $H_{5,q}$ in this case.
Figure 3.11. A 5-coloring for $H_{5,q}$ when $q$ is odd.
Chapter 4

Castelnuovo-Mumford Regularity

In this chapter, we will discuss the Castelnuovo-Mumford regularity of square-free monomial ideals. Our focus is on bounds and exact values for the regularity in terms of combinatorial data from associated simplicial complexes and hypergraphs.

4.1 Combinatorial Bounds for Regularity

In this section, we examine various bounds for the regularity of a squarefree monomial ideal in terms of combinatorial data from associated simplicial complex and hypergraph.

For edge ideals, the matching number and induced matching number provide nice upper and lower bounds for the regularity. Let’s recall the definitions of matching number and induced matching number.

Let $H$ be a hypergraph. Let $Y \subseteq V(H)$ be a subset of the vertices in $H$. The induced subhypergraph of $H$ on $Y$, denoted by $H[Y]$, is the hypergraph with vertex set $Y$ and edge set \{$e \in E(H) \mid e \subseteq Y$\}.

**Definition 4.1.1.** Let $H$ be a simple graph.

1. A collection $C$ of edges in $H$ is called a matching if the edges in $C$ are pairwise disjoint. The maximum size of a matching in $H$ is called its matching number.
A collection $C$ of edges in $H$ is called an \textit{induced matching} if $H$ is a matching, and $C$ consists of all edges of the induced subhypergraph of $H[\cup_{e \in C} e]$ of $H$. The maximum size of an induced matching in $H$ is called its \textit{induced matching number}, denoted by $\nu(H)$.

**Example 4.1.2.** Let $G$ be graph in Figure 2.2. $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ is a matching, but not an induced matching. The induced matching number of $G$ is 2.

![Figure 4.1. A graph $G$ with $\nu(G) = 2$.](image)

The following result due to Katzman [47] provides a lower bound for edge ideals.

**Theorem 4.1.3.** ([47, Lemma 2.2]) Let $G$ be a simple graph and let $\nu(G)$ be the induced matching number. Then

$$\text{reg}(I(G)) \geq \nu(G) + 1.$$ 

This result is generalized for all simple hypergraphs in [35].

**Theorem 4.1.4.** ([35, Corollary 3.9]) Let $H$ be a simple hypergraph. Suppose that $\{e_1, \ldots, e_s\}$ forms an induced matching in $H$. Then

$$\text{reg}(I(H)) \geq \sum_{i=1}^{s} (|e_i| - 1) + 1.$$
Remark 4.1.5. If $H$ consists disjoint edges, then the bound in Theorem 4.1.4 becomes an equality.

Example 4.1.6. The regularity of a hypergraph can be arbitrarily larger than the lower bound in Theorem 4.1.4. For $s > 1$, consider the hypergraph $H_s$ with edges $\{x, y_i, z_i\}$ (for $i = 1, \ldots, s$) on the vertex set $\{x, y_1, \ldots, y_s, z_1, \ldots, z_s\}$. The induced matching number of $H_s$ is 1. On the other hand, it is straightforward to compute that $\text{reg}(I(H_s)) = s + 2$.

Another invariant related to the induced matching number is the (minimax) matching number. And it provides an upper bound for edge ideals of graphs.

Theorem 4.1.7. ([5, Theorem 6.7], [48, Theorem 11]) Let $G$ be a simple graph. Let $\beta(G)$ be the minimum size of a maximal matching in $G$. Then

$$\text{reg}(I(G)) \leq \beta(G) + 1.$$ 

Remark 4.1.8. If $G$ consists of disjoint edges, then the bound in Theorem 4.1.7 becomes an equality.

Example 4.1.9. Using the result in Theorem 4.1.7 to bound the regularity of hypergraphs is no longer true. We still consider the hypergraph in Example 4.1.6. The minimax matching number of $H_s$ is 1. On the other hand, $\text{reg}(I(H_s))$ can be taken to be arbitrarily large.

The result in Theorem 4.1.7 was generalized for hypergraphs with replacing the notion of a matching by a 2-collage.

Definition 4.1.10. Let $H$ be a simple hypergraph. A subset $C$ of the edges in $H$ is called a 2-collage if for each edge $E$ in $H$, there exists a vertex $v \in E$ such that $E \setminus \{v\}$ is contained in some edge of $C$. 

Theorem 4.1.11. ([38, Theorem 1.2]) Let $H$ be a simple hypergraph, and let $\{E_1, \ldots, E_s\}$ be a 2-collage in $H$. Then

$$\text{reg}(I(H)) \leq \sum_{i=1}^{s} (|E_i| - 1) + 1.$$ 

4.2 Regularity of Edge Ideals of Vertex Decomposable Graphs

In this section, we provide upper bounds for the regularity of vertex decomposable graphs in terms of induced matching number and the number of cycles.

Theorem 4.2.1. If $G$ is a vertex decomposable graph, then

$$\text{reg}(I(G)) \leq \nu(G) + 1 + \min\{W_5(G), C_4(G) + C_5(G)\}$$

where $W_5(G)$ denotes the number of cycles of length 5 in $G$, $C_4(G)$ and $C_5(G)$ denote the number of induced 4-cycles and induced 5-cycles in $G$.

For a vertex $x$ of $G$, the neighborhood of $x$ in $G$ is the induced subgraph of $G$ consisting of all vertices adjacent to $x$ and all edges connecting two such vertices. We denote the neighborhood of $x$ by $N(x)$ and $N(x) \cup \{x\}$ by $N[x]$. With these notations, we can define a vertex decomposable graph in the following way and it is equivalent to Definition 2.3.4. We will use this equivalent definition in our proofs of the following section.

Definition 4.2.2. A graph $G$ is called vertex decomposable if $G$ is totally disconnected, or if

1. there is a vertex $x$ s.t. $G \backslash \{x\}$ and $G \backslash N[x]$ are both vertex decomposable, and
2. no independent set in $G \setminus N[x]$ is a maximal independent set in $G \setminus \{x\}$.

To show Definition 4.2.2 is indeed equivalent to Definition 2.3.4 we prove the following lemma.

**Lemma 4.2.3.** Let $G$ be a graph with its independence complex $\Delta$. Then any facet of $\operatorname{del}_\Delta(x)$ is a facet of $\Delta$ if and only if no independent set in $G \setminus N[x]$ is a maximal independent set in $G \setminus \{x\}$.

**Proof.** It is easy to see that the independence complex of $G \setminus \{x\}$ is $\operatorname{del}_\Delta(x)$ and the independence complex of $G \setminus N[x]$ is $\operatorname{link}_\Delta \{x\}$.

Suppose there is an independent set $S = \{x_1, x_2, \ldots, x_n\} \subseteq G \setminus N[x]$ such that $S$ is a maximal independent set in $G \setminus \{x\}$. Then $S$ is a facet of $\operatorname{del}_\Delta(x)$. This implies $S$ is a facet of $\Delta$ since any facet of $\operatorname{del}_\Delta(x)$ is a facet of $\Delta$. Thus, $S$ is a maximal independent set in $G$. But $S \cup \{x\}$ is an independent set in $G$, which is a contradiction.

On the other hand, suppose $S = \{x_1, x_2, \ldots, x_n\}$ is a facet of $\operatorname{del}_\Delta(x)$. Then $S$ is a maximal independent set in $G \setminus \{x\}$. Thus, there exists some $x_i \in S$ such that $x_i \in N(x)$. Otherwise, $S$ is an independent set in $G \setminus N[x]$ that is a maximal independent set in $G \setminus \{x\}$. So $S$ is a maximal independent set in $G$ and $S$ is a facet of $\Delta$.

\[ \square \]

The following inductive equality is one of the key ingredients in our proof.

**Theorem 4.2.4.** ([38, Theorem 1.5]) Let $H$ be a simple hypergraph with edge ideal $I = I(H)$. Suppose that $H$ is vertex decomposable and $x$ is the initial in its shedding order. Then

\[
\operatorname{reg}(I) = \max\{\operatorname{reg}(I : x) + 1, \operatorname{reg}(I, x)\}.
\]

This theorem allows us to do inductions on the number of vertices by deleting the shedding vertices. In fact, the following inductive bound holds for any hypergraph.
For a subset $V$ of the vertices in a hypergraph $H$, let $H : V$ and $H + V$ denote the hypergraphs corresponding to the squarefree monomial ideals $I(H) : x^V$ and $I(H) + x^V$, respectively.

**Theorem 4.2.5.** (1) Let $\Delta$ be a simplicial complex and let $\sigma$ be a face of dimension $d - 1$ in $\Delta$. Then

$$\text{reg}(\Delta) \leq \max\{\text{reg}(\text{link}_\Delta(\sigma)) + d, \text{reg}(\text{del}_\Delta(\sigma))\}.$$ 

(2) Let $H$ be a simple hypergraph and let $V$ be a collection of $d$ vertices in $H$. Then

$$\text{reg}(H) \leq \max\{\text{reg}(H : V) + d, \text{reg}(H + V)\}.$$ 

Before proving our main result, Theorem 4.2.1, we shall prove the following two lemmas which are key ingredients to our induction process used in Theorem 4.2.2.

**Lemma 4.2.6.** Suppose that $G$ is a vertex decomposable graph containing $n$ 5-cycles. Let $x$ be a shedding vertex and $G'' = G \setminus N[x]$. If $W_5(G'') = n$, then $\nu(G'') + 1 \leq \nu(G)$.

**Proof.** First we consider the case when $x$ has only one neighbor, say $y$. If $x \neq z \in N(y)$, then every maximal independent set containing $z$ in $G \setminus N[x]$ is a maximal independent set in $G \setminus \{x\}$. This contradicts the fact that $x$ is a shedding vertex. So $N(y) = x$, that is, $\{x, y\}$ is a single edge. In this case, $\nu(G'') + 1 = \nu(G)$. Now we suppose $N(x) = \{y_1, \ldots, y_t\}$, $t \geq 2$. We claim that there is a vertex $y_i \in N(x)$ such that $N[y_i] \subseteq N[x]$. We prove this claim by contradiction. A picture may help illustrate the following argument.
Assume that for each $1 \leq i \leq t$, there is a vertex $w_i \in N(y_i) \cap (G \setminus N[x])$. If \{w_i, w_j\} is an edge for some $1 \leq i, j \leq t$, then $x - y_i - w_i - w_j - y_j - x$ is a $W_5$, which is a contradiction. So \{w_1, \ldots, w_t\} is an independent set in $G \setminus N[x]$. Thus, every maximal independent set containing \{w_1, \ldots, w_t\} is a maximal independent set in $G \setminus \{x\}$. This contradicts the fact that $x$ is a shedding vertex. The claim is proved.

For every induced matching $C$ in $G''$, $C \cup \{x, y_i\}$ is an induced matching in $G$. So, $\nu(G'') + 1 \leq \nu(G)$.

\begin{lemma}
Let $G$ be a vertex decomposable graph with a shedding vertex $x$. Let $G'' = G \setminus N[x]$. If $C_4(G'') + C_5(G'') = C_4(G) + C_5(G)$, then $\nu(G'') + 1 \leq \nu(G)$.
\end{lemma}

\begin{proof}
We claim that there is a vertex $y \in N(x)$ such that $N[y] \subseteq N[x]$. We prove this claim by contradiction. Suppose that for any $y \in N(x)$, $N[y] \cap (G \setminus N[x]) \neq \emptyset$. Let $S_1 = N(x)$ and $S_2 = N(S_1) \setminus S_1$. Then $S_2 \neq \emptyset$. It suffices to find an independent set in $G \setminus N[x]$ such that every vertex in $N(x)$ is adjacent to a vertex in that independent set.

Assume that this is not the case, that is, for every independent set in $G \setminus N[x]$, there is some vertex in $N(x)$ that is not adjacent to any vertex in that independent set. A picture may illustrate the following argument.

[Diagram of graph with vertices and edges]
Let $I \subset S_2 \subset G \setminus N[x]$ be an independent set such that the number of its neighbors in $S_1$ is of maximum size. Then there exists a vertex $v \in S_1$ such that \{v, w\} is not an edge for any $w \in I$. But $v$ must have a neighbor that is not a neighbor of $x$. Otherwise, $N[v] \subset N[x]$. Let $v_1 \in S_2 \setminus I$ be a neighbor of $v$. Then there is some $v_2 \in I$ such that \{v_1, v_2\} is an edge. Otherwise, the independent set $I \cup \{v_1\}$ has more neighbors in $S_1$ since $v \in N(v_1)$ but $v \notin N(I)$.

If for any $v_2 \in N(v_1) \cap I$, \{v_2, v_3\} is an edge for some $v_3 \in S_1$, then \{v_1, v_3\} must be an edge. Suppose \{v_1, v_3\} is not an edge. If \{v, v_3\} is an edge, then $v - v_1 - v_2 - v_3$ is a $C_4$. Otherwise, $x - v - v_1 - v_2 - v_3$ is a $C_5$. So, if we replace $N(v_1) \cap I$ by $v_1$, then we have an independent set contained in $S_2$ that is adjacent with more vertices in $S_1$. This contradicts our assumption. So there exists an independent set such that every vertex in $S_1$ is adjacent to a vertex in that independent set. But this contradicts the fact that $x$ is a shedding vertex. The claim is proved. For every induced matching $C$ in $G''$, $C \cup \{x, y\}$ is an induced matching in $G$. So, $\nu(G'') + 1 \leq \nu(G)$. \hfill \□

Now we are ready to prove Theorem 4.2.1.

\textit{Proof of Theorem 4.2.1}. We first prove that $\text{reg}(I(G)) \leq \nu(G) + 1 + W_5(G)$. We do induction on the number of vertices. If $|V(G)| = 2$, then $G$ is totally disconnected or a single edge. Hence, $\text{reg}(I(G)) \leq \nu(G) + 1 + W_5(G)$. Suppose $G$ is a vertex decomposable graph containing $n W_5$s with $|V(G)| > 2$ and the result holds for each vertex decomposable graph with smaller values of $|V(G)|$. Since $G$ is vertex
decomposable, there exists a shedding vertex $x$ such that both $G' = G\{x\}$ and $G'' = G\setminus N[x]$ are vertex decomposable and

$$\text{reg}(G) = \max\{\text{reg}(G'), \text{reg}(G'') + 1\}.$$ 

**Case 1:** $G'$ and $G''$ contain less than $n$ 5-cycles. Then

$$\text{reg}(G'') \leq \nu(G) + 1 + n - 1 = \nu(G) + n$$

and

$$\text{reg}(G') \leq \nu(G) + 1 + n - 1 = \nu(G) + n.$$ 

In this case,

$$\text{reg}(G) \leq \nu(G) + 1 + n.$$ 

**Case 2:** $G'$ contains $n$ cycles of length 5 but $G''$ contains less than $n$ 5-cycles. Then

$$\text{reg}(G'') \leq \nu(G) + 1 + n - 1 = \nu(G) + n$$

and

$$\text{reg}(G') \leq \nu(G) + 1 + n.$$ 

In this case,

$$\text{reg}(G) \leq \nu(G) + 1 + n.$$ 

**Case 3:** $G'$ contains $n$ cycles of length 5 and $G''$ contains $n$ cycles of length 5. By Lemma 3.2, $\nu(G'') + 1 \leq \nu(G)$. So

$$\text{reg}(G'') \leq \nu(G'') + 1 + n \leq \nu(G) + n.$$
and
\[ \text{reg}(G') \leq \nu(G) + 1 + n. \]

In this case,
\[ \text{reg}(G) \leq \nu(G) + 1 + n. \]

The same argument with replacing \( W_5(G) \) by \( C_4(G) + C_5(G) \) allows us to prove that \( \text{reg}(I(G)) \leq \nu(G) + 1 + C_4(G) + C_5(G) \).

As a corollary we can recover results of Khosh-Ahang and Moradi in [39] and Biyikoglu and Civan in [40] as follows.

**Corollary 4.2.8.** Let \( G \) be a vertex decomposable graph.

1. ([39, Theorem 2.4]) If \( G \) contains no 5-cycles, then
\[ \text{reg}(G) = \nu(G) + 1, \]
where \( \nu(G) \) is the induced matching number of \( G \).

2. ([40, Theorem 24]) If \( G \) contains no induced 5-cycles or 4-cycles, then
\[ \text{reg}(G) = \nu(G) + 1, \]
where \( \nu(G) \) is the induced matching number of \( G \).

**Proof of Corollary 4.2.8.** By Lemma 2.2 in [47], \( \text{reg}(G) \geq \nu(G) + 1 \). The corollary follows from Theorem 4.2.1.

In fact, for a number of classes of graphs, the regularity can be computed by its induced matching number.

**Theorem 4.2.9.** Let \( G \) be a simple graph. Let \( \nu(G) \) be the induced matching number...
of $G$. Then

$$\text{reg}(G) = \nu(G) + 1$$

in the following cases:

1. $G$ is a sequentially Cohen-Macaulay bipartite graph (see [62]);
2. $G$ is an unmixed bipartite graph (see [53]);
3. $G$ is a very well-covered graph (see [57]);
4. $G$ is a $W_5$-free vertex decomposable graph (see [39]);
5. $G$ is a $(C_4, C_5)$-free vertex decomposable graph (see [40]).

**Remark 4.2.10.** Chordal and sequentially Cohen-Macaulay bipartite graphs are vertex decomposable. Also, bipartite graphs are $W_5$-free. Therefore, in the above theorem, (1) follows from (4).

**Example 4.2.11.** The upper bound $\nu(G) + 1 + W_5(G)$ in Theorem 4.2.2 does not apply for any graph in general. Suppose $G$ consists of the following two disconnected components.

![Figure 4.2. A graph with reg(G) > \nu(G) + 1 + W_5(G).](image)
$G$ is not vertex decomposable. $\nu(G) = 2 + 2 = 4$, $W_5(G) = 1$, but $\text{reg}(G) = 7 > \nu(G) + 1 + 1$.

**Example 4.2.12.** The upper bound in $\nu(G) + 1 + C_4(G) + C_5(G)$ in Theorem 4.2.2 does not apply for any graph in general. Suppose $G$ consists of the following two disconnected components.

![Graph](image.png)

Figure 4.3. A graph $G$ with $\text{reg}(G) > \nu(G) + 1 + C_4(G) + C_5(G)$.

$G$ is not vertex decomposable. $\nu(G) = 4$, $C_5(G) + C_4(G) = 1$, but $\text{reg}(G) = 7 > \nu(G) + 1 + 1$.

**Example 4.2.13.** Given any $0 \leq t \leq s$, we can construct a vertex decomposable graph $G$ such that $C_4(G) + C_5(G) = s$ and $\text{reg}(G) = \nu + 1 + t$.

If $s = 2n$, then we take $G$ to be $n$ disconnected copies as follows:

![Graph](image.png)

Then $\nu(G) = 2n$, $\text{reg}(G) = 2n + 1 = \nu(G) + 1$. 
If we replace one copy by the following graph,

then $C_4(G) + C_5(G)$ and $\nu(G)$ remain the same, but $\text{reg}(G)$ increases by one.

And if we replace one copy by the following graph,

then $C_4(G) + C_5(G)$ and $\nu(G)$ remain the same, but $\text{reg}(G)$ increases by two. Since all these disconnected components are vertex decomposable, we can construct a vertex decomposable graph $G$ such that $\text{reg}(G) = \nu(G) + 1 + t$ with these components.

If $s = 2n + 3$, then we take $G$ to be $n$ disconnected copies as follows:

and one copy as follows:
then $\nu(G) = 2n + 3$, $\text{reg}(G) = 2n + 4 = \nu(G) + 1$. If we replace one copy by the following graph,

then $C_4(G) + C_5(G)$ and $\nu(G)$ remains the same, but $\text{reg}(G)$ increases by one. And if we replace one copy by the following graph,

then $C_4(G) + C_5(G)$ and $\nu(G)$ remain the same, but $\text{reg}(G)$ increases by two. We can construct a vertex decomposable graph $G$ such that $\text{reg}(G) = \nu(G) + 1 + t$ with these components.

In addition, Woodrofe in [24] proved that if $G$ is a graph with no induced cycles of length other than 3 or 5, then $G$ is vertex decomposable. Now we consider the graphs containing only $C_3$ and $C_5$ and we provide an upper bound and a lower bound for the regularity of this class of graphs.
**Definition 4.2.14.** Let $G$ be a simple graph. A leaf in $G$ is a vertex of degree 1.

**Theorem 4.2.15.** If $G$ is a graph containing only $C_3$ or $C_5$, then

$$\nu + 1 + c \leq \text{reg}(G) \leq \nu + 1 + C_5(G) - b,$$

where $c$ is the number of isolated $C_5$ and $b$ is the number of induced 5-cycles whose neighbors contains exactly one leaf.

**Example 4.2.16.** Suppose $G$ is a graph consisting of two disconnected components as follows.

![Figure 4.4](attachment:image.png)

We see that $G$ has one isolated $C_5$ and one $C_5$ whose neighbors contains exactly one leaf. And the induced matching number of $G$ is 4. So, according to Theorem 4.2.16, we have $6 \leq \text{reg}(G) \leq 7$. Using Macaulay2, we compute that $\text{reg}(G) = 6$.

**Proof of Theorem 4.2.16.** Let $G = G_1 \cup G_2$, where $G_1$ represents $c$ isolated 5-cycles and $G_2$ represents the remaining component. For one isolated $C_5$, $\nu(C_5) = 1$ and $\text{reg}(C_5) = 3$. Thus, for $c$ isolated $C_5$’s,

$$\nu(G_1) = c$$

and according to [11, Lemma 8], for disjoint union of $c$ $C_5$’s, we have

$$\text{reg}(G_1) = \nu(G_1) + 1 + c.$$
For the remaining component, \( \text{reg}(G_2) \geq \nu(G_2) + 1 \). So we have

\[
\text{reg}(G) = \text{reg}(G_1) + \text{reg}(G_2) - 1 \geq \nu(G) + 1 + c.
\]

Let \( x \) be a vertex of some \( C_5 \) such that \( x \) is adjacent to a leaf \( y \). Then \( x \) is a shedding vertex because \( N[y] \subseteq N[x] \). Let \( G' = G \setminus x \) and \( G'' = G' \setminus N[x] \). We do induction on the number of vertices. For \( |V(G)| = 2 \), \( G \) is totally disconnected or a single edge. Thus, \( \text{reg}(G) = 1 = \nu(G) + 1 \) or \( \text{reg}(G) = 2 = \nu(G) + 1 \). Suppose that \( G \) is a graph containing only \( C_3 \) or \( C_5 \) with \( |V(G)| > 2 \) and the result holds for each graph containing only \( C_3 \) or \( C_5 \) with smaller values of \( |V(G)| \).

Then we have

\[
\text{reg}(G) \leq \max\{\nu(G') + 1 + C_5(G') - b(G'), \nu(G'') + 1 + C_5(G'') - b(G'') + 1\}.
\]

Since \( C_5(G) - C_5(G') = b(G) - b(G') \) and \( \nu(G') \leq \nu(G) \), we have

\[
\nu(G') + 1 + C_5(G') - b(C_5') \leq \nu(G) + 1 + C_5(G) - b(G).
\]

And since \( C_5(G) - C_5(G'') \geq b(G) - b(G'') \) and \( \nu(G'') + 1 \leq \nu(G) \), we have

\[
\nu(G'') + 1 + C_5(G'') - b(C_5'') \leq \nu(G) + 1 + C_5(G) - b(G).
\]

So,

\[
\text{reg}(G) \leq \nu(G) + 1 + C_5 - b.
\]
4.3 Regularity of Edge Ideals of Hypergraphs

We have seen in the previous section that one can bound the regularity in terms of the induced matching number. But the problem is more subtle when moving to hypergraphs. Indeed, Example 4.1.9 shows that for hypergraphs, the matching number is no longer the right invariant to bound the regularity. As our proof is based on mathematical induction, we need to find combinatorial invariants of hypergraphs that respect induction processes.

We use the number of cycles to provide upper bounds for regularity of vertex decomposable graphs. Here we generalize the notion of cycles for hypergraphs and use it to give an upper bound on regularity of a special class of vertex decomposable hypergraphs.

**Definition 4.3.1.** Let $H$ be a simple hypergraph. A $n$-cycle in $H$ is defined to be $x_1 E_1 \cdots x_n E_n$, where $x_i$s are distinct vertices, $E_i$s are distinct edges, and $x_1, x_n \in E_n$, $x_i, x_{i+1} \in E_i$ for $1 \leq n - 1$. We consider $x_1' E_1 \cdots x_n' E_n$ and $x_1 E_1 \cdots x_n E_n$ as the same $n$-cycle.

Here we denote a $n$-cycle of a hypergraph by $W_n$.

**Example 4.3.2.** Let $H = \{\{x_1, x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5, x_6\}, \{x_1, x_6\}\}$ be a hypergraph. Then $H$ is a 4-cycle.

We also define another invariant of a hypergraph which we will use to provide an upper bound on regularity.

**Definition 4.3.3.** Let $H$ be a simple hypergraph. We define $b(H)$ by

$$b(H) = \max\{|\bigcup_{i=1}^k E_i| - k\mid \text{ the induced subhypergraph of } H \text{ on } \bigcup_{i=1}^k E_i \text{ only contains } E_1, \ldots, E_k\}.$$
We give an upper bound for the regularity of a special class of vertex decomposable hypergraph.

**Theorem 4.3.4.** Suppose that \( H \) is a vertex decomposable hypergraph and for any two edges \( E_i \) and \( E_j \), \( |E_i \cap E_j| \leq 1 \). Then

\[
\text{reg}(I(H)) \leq W_5 + b(H) + 1.
\]

where \( W_5(H) \) is the number of 5-cycles defined in Definition 4.3.1 and \( b(H) \) is the invariant defined in Definition 4.3.3.

Let \( H' \) denote the hypergraph corresponding to the squarefree monomial ideal \( (I(H),x) \). Then it is easy to see that \( H' \) is a hypergraph whose edges are the edges of \( H \) not containing \( x \). Let \( H'' \) be the hypergraph corresponding to the squarefree monomial ideal \( (I(H) : x) \). Then it is also easy to see that \( H'' \) is the hypergraph whose edges are the minimal elements of \( \{ E \setminus \{ x \} \mid E \text{ is an edge of } H \} \).

Before proving Theorem 4.3.4, we shall prove a lemma first.

Let \( W_5'' = W_5(H'') \), \( W_5' = W_5(H') \) and \( W_5 = W_5(H) \). Let \( b = b(H) \), \( b' = b(H') \) and \( b'' = b(H'') \).

**Lemma 4.3.5.** Let \( H \) be a vertex decomposable hypergraph with the condition that for any two edges \( E_i \) and \( E_j \), \( |E_i \cap E_j| \leq 1 \). Let \( x \) be the shedding vertex of \( H \). If \( W_5 = W_5'' \), then \( b'' \leq b - 1 \).

**Proof.** Suppose \( E_1, \ldots, E_m \) are all the edges of \( H \) containing \( x \) and \( e_1, \ldots, e_n \) are all the edges of \( H \) not containing \( x \). Let \( E_i' = E_i \setminus \{ x \} \), \( 1 \leq i \leq m \). Then the edges of \( H'' \) are minimal elements of \( \{ E_1', \ldots, E_m', e_1, \ldots, e_n \} \).

We claim that if \( \{ e_1, \ldots, e_k \} \) is a subset of edges of \( H'' \) such that \( \cup_{i=1}^k e_i \) only contains \( e_1, \ldots, e_k \), then there exists an edge \( E_i \) of \( H \) such that \( \{ e_1, \ldots, e_k, E_i \} \) is a
subset of edges in $H$ such that $(\cup_{i=1}^{k}e_{i}) \cup E_{i}$ only contains $e_{1}, \ldots, e_{k}$ and $E_{i}$. We prove the claim by contradiction.

Suppose that for each edge $E_{i}$ of $H$, there is an edge $F_{i}$ of $H$ such that $F_{i} \subseteq (\cup_{i=1}^{k}e_{i}) \cup E_{i}$ and $F_{i} \neq e_{1}, \ldots, e_{k}, E_{i}$. Then $F_{i}$ cannot be an edge of $H''$. First we show that $x \notin F_{i}$ for all $i$. If $F_{i}$ contains $x$, then $F_{i} = E_{j}$ for some $1 \leq j \leq m$.

Thus, $E_{j} \setminus \{x\} = E_{j} \setminus E_{i} \subseteq \cup_{i=1}^{k}e_{i}$. This is a contradiction since $E_{j} \setminus \{x\}$ is an edge of $H''$. So $x \notin F_{i}$ for all $i$.

Then we show that for $\forall i \neq j, F_{i} \neq F_{j}$. Suppose that there exists some $E_{i}$ and $E_{j}$ such that $F_{i} = F_{j}$. Then

$$F_{i} \setminus (E_{i} \cap E_{j}) = (F_{i} \setminus E_{i}) \cup (F_{i} \setminus E_{j}) = (F_{i} \setminus E_{i}) \cup (F_{j} \setminus E_{j}) \subseteq \cup_{i=1}^{k}e_{i}.$$ 

Since $E_{i} \cap E_{j} = \{x\}$ and $x \notin F_{i}$, $F_{i} \subseteq \cup_{i=1}^{k}e_{i}$. Also, since $\cup_{i=1}^{k}e_{i}$ only contains $e_{1}, \ldots, e_{k}$ in $H''$, $F_{i}$ is not an edge of $H''$. Then there exists some $E'_{i} \subseteq F_{i} \subseteq \cup_{i=1}^{k}e_{i}$, which is a contradiction since $E'_{i}$ is an edge of $H''$.

Now let $S = \cup_{i=1}^{m}(F_{i} \setminus E_{i})$. We show that $S$ is an independent set of $H''$. Suppose not. Then there exists some edge $E$ of $H''$ such that $E \subseteq S \subseteq \cup_{i=1}^{k}e_{i}$. Then $E$ must be $e_{l}$ for some $1 \leq l \leq k$. This implies that $e_{l} = E \subseteq \cup_{i=1}^{m}(F_{i} \setminus E_{i})$. Thus, $E_{l}$ must intersect with at least two distinct edges $F_{i}$ and $F_{j}$. Otherwise, $e_{l} \subseteq F_{i} \setminus E_{i} \subseteq F_{i}$ for some $1 \leq i \leq m$.

Now we claim that $e_{l}F_{i}E_{i}E_{j}F_{j}e_{l}$ is a 5-cycle in $H$. Let $e_{l} \cap F_{i} = x_{1}$ and $e_{l} \cap F_{j} = x_{2}$. Since $F_{i} \cap E_{i} \neq F_{i} \cap (\cup_{i=1}^{k}e_{i})$ and $F_{j} \cap E_{j} \neq F_{j} \cap (\cup_{i=1}^{k}e_{i})$, we have

$$F_{i} \cap F_{i} = x_{3} \neq x_{1} \text{ and } F_{j} \cap E_{j} = x_{4} \neq x_{2}.$$ 

If $x_{3} = x_{4}$, then $E_{i} \cap E_{j}$ contains $x$ and $x_{3}$, which is a contradiction. So $x_{3}$ and $x_{4}$ are distinct.
Now we show that $x_1 \neq x_4$ and $x_2 \neq x_3$. Indeed, we show that $e_l \cap F_i \neq E_j \cap F_j$ for all $1 \leq l \leq k$. Suppose that there is some $e_l$ such that $e_l \cap F_i = E_j \cap F_j = \{y\}$. Then $y \notin E_i$ since $y \in E_j$. But $y \in F_i$. So

$$E_j \cap F_j = \{y\} \subseteq F_i \setminus E_i \subseteq \bigcup_{i=1}^k e_i.$$  

Also, $F_j \setminus (E_j \cap F_j) = F_j \setminus E_j \subseteq \bigcup_{i=1}^k e_i$. We have $F - j \subseteq \bigcup_{i=1}^k e_i$, which is a contradiction. So,

$$e_l \cap F_i \neq E_j \cap F_j \quad \forall 1 \leq l \leq k.$$  

Thus, $x_1 \neq x_4$ and $x_2 \neq x_3$. We have proved that $e_l x_1 F_i x_3 E_i x E_j x_4 F_j x_5 e_l$ is a 5-cycle in $H$. But it is not a 5-cycle in $H''$, which contradicts the assumption that $W_5 = W''_5$. So, $S$ is an independent set in $H''$.

Let $S_1$ be a maximal independent set of $H''$ containing $S$. Then $S_1$ is also a maximal independent set in $H'$. If not, then there is some maximal independent set $S_2$ in $H'$ containing $S_1$. Since $x$ is a shedding vertex, $S_2$ must contain some $E_i$s for $1 \leq i \leq m$. And,

$$F_i \setminus (E_i \cap F_i) = F_i \setminus E_i \subseteq S \subseteq S_1 \subseteq S_2.$$  

But $E_i \cap F_i \subseteq E'_i \subseteq S_2$. So $F_i \subseteq S_2$, which is a contradiction since $F_i$ is an edge in $H'$. Now, $S_1$ is a maximal independent set in $H''$ and also a maximal independent set in $H'$. This contradicts the fact that $x$ is a shedding vertex. The claim is proved.

Since $E'_i$ is an edge of $H''$, $E'_i \not\subseteq \bigcup_{i=1}^k e_i$. So, we have

$$\left|\bigcup_{i=1}^k e_i \cup E_i\right| - (k + 1) \geq \left|\bigcup_{i=1}^k e_i\right| + 2 - (k + 1) = \left|\bigcup_{i=1}^k e_i\right| - k + 1.$$
We are ready to prove Theorem 4.3.4.

Proof of Theorem 4.3.4. We do induction on the number of vertices. If $|V(H)| = 1$, then it is easy to see the result holds. Hence, $\text{reg}(I(H)) \leq W_5(H) + 1 + b(H)$. Suppose $H$ is a vertex decomposable hypergraph that satisfies the condition in the assumption and contains $n$ $W_5's$ with $|V(H)| > 1$, and the result holds for each such vertex decomposable graph with smaller value of $|V(H)|$. Since $H$ is vertex decomposable, there exists a shedding vertex $x$ such that both $G' = G\{x\}$ and $G'' = G\{N[x]\}$ are vertex decomposable and

$$
\text{reg}(H) = \max\{\text{reg}(H'), \text{reg}(H'') + 1\}.
$$

By induction, $\text{reg}(H) \leq \max\{W_5' + b' + 1, W_5'' + b'' + 2\}$. It is easy to see that $W_5' \leq W_5$ and $b' \leq b$. So, $\text{reg}(H') \leq W_5 + b + 1$.

Suppose $E_1, \ldots, E_m$ are all the edges of $H$ containing $x$ and $e_1, \ldots, e_n$ are all the edges of $H$ not containing $x$. Let $E'_i = E_i \{x\}$, $1 \leq i \leq m$. Then the edges of $H''$ are minimal elements of $\{E'_1, \ldots, E'_m, e_1, \ldots, e_n\}$.

If there is some $E_i = \{x, y\}$, then all the $e_i$s containing $y$ are not edges of $H''$.

Now we assume that the edges of $H'$ are $E'_1, \ldots, E'_m, e_1, \ldots, e_s$, where $s \leq n$. If $\{E'_1, \ldots, E'_i, e_1, \ldots, e_j\}$ is a subset of edges in $H''$ such that the induced subhypergraph of $H''$ on them only contains $E'_1, \ldots, E'_i, e_1, \ldots, e_j$, then $\{E_1, \ldots, E_i, e_1, \ldots, e_j\}$ is a subset of edges of $H$ such that the induced subhypergraph on $H$ on them only contains $E_1, \ldots, E_i, e_1, \ldots, e_j$.

Suppose not. Then it must contain some $E_l$ or $e_k$. If it contains some $E_l$, then $\bigcup_{k=1}^j E'_k \cup \bigcup_{k=1}^j e_k$ contains $E'_l$, which is a contradiction since $E'_l$ is an edge of $H''$. If it contains some $e_k$, then $s < k \leq n$ and $e_k$ must contain some $E'_j$ for $l < j \leq m$. 

Since $|e_k \cap E_j'| \leq 1$, we have $|E_j'| = 1$, say $E_j' = \{y\}$. Then

$$y \in (\cup_{k=1}^j E_k') \cup (\cup_{k=1}^j e_k).$$

Since $y \notin \cup_{k=1}^j E_k'$, we have $y$ must be contained in $\cup_{k=1}^j e_k$. Thus, $y \in e_k$ for some $1 \leq k \leq j$, which is a contradiction since all the $e_k$s containing $y$ are not edges of $H''$.

So, we have $b'' \leq b$. Also, it is easy to see $W_5'' \leq W_5$. If $W_5'' \leq W_5 - 1$, then

$$\text{reg}(H'') + 1 \leq W_5'' + b'' + 2 \leq W_5 + b + 1.$$

If $W_5'' = W_5$, according to Lemma 4.3.5, we have $b'' \leq b - 1$. So,

$$\text{reg}(H'') + 1 \leq W_5'' + b'' + 2 \leq W_5 + b + 1.$$

\[\square\]

**Example 4.3.6.** Let $H$ consists of edges $\{x_1, x_2, x_3\}$, $\{x_1, x_4, x_5\}$, and $\{x_1, x_6, x_7\}$. $H$ satisfies the condition in Theorem 4.3.4. $W_5(H) = 0$ and $b(H) = 4$. And we compute $\text{reg}(H) = 5$. In this case, we have $\text{reg}(H) = W_5(H) + b + 1$. 
Example 4.3.7. The result in Theorem 4.3.4 does not hold for any vertex decomposable hypergraph. Let $H$ consists of edges $\{x_1, x_2, x_3\}$, $\{x_1, x_3, x_4\}$, $\{x_2, x_4, x_6\}$, and $\{x_2, x_4, x_5\}$. We notice that $H$ does not satisfy the condition that the intersection of any two edges has cardinality no larger than 1. $W_5(H) = 0$ and $b(H) = 2$. We compute $\text{reg}(H) = 4 > W_5(H) + b(H) + 1$. 

Figure 4.5. The hypergraph in Example 4.3.6

Figure 4.6. The hypergraph in Example 4.3.7
Chapter 5

Projective Dimension

Projective dimension is another important invariant that measures the complexity of a module and we have seen that Theorem 2.6.3 relates the regularity with projective dimension via Alexander dual. In this chapter, we study projective dimension of squarefree monomial ideals. In particular, we provide an upper bound for projective dimension of hypergraphs in terms of a domination parameter.

5.1 Graph Domination Parameters and Bounds for Projective Dimension

In this section, we introduce several domination parameters and see how they can provide bounds for projective dimension.

We first recall a catalog of basic domination parameters. Let $G$ be a graph, and recall that a subset $A \subseteq V(G)$ is dominating if every vertex of $V(G) \setminus A$ is a neighbor of some vertex in $A$, that is, $N(A) \cup A = V(G)$.

(1) $\gamma(G) = \min\{|A| : A \subseteq V(G) \text{ is a dominating set of } G\}$.

(2) $i(G) = \min\{|A| : A \subseteq V(G) \text{ is independent and a dominating set of } G\}$.

(3) $\gamma_0(G) = \gamma_0(V(G), G)$. That is, $\gamma_0(G)$ is the least cardinality of a subset $A \subseteq V(G)$ such that every vertex of $G$ is adjacent to some $a \in A$. 

(4) \( \tau(G) = \max\{\gamma_0(A, G) : A \subseteq V(G) \text{ is independent}\} \).

The following proposition compares the domination parameters.

**Proposition 5.1.1.** ([42, Proposition 4.1]) *For any graph* \( G \), *\( i(G) \leq \gamma(G) \) and \( \tau(G) \leq \gamma(G) \).*

In [42], Dao and Schweig proved several bounds for the projective dimension of an arbitrary graph.

**Theorem 5.1.2.** ([42, Theorem 4.4]) *Let* \( G \) *be a graph without isolated vertices. Then*

\[
pd(G) \leq n - \tau(G),
\]

*where* \( n \) *is the number of vertices of* \( G \).

A lower bound involving \( i(G) \) was proved in the following theorem.

**Theorem 5.1.3.** ([42, Proposition 4.7]) *Let* \( G \) *be a graph on* \( n \) *vertices. Then*

\[
pd(G) \geq n - i(G).
\]

Also, they introduced a new graph domination parameter \( \epsilon \) and used that to give another upper bound. If \( E \) is the set of edges of \( G \), we say a subset \( F \) of \( E \) is *edgewise dominant* if any \( v \in G \) is adjacent to an endpoint of some edge \( e \in F \). We define

\[
\epsilon(G) = \min\{|F| : F \subseteq E \text{ is edgewise dominant}\}.
\]

**Theorem 5.1.4.** ([42, Theorem 4.3]) *Let* \( G \) *be a graph. Then*

\[
pd(G) \leq n - \epsilon(G),
\]

*where* \( n \) *is the number of vertices in* \( G \).
Example 5.1.5. The upper bounds in Theorem 5.1.2 and Theorem 5.1.4 are incomparable. Let $G$ be a 5-cycle. Note that $\epsilon(G) = 2$ and $\tau(G) = 1$.

\begin{center}
\begin{tikzpicture}
\node[vertex] (v1) at (0,0) {};
\node[vertex] (v2) at (1,0) {};
\node[vertex] (v3) at (2,0) {};
\node[vertex] (v4) at (1,1) {};
\node[vertex] (v5) at (1,-1) {};
\draw (v1) -- (v2) -- (v3) -- (v4) -- (v5) -- (v1);
\end{tikzpicture}
\end{center}

Figure 5.1. A graph $G$ with $\epsilon(G) > \tau(G)$.

Let $P_n$ denote the path on $n$ vertices. Then $\epsilon(P_4) = 1$ and $\tau(P_4) = 2$.

\begin{center}
\begin{tikzpicture}
\node[vertex] (v1) at (0,0) {};
\node[vertex] (v2) at (1,0) {};
\node[vertex] (v3) at (2,0) {};
\node[vertex] (v4) at (3,0) {};
\node[vertex] (v5) at (4,0) {};
\draw (v1) -- (v2) -- (v3) -- (v4) -- (v5);
\end{tikzpicture}
\end{center}

Figure 5.2. A graph $G$ with $\epsilon(G) < \tau(G)$.

As a corollary, we have

**Corollary 5.1.6.** ([42, Corollary 4.2]) Let $G$ be a graph. Then

$$n - i(G) \leq \text{pd}(G) \leq n - \max\{\epsilon(G), \tau(G)\}.$$ 

Moreover, strong bounds and an exact formula on projective dimension were proved in [42] for any hereditary class of graphs satisfying a certain dominating set condition.

**Theorem 5.1.7.** ([42, Theorem 5.1]) Let $f(G)$ be either $\gamma(G)$ or $i(G)$. Suppose $G$ is a hereditary class of graphs such that whenever $G \in \mathcal{G}$ has at least one edge, there is some vertex $v$ of $G$ with $f(G) \leq f(G \setminus v)$. Then for any $G \in \mathcal{G}$ with $n$ vertices,

$$\text{pd}(G) \leq n - \gamma(G).$$
If \( f(G) \) is \( i(G) \), then for any \( G \in (G) \) with \( n \) vertices, we have the equality

\[
\text{pd}(G) = n - i(G).
\]

A graph is **chordal** if every cycle of length at least 4 has a chord. The class of chordal graphs satisfies the conditions of Theorem 5.1.7, so we have the following corollary.

**Corollary 5.1.8.** ([42, Corollary 5.6]) If \( G \) is a chordal graph, then

\[
\text{pd}(G) = n - i(G),
\]

where \( n \) is the number of vertices in \( G \).

In fact, the formula for projective dimension in Corollary 5.1.8 holds for all sequentially Cohen-Macaulay graphs. This follows from a result on Cohen Macaulay complexes [63] and a theorem [50, Theorem 1.2] which shows that chordal graphs are sequentially Cohen-Macaulay.

Since all chordal graphs are perfect, one may ask if the same bound holds for perfect graphs. However, this is not true. Let \( G \) be a 4-cycle. Then \( \text{pd}(G) = 3 \), but \( n - i(G) = 4 - 2 = 2 \).

### 5.2 Generalization of Domination Parameters

Let \( I \subseteq R = k[x_1, \ldots, x_n] \) be a squarefree monomial ideal. By the Auslander-Buchsbaum formula, we have \( \text{pd}(R/I) \leq n \). The only nontrivial bound follows from a general result due to Faltings. Let \( \text{cd}(I) \) denote the cohomological dimension of an ideal \( I \).
**Theorem 5.2.1** ([51]). For any ideal \( I \subseteq R = k[x_1, \ldots, x_n] \),

\[
\text{cd}(I) \leq n - \left\lfloor \frac{n - 1}{\text{BigHeight}(I)} \right\rfloor.
\]

Since \( \text{cd}(I) = \text{pd}(R/I) \) for any squarefree monomial ideal, the above bound due to Falings applies to \( \text{pd}(H) \) for any simple hypergraph \( H \).

In last section, we have seen that two domination parameters provide upper bounds for projective dimension of graphs. In this section, we generalize these domination parameters for hypergraphs and use them to provide upper bounds for the projective dimension of any squarefree monomial ideal.

In [43], Dao and Schweig generalized the parameter \( \epsilon \) from graphs to hypergraphs and use it to provide a new upper bound on projective dimension.

**Definition 5.2.2.** Let \( H \) be a simple hypergraph. We call \( F \subseteq E(H) \) *edgewise dominant* if every vertex in \( V(H) \) not contained in some edge of \( F \) or contained in a trivial edge has a neighbor contained in some edge of \( F \).

We define \( \epsilon(H) \) by

\[
\epsilon(H) = \min\{|F| : F \subseteq E(H) \text{ is edgewise dominant}\}.
\]

An upper bound involving \( \epsilon \) was proved in the following theorem.

**Theorem 5.2.3.** ([43, Theorem 3.2]) For any simple hypergraph \( H \),

\[
\text{pd}(H) \leq n - \epsilon(H),
\]

where \( n \) is the number of vertices of \( H \).

We generalize the other domination parameter \( \tau \) from graphs to hypergraphs and use it to give a new upper bound.
**Definition 5.2.4.** For a simple hypergraph $H$, let $\beta(S, H) = \min\{|A| | A \subseteq V(H), S \subseteq N(A)\}$. We define $\alpha(H) := \max\{\beta(S, H) | S \text{ is an independent set of } H\}$.

**Theorem 5.2.5.** If $H$ is a hypergraph, then

$$\text{pd}(H) \leq n - \alpha(H)$$

where $n$ is the number of vertices of $H$.

**Remark 5.2.6.** In [63], Dao and Schweig also generalized the parameter $\tau$ from graphs to hypergraphs and give a similar upper bound.

The next example shows that the difference between projective dimension $\text{pd}(H)$ and the upper bound $n - \alpha(H)$ can be made arbitrarily large.

**Example 5.2.7.** For $k > 1$, consider the hypergraph $H_k = \{\{x, y_1, z_1\}, \ldots, \{x, y_k, z_k\}\}$. It is easy to see that the number of vertices in $H_k$ is $2k + 1$ and $\alpha(H) = 2$. On the other hand, we compute that $\text{pd}(H_k) = k$.

The following examples show that the upper bounds in Theorem 5.2.2 and 5.2.4 are incomparable.

![Figure 5.3. A hypergraph $H$ with $\epsilon(H) < \alpha(H)$.](image)
Example 5.2.8. Consider $H = \{\{x_1, x_2, x_3\}, \{x_2, x_4, x_5\}, \{x_3, x_5, x_6\}\}$ as in Figure 5.3.

Note that $\alpha(H) = \beta(\{x_1, x_4, x_5, x_6\}; H) = |\{x_2, x_3\}| = 2$. But $F = \{\{x_1, x_2, x_3\}\}$ is edgewise dominant, so $\epsilon(H) = 1$. We have $\text{pd}(H) \leq 6 - \alpha(H) = 4$.

Example 5.2.9. Consider $H = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_1\}\}$ Note that $\epsilon(H) = |\{\{x_1, x_2\}, \{x_3, x_4\}\}| = 2$ and $\alpha(H) = 1$. So $\text{pd}(H) \leq 5 - \epsilon(H) = 3$.

As a corollary, we have the following:

**Corollary 5.2.10.** If $H$ is a simple hypergraph, then

$$\text{pd}(H) \leq n - \max\{\alpha(H), \epsilon(H)\}$$

where $n$ is the number of vertices of $H$.

### 5.3 Proof of Main Result

Before proving Theorem 5.2.5, we shall prove the following lemma.

**Lemma 5.3.1.** Let $H$ be a simple hypergraph. If $S$ is an independent set of $H$ realizing $\alpha(H)$ and $A \subseteq V(H)$ is a subset such that $\alpha(H) = |A|$ and $S \subseteq N(A)$, then there exists a vertex $w \in A \setminus S$.

**Proof.** Let $S = \{x_1, \ldots, x_n\}$ be an independent set of $H$ realizing $\alpha$. Let $A \subseteq V(H)$ be such that $|A| = \alpha$ and $S \subseteq N(A)$. If $A$ contains some vertex $w \notin S$, then we pick $w$. If this is not the case, that is, $A$ is a subset of $S$, we claim that such a vertex can be always picked by replacing either $S$ or $A$.

We start with the case when $A \subseteq S$. Without loss of generality, let $A = \{x_1, \ldots, x_k\}, k \leq n$. If there exists some $x_i \in A$ such that $x_j \in N(x_i), x_j \notin N(A \setminus x_i)$,
and $S \setminus x_j \subseteq A \setminus x_i$ for some $j$, then the edge containing $x_i$ and $x_j$ must contain some $w \notin S$ since $S$ is an independent set. Thus, we can replace $x_i$ by $w$.

Now we suppose $\forall x_i \in A$, there exist more than one vertices in $S$ that are not contained in the neighbors of $A \setminus x_i$, say $\{x_{i1}, \ldots, x_{ij}\} \subseteq S$. If there exists some $w \notin S$ such that $\{x_{i1}, \ldots, x_{ij}\} \subseteq N(w)$, then we can replace $x_i$ by $w$. If this is not the case, then $\forall w \notin S$, $\{x_{i1}, \ldots, x_{ij}\} \notin N(w)$.

Let $w_{i1}$ be such that $x_{i1} \in N(w_{i1})$. Then $S_1 = S \setminus \{x_{i2}, \ldots, x_{ij}\} \subseteq N((A \setminus x_1) \cup w_{i1}) = N(w_{i1}, x_2, x_3, \ldots, x_k)$. If $\beta(S_1, H) = (A \setminus x_1) \cup w_{i1}$, then $\alpha = \beta(S_1, H)$. Thus, we can replace $S$ by $S_1$. If $\beta(S_1, H) < \left|(A \setminus x_1) \cup w_{i1}\right| = \alpha$, then $\beta(S_1, H) = \alpha - 1$. Otherwise, $\alpha = \beta(S, H) < \alpha - 1 + 1 = \alpha$,

which is a contradiction.

Let $A_1 \subseteq H$ be s.t. $\beta(S_1, H) = |A_1|$ and $S_1 \subseteq N(A_1)$. If $A_1$ contains some $w \notin S$, then $S \subseteq N(A_1 \cup x_1)$. We can replace $A$ by $A_1$ and pick $w$.

If $A_1$ contains no $w \notin S$, say $A_1 = \{x_2', \ldots, x_{k'}\}$. WLOG, we can assume that $x_2' = x_{k+1}$ and $x_{i1} \in N(x_{k+1})$. Then $A_1 \cup x_1$ is a subset of vertices s.t. $|A_1 \cup x_1| = \alpha$ and $S \subseteq N(A_1 \cup x_1)$. We see that the number of vertices of $S$ that are only contained in the neighbors of $x_1$ is smaller since $S \setminus \{x_{i2}, \ldots, x_{ij}\} \subseteq N(A_1)$.

We continue the process until there is only one such vertex. Without the loss of generality, let $x_{ij}$ be that vertex. Since $x_{ij} \in N(w_{ij})$, we can replace $x_1$ by $w_{ij}$. So, the claim is proved.

Now we are ready to prove Theorem 5.2.5.

**Proof of Theorem 5.2.5.** Let $S$ be an independent set of $H$ realizing $\alpha(H)$ and $A \subseteq V(H)$ be a subset such that $\alpha(H) = |A|$ and $S \subseteq N(A)$. By Lemma 4.8, we pick a
vertex \( w \in A \setminus S \).

Suppose that \( H \) consists of edges \( E_1 \ldots, E_n, e_1, \ldots, e_m \), where \( w \in E_i \) and \( w \notin e_j \) for \( 1 \leq i \leq n \), and \( 1 \leq j \leq m \). Let \( H' \) consist of all the edges in \( H \) not containing \( w \) and \( H'' \) consist of the minimal elements(w.r.t inclusion) of \( \{E_1 \setminus w, \ldots, E_n \setminus w, e_1, \ldots, e_m\} \).

Then the associated simplicial complexes of \( H' \) and \( H'' \) are respectively \( \text{del}_{\Delta H}(w) \) and \( \text{link}_{\Delta H}(w) \). The above observation follows directly from the definitions. By Theorem 3.4 in [5], we have

\[
\text{pd}(H) \leq \max\{\text{pd}(H') + 1, \text{pd}(H'')\}.
\]

We proceed induction on the number of vertices. Suppose \( |V(H)| = 1 \). Then \( H \) is either a single vertex or a single edge. Thus, \( \text{pd}(H) = 0 \) or 1.

Suppose that the result holds for each hypergraph with smaller values of \( |V(H)| \). If \( S \subseteq H' \), then \( S \) is also an independent set of \( H' \). Let \( \alpha' = \alpha(H') \) and \( n' = |V(H')| \). Let \( A' \subseteq H' \) be s.t. \( |A'| = \beta(S, H') \) and \( S \subseteq N(A') \). Then \( A' \) is also a subset of vertices in \( H \) s.t. \( S \) is contained in the neighbors of \( A' \). Thus, \( \alpha = \beta(S, H) \leq |A'| = \beta(S, H') \leq \alpha' \).

Since \( n - n' \geq 1 \), we have

\[
1 + \text{pd}(H') \leq n' - \alpha' + 1 \leq n - \alpha,
\]

If \( S \not\subseteq H' \), then let \( B = S \setminus H' \). Since \( S \) is an independent set in \( H \), \( S \cap H' = S \setminus B \) is an independent set in \( H' \). Let \( A' \subseteq H' \) be s.t. \( \beta(S \setminus B, H') = |A'| \) and \( S \setminus B \subseteq N(A') \). Then \( \alpha = \beta(S, H) \leq |A'| + |B| \leq \alpha' + |B| \). Since \( w \notin S \), \( w \notin B = S \setminus H' \). Then \( n \geq n' + |B \cup \{w\}| = n' + |B| + 1 \).
Therefore,

\[ \text{pd}(H') + 1 \leq n' - \alpha' + 1 \leq n - \alpha. \]

Now we show that \( \text{pd}(H'') \leq n - \alpha \). Let \( n'' = |V(H'')| \) and \( \alpha'' = \alpha(H'') \).

Let \( B = S \setminus H'' \). If \( S \) does not contain any \( E_i \setminus w \) for \( 1 \leq i \leq n \), then \( S \cap H'' \) is an independent set in \( H'' \). Let \( A'' \subseteq H'' \) be s.t. \( \beta(S \cap H'', H'') = |A''| \) and \( S \cap H'' \subseteq N(A'') \). Then \( \alpha = \beta(S, H) \leq |A''| + |B| \leq \alpha'' + |B| \). Since \( n - n'' \geq |B| \), we have

\[ \text{pd}(H'') \leq n'' - \alpha'' \leq n - \alpha. \]

If \( S \) contains some \( E_i \setminus w \), then for each \( E_i \setminus w \) contained in \( S \), pick \( x_i \in E_i \setminus w \).

Let \( K = S \setminus \{x_i's\} \). Then \( K \cap H'' \) is an independent set in \( H'' \). Let \( A'' \subseteq H'' \) be s.t. \( \beta(K \cap H'', H'') = |A''| \) and \( K \cap H'' \subseteq N(A'') \). For each vertex in \( B \), choose a neighbor and let \( Y \) consist of all these neighbors. Since \( \{x_i's\} \subseteq N(w) \), \( S \subseteq N(A'' \cup Y \cup w) \).

Then

\[ \alpha = \beta(S, H) \leq |A''| + |B| + 1 \leq \alpha'' + |B| + 1. \]

Since \( n - n'' \geq |B| + 1 \), we have

\[ \text{pd}(H'') \leq n'' - \alpha'' \leq n - \alpha. \]
References


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Biography

Mengyao Sun was born and raised in Hefei, Anhui, China. She graduated from Anhui Normal University, Wuhu, China in 2011 with a Bachelor of Science, majoring in Mathematics. In the fall of 2011, she joined the Department of Mathematics at Tulane University to pursue a Doctor of Philosophy in Mathematics under the guidance of Dr. Tăi Huy Hà.