Numerical Models of the Effects of Fishing on Shark Populations

An Abstract
Submitted On The Twenty-Third Day Of May, 2014
To The Center for Computational Science
Of The Graduate School Of
Tulane University
In Partial Fulfillment Of The Requirements
For The Degree Of
Master of Science
By

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Abstract

A spatiotemporal system of partial differential equations is implemented for describing a marine predator-prey system of shark and prey fish. The model is developed to account for predator migration and for harvesting of both predator and prey animals. The Finite Difference Method is employed to develop a numerical model to describe the behavior of the system in space over time. The dynamics of the system for different initial conditions for predator and prey populations and harvesting rates of both predators and prey using the numerical scheme. The resulting dynamics of the system from adding a predator sanctuary (an area within which the predator cannot be harvested) are also examined. It is hoped that this paper will illustrate that model behaves as a predator-prey system is expected to behave under the tested conditions.

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Acknowledgments

The author would like to acknowledge the outstanding patience and invaluable assistance of Dr. Ricardo Cortez, Professor Caroline Taylor and Professor Hideki Fujioka with the research presented in this thesis. The author would also like to thank Dr. John Caruso for his contributions, and Stephanie Borrego for her tireless effort and assistance.

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Chapter 1

Introduction

1.1 Introduction to the Problem

The mathematical modeling of shark populations is not in and of itself a complicated or difficult task. Indeed, university-level mathematics students are introduced to rudimentary shark population models in their first course in ordinary differential equations as an introduction to Lotka-Volterra predator prey systems. However, while these rudimentary models serve as excellent examples of population modeling, several key ecological factors are usually left out for the sake of simplicity. This means that factors like the migratory patterns of sharks and their prey animals as well as the effects of human activity like fishing are not accounted for in these models. It will be the purpose of this thesis to present a mathematical model that accounts for these missing factors and therefore, give a more accurate representation of shark populations in the real world. To this end, a system of differential equations containing spatiotemporal dynamics based on the work of Garvie and Trenchea[1] will be introduced and solved using the Finite Difference Method.

It is hoped that the work presented in this thesis will serve as a validation of the work of the many marine biologists that have made the ecological study of sharks and shark populations their scientific focus. It also hoped that it may help to serve

as a point of advocacy for the conservation of sharks worldwide, as well as show the importance of marine wildlife sanctuaries.

1.2 Background Information

Sharks are generally regarded as the quintessential apex predator. Over millions of years the body of a shark has evolved to have virtually perfect hunting anatomy, especially their jaws and multiple rows of teeth [2]. Another key evolutionary trait that sharks possess are heightened senses of smell, sight, hearing and electroreception give them a considerable edge when hunting for prey [2]. The aerodynamics of their bodies also make them ideal aquatic animals, as it allows them to travel very efficiently through water [2]. Most species of shark are carnivores and have few natural predators other than other sharks. This is another key characteristic of an apex predator.

In spite of being excellent predators that under ideal circumstances would never approach extinction, worldwide shark populations are on the decline [3, 4]. There are several key factors that scientists believe to play into the recent, sudden decline in shark populations. Firstly, sharks, unlike types of fish, generally achieve sexual maturity later in life and produce a small number of offspring. This means that they are particularly vulnerable over the long term to human activity, in particular to wasteful fishing practices like bottom trawling or shark finning [3].

While virtually all species of shark are vulnerable to these fishing practices, the model presented in this thesis will focus on the species *Galeorhinus galeus*, also commonly known as the school shark or soupfin shark. A few key characteristics of this species of shark that make it a good choice for our model are:

- It is harvested by humans for its flesh, liver and fins [2]
- Some of its main prey animals like squid and sardines are also harvested by

humans [2]

- It is an oviviporous species of shark, meaning that eggs are fertilized and fetus develops in the womb of the female shark with a long gestation period (about one year) [2]
- A single female produces relatively few offspring (28-38 depending on the size of the shark) [2]
- The species has a history of being overfished [2].

In order to account for all of these characteristics, our model will have to utilize partial differential equations instead of the simple ordinary differential equations that are normally used to model predator-prey systems. This will be discussed in greater detail in the next section.

1.3 The Mathematics

As mentioned in Section 1, modeling a predator-prey system is not by itself especially difficult. The most rudimentary predator-prey system model model usually consists of a two body system of ODE's. Notable examples of such predator-prey systems are the Lotka-Volterra (LV) and Rosenzweig-MacArthur (RM) models [5, 6]. The model used in this thesis is more complex, incorporating spatial factors as well the temporal factors that are present in the more basic models. This means that unlike the LV and RM models, the system of equations used in this thesis will need to use partial differential equations which are more difficult to solve. In order to approximate the solution, a numerical scheme will be developed using the Finite Difference method.

The model used in this thesis will be based on the general form of a coupled pair of reaction-diffusion equations [1]. This will allow the model to specifically account

for population density of the sharks and their prey fish in different areas of water. Even though the movement of the animals does not necessarily follow a diffusion pattern, reaction-diffusion equations should still provide a good idea of the population dynamics. Modifications to the equations will also be implemented to more accurately model the movement of the two groups of animals. It will also be assumed that no predators or prey ever enter or leave the system except through birth and death. How this is achieved mathematically is covered in Chapter 2.

The equations will also eventually be modified to include a separate death rate for both the predators and prey in the system due to human activity. For the purposes of this thesis we will assume that the human population does not affect the model. This will allow us to account for human activity without adding a third body to the problem and thereby complicating it unnecessarily.

Chapter 2

System of Equations and Numerical Scheme

2.1 The System of Equations

As mentioned in the last section of Chapter 1, the system of equations upon which our model will be based is a system of reaction-diffusion equations that outline a predator-prey system. For the purposes of this project, the system we will be dealing with will be only for time $t > 0$ and location x such that $x \in [0, 1]$. The general form of the system of equations is [1]:

$$\frac{\partial u}{\partial t} = \delta_1 \triangle u + ru(1 - \frac{u}{w}) - pvh(ku) \tag{2.1a}$$

$$\frac{\partial v}{\partial t} = \delta_2 \triangle v + qvh(ku) - sv \tag{2.1b}$$

$$u(x, 0) = u_0(x) \text{ and } v(x, 0) = v_0(x) \tag{2.1c}$$

$$\frac{\partial u}{\partial \mathbf{v}} = \frac{\partial v}{\partial \mathbf{v}} = 0 \text{ at } x = 0 \text{ and } x = 1 \tag{2.1d}$$

where $u = u(x,t)$ and $v = v(x,t)$ are the population densities of the prey and predator animals respectively. Here $\mathbf{v}$ represents the outward normal to the boundary of the vector space x. δ_1 and δ_2 are diffusion constants for the prey and predator respectively. Here The constant k determines how fast the consumption rate saturates as the prey density increases. q and r denote maximal per-capita predator and prey birth rates, respectively. s is the per-capita predator death rate and w is the prey carrying capacity [1]. $h = h(z)$ is the functional response and satisfies

$$h(0) = 0 \text{ and } \lim_{z\to\infty} h(z) = 1$$

and is strictly increasing in the domain $[0, \infty)$. Here, h depends upon u and k such that

$$h(ku) = \tfrac{ku}{1+ku}$$

All parameters in Equations (2.1a) and (2.1b) are strictly positive. The imposition of Neumann Boundary Conditions will keep the rate of change of the predator and prey population densities at zeros at the boundaries of x. the local growth of the prey is logistic and the predator shows the "Holling type II functional response" [1].

While the model above sufficiently represents predator-prey population dynamics in a bounded space, it assumes that the animals' movement patterns can be described by diffusive movement alone. In the real world, this is only partially the case. If our model is to accurately represent real-world movement patterns, then migrations of species must be accounted for mathematically. For our purposes, we will assume for now that only the predator animal is migratory. Since the predator

animals in this model are assumed to migrate, we need to add a term to the predator equation to account for migratory patterns. To do this we add a partial derivative of v with respect to x modified by a simple oscillating time function $m(t)$ to the predator equation. In doing so, our model now has the general form:

$$\frac{\partial u}{\partial t} = \delta_1 \triangle u + ru(1 - \frac{u}{w}) - pvh(ku) \tag{2.2a}$$

$$\frac{\partial v}{\partial t} = \delta_2 \triangle v + qvh(ku) - sv + \frac{\partial v}{\partial x} m(t) \tag{2.2b}$$

$$u(x, 0) = u_0(x) \text{ and } v(x, 0) = v_0(x) \tag{2.2c}$$

$$\frac{\partial u}{\partial \mathbf{v}} = \frac{\partial v}{\partial \mathbf{v}} = 0 \text{ at } x = 0 \text{ and } x = 1 \tag{2.2d}$$

Here, $m(t)$ represents the speed of migration of the predators accross the space x.

Further, since we will be modeling a predator-prey in which both the predators and the prey are subject to harvesting (in this case by humans) we need to add a term to Equations (2.2a) and (2.2b) to account for this as well [7]. Since the harvesting of school sharks and anchovies depends largely on the location of human fishing vessels, that harvesting function (H) should depend on vector position in our bounded area. It is also reasonable to assume that the harvesting function should be different for both predators and prey, since sharks and anchovies are harvested at different rates in the real world. Adding the harvesting functions to the model, we can now rewrite the system of equations as

$$\frac{\partial u}{\partial t} = \delta_1 \triangle u + ru(1 - \frac{u}{w}) - pvh(ku) - H_u(x) \tag{2.3a}$$

$$\frac{\partial v}{\partial t} = \delta_2 \triangle v + qvh(ku) - sv + \frac{\partial v}{\partial x} m(t) - H_v(x) \tag{2.3b}$$

$$u(x,0) = u_0(x) \text{ and } v(x,0) = v_0(x) \tag{2.3c}$$

$$\frac{\partial u}{\partial \mathbf{v}} = \frac{\partial v}{\partial \mathbf{v}} = 0 \text{ at } x = 0 \text{ and } x = 1 \tag{2.3d}$$

Here $m(t)$ will be defined as a simple oscillating function of time (i.e. $b\sin(at)$) to allow for migration of the predators in both the positive and negative direction. $H_u(x)$ and $H_v(x)$ will be piecewise functions of the form $H_u(x) = \alpha_1$, $H_v(x) = \alpha_2$ where α_1 and α_2 are scalar constants such $\alpha_1 = \alpha_2 = 0 \ \forall x \notin [m,n] \in (0,\infty)$.

Now, we will tackle the issue of solving the system for u and v. Since solving the system directly would prove difficult, it will be necessary for our purposes to develop a numerical scheme to evaluate solutions.

2.2 Numerical Scheme

Many numerical methods exist for solving systems of PDE's like the one presented in the last section. For this particular system, we will make use of the Finite Difference Method to approximate solutions to the equations.

To start, we need to break up time and space into small intervals for which we can solve the system numerically. For the time variable we choose some small interval Δt such that $M = \frac{T}{\Delta t}$ and $t_i = \frac{iT}{M}$ for $i = 0, 1, \ldots, M$ where T is the final

time. Similarly, since $x \in [0,1]$ we choose some N and let $x_j = \frac{j}{N}$ for $j = 0, 1, \ldots, N$. If $u_j^i = u(t_i, x_j)$ and $v_j^i = v(t_i, x_j)$ (for $i \in (0, M)$ and $j \in (0, N)$) are solutions to u and v respectively at time index i and space index j, then the general form for a numerical scheme for our model can be written as

$$u_j^{i+1} = u_j^i + \frac{\delta_1 \Delta t}{\Delta x^2}(u_{j+1}^i - 2u_j^i + u_{j-1}^i) + \Delta t f_u(u, v) \tag{2.4a}$$

$$v_j^{i+1} = v_j^i + \frac{\delta_2 \Delta t}{\Delta x^2}(v_{j+1}^i - 2v_j^i + v_{j-1}^i) + \Delta t f_v(u, v) \tag{2.4b}$$

where f_u and f_v are parts of Equations (2.3a) and (2.3b) that follow the Laplacian term [8] and δ_1 and δ_2 are the diffusion constants of u and v respectively. Since this model is in a bounded space, the numerical scheme must account for the boundary conditions of the model. Since we are using an explicit method in such a space, it is necessary to create two fictional grid points for both equations at the left and right boundaries (x_{-1} and x_{N+1}). Since we have imposed Neumann boundary conditions (i.e. $\frac{\partial u}{\partial x}(x_0) = \frac{u_1^i - u_{-1}^i}{2\Delta x} = 0$), we can assume that the fictitious grid points will have the same value as the grid point on the inside of the boundary point [8].

For example, this means that u_{-1}^i and u_{N+1}^i would have the same value as u_1^i and u_{N-1}^i respectively and would make the numerical boundary conditions for u

$$u_0^{i+1} = u_0^i + \frac{\delta_1 \Delta t}{\Delta x^2}(2u_2^i - 2u_0^i) + \Delta t f_u$$
$$u_N^{i+1} = u_N^i + \frac{\delta_1 \Delta t}{\Delta x^2}(2u_{N-1}^i - 2u_N^i) + \Delta t f_u$$

Before we can write the numerical scheme out in its final form, we should first address how the term $\frac{\partial v}{\partial x} m(t)$ in the predator equation is dealt with as it is not entirely intuitive. Since the predators migrate back and forth across the domain of x according to the function $m(t)$, m has to positive for some values of t and negative

for others. This creates an issue for our numerical scheme as the approximation of $\frac{\partial v}{\partial x}$ changes based on whether $m(t_i)$ is positive or negative. To account for this we need to implemented a method known as upwind scheme. This allows to write the numerical solution for $\frac{\partial v}{\partial x}m(t)$ as

$$(\frac{\mathbf{sign}(m(t_i))+1}{2}\frac{v^i_{j+1}-v^i_j}{\Delta x} + \frac{1-\mathbf{sign}(m(t_i))}{2}\frac{v^i_j-v^i_{j-1}}{\Delta x})m(t_i)$$

Writing out f_u and f_v in their full form, we can write the numerical scheme for solving the system of equations as

$$\begin{aligned} u^{i+1}_j &= u^i_j + \frac{\delta_1 \Delta t}{\Delta x^2}(u^i_{j+1} - 2u^i_j + u^i_{j-1}) \\ &\quad + \Delta t \left(r u^i_j (1 - \frac{u^i_j}{w}) - p v^i_j h(k u^i_j) - H_u(x_j) \right) \end{aligned} \tag{2.5a}$$

$$\begin{aligned} v^{i+1}_j &= v^i_j + \frac{\delta_2 \Delta t}{\Delta x^2}(v^i_{j+1} - 2v^i_j + v^i_{j-1}) + \Delta t \left(q v^i_j h(k u^i_j) - s v^i_j \right) \\ &\quad + \Delta t \left(\frac{\mathbf{sign}(m(t_i))+1}{2}\frac{v^i_{j+1}-v^i_j}{\Delta x} + \frac{1-\mathbf{sign}(m(t_i))}{2}\frac{v^i_j-v^i_{j-1}}{\Delta x})m(t_i) - H_v(x_j) \right) \end{aligned} \tag{2.5b}$$

Now that we have a numerical method for our model, the next step is to choose values for the physical parameters in the model and run the model in MATLAB to see how the system behaves under different conditions. This process will be covered in detail in the next chapter. The values chosen for the physical parameters in this thesis are experimental and are meant to illustrate the dynamics of the model. Further research may dictate that the values for the parameters be modified to more accurately

reflect real-world parameters.

Chapter 3

Examining the Behavior of the System

3.1 Predators and Prey Alone

Part of the process of choosing parameter values for a mathematical model involves performing experiments with the predator or prey equation in the system "turned off" or with the initial conditions of that particular equation set to be zero at all initial positions. In such experiments, the outcome can be easily predicted, and if the experimental result is different from the desired result, the parameter values can be modified until the desired outcome is achieved. This section details a series of such experiments for the system of equations introduced in the last chapter. While this step may seem trivial, it is important to establish the model behaves correctly for conditions for which the desired result is already known. Table 3.1 shows a list of the parameters used in all experiments that do not change.

Prey Alone

The first experiment we conduct is to set the initial predator population density at all locations along the vector space to 0, leaving the prey free disperse along the space from their initial positions without being eaten. This means that the only

Parameter	Symbol	Value	Units
Maximal Consumption Rate of Prey	p	3	(Number of Prey Consumed/Unit Time)
Rate of Saturation of Consumption Rate	k	1	(Scalar)
Maximal Per Capita Birth Rate of Prey	r	5	(Prey Born/Unit Time)
Maximal Per Capita Birth Rate of Predator	q	1.5	(Predators Born/Unit Time)
Per Capita Death Rate of Predator	s	1	(Deaths / Unit Time)
Carrying Capacity of Prey	w	15	(Scalar)
Diffusion Coefficients	δ_1	0.05	(Scalar)
	δ_2	0.03	(Scalar)

Table 3.1: The values of these parameters are the same for all experiments.

parameters influencing the population density of the prey at any point in x will be the diffusion constant δ_1, the maximal per capita birth rate r, and the carrying capacity w. Since the growth of the prey population in our model is logistic, the expected outcome of this experiment is for the prey population density to grow until the carrying capacity is reached. Figure 3.1 shows the behavior of the model under these conditions.

Predators Alone

The second experiment examines the behavior the model when the initial prey population density at all locations along x is set to 0. Since positive predator population growth is determined entirely by consumption of prey, the expected result for this experiment is the predator population shrinking rapidly after dispersal and eventually reaching zero at all locations. Figure 3.2 shows the results of the experiment.

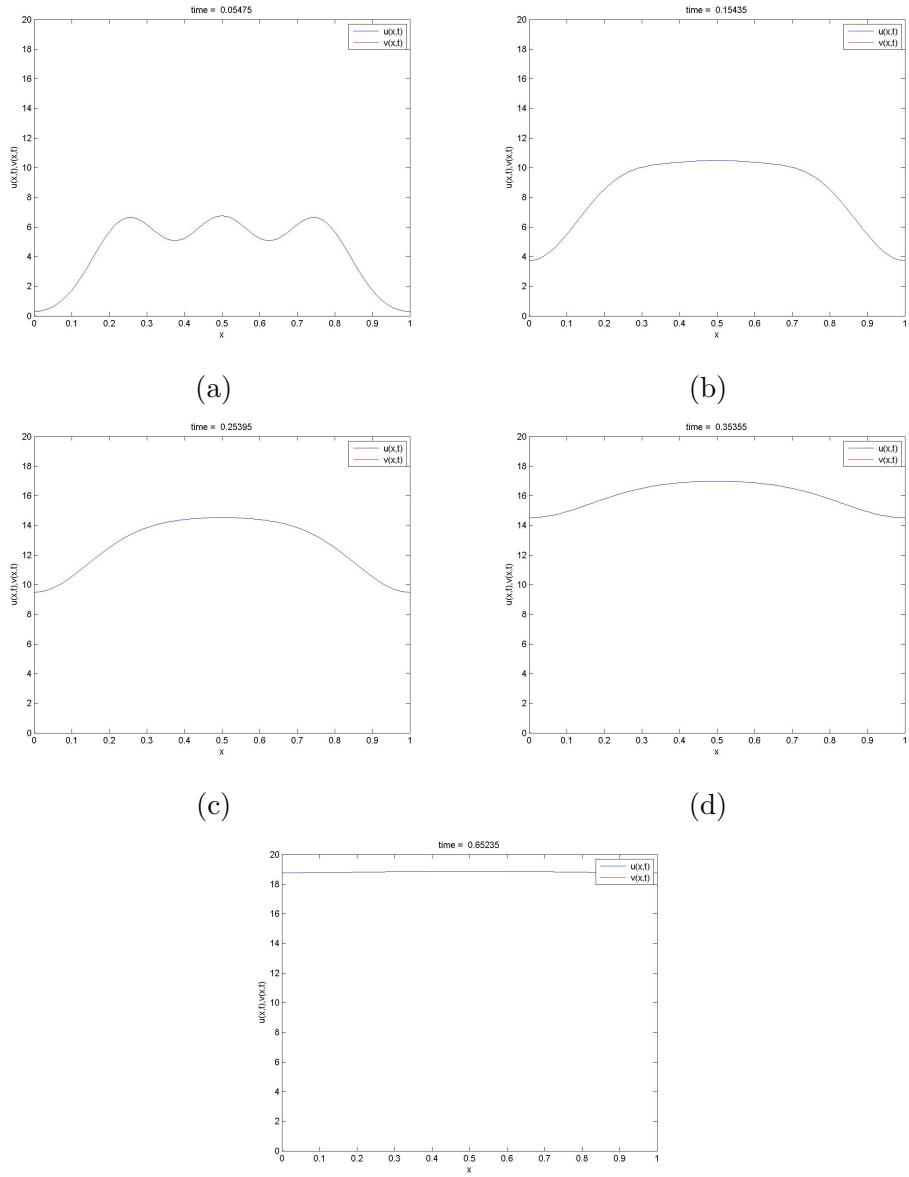


Figure 3.1: A-G show the behavior of the model over a hypothetical 1 month time period with initial predator population density and harvesting set to 0 at all locations on x. From an initial normal distribution with three peaks, the prey fish (population density shown in blue) diffuse evenly over x and grow consistently toward the carrying capacity. This behavior is expected from the model under these conditions, since the initial predator population and harvesting rates have been set to 0.

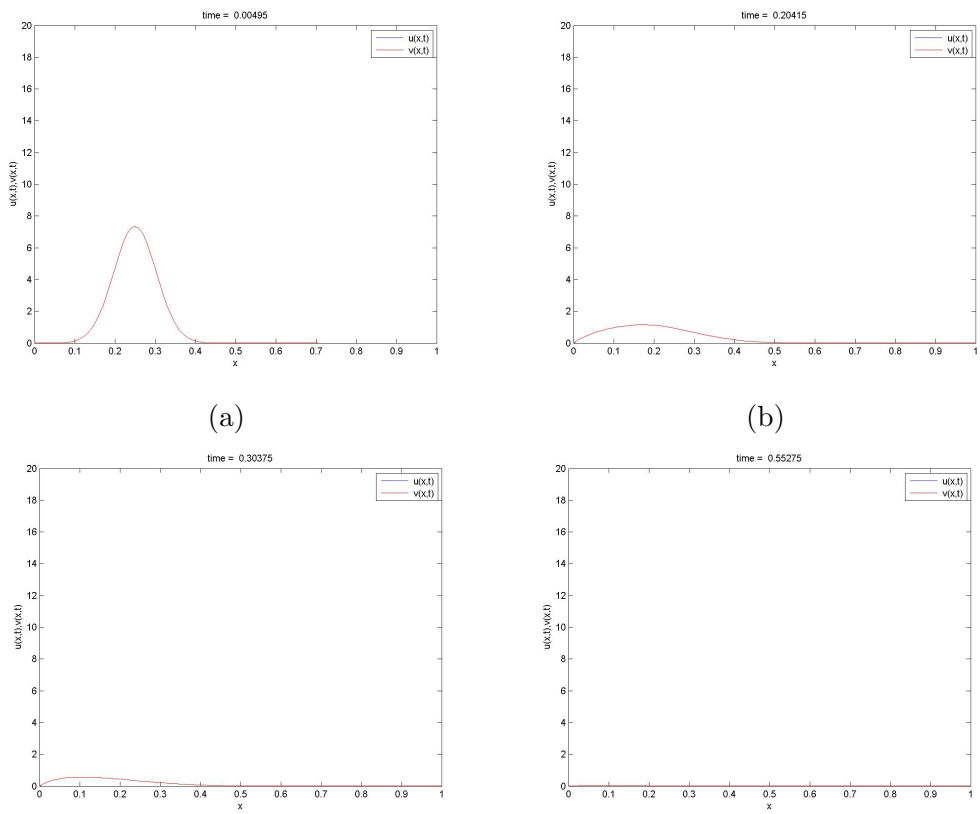


Figure 3.2: A-D show the behavior of the model from over a hypothetical 1 month time period with initial prey population density and harvesting set to 0 at all locations on x. From an initial normal distribution with a peak at the left of x, the predators (represented in red) disperse and migrate across x. With no prey animals to sustain positive population growth, the populations density of the predators eventually shrinks to 0 at all locations on x.

3.2 Predators and Prey Without Harvesting

Constant and Non-Constant Predator and Prey Distributions

Before we examine the dynamics of the full model, it is important to examine the model without its spatial components to see if there exist any equilibrium points other than $u = 0$ and $v = 0$. To do this, we set the derivatives of x in both equations equal to zero so that we are left with a system ODE's:

$$u'(t) = ru(1 - u/w) - pvh(ku)$$
$$v'(t) = (qh(ku) - s)v$$

Obviously there exists an equilibrium point at $(u, v) = (0, 0)$, but another can be found by setting $qh(ku) = s$ in the second equation. Once this is done, the equation can be reduced to

$$u(t) = \frac{\left(\frac{s}{kq}\right)}{\left(\frac{s}{q}\right) - 1} = \frac{s}{(k(q - s))}$$

If this value for u is plugged back into the first equation we can find the equilibrium value for v:

$$v(t) = \frac{\left(r\left(1 + \frac{s}{(q - s)}\right)\left(\frac{s}{k(q - s)} - w\right)\right)}{kpw}$$

If we plug in our parameter values, we find that the equilibrium value of u is 2 and the equilibrium value of v is $\frac{13}{2}$. This means that if an initial population density of the predators an d prey distribution are set to these values, the result should be that the population density of both predators and prey do not change at any location on x. To see if the system behaves this way with our parameters, we can the trapezoid rule trapezoid rule to integrate $u(x, t)$ and $v(x, t)$ to find the total number of predators and prey in the system at any time t. Figure 3.3 shows that the result result of

this experiment. We expect these solution to be flat lines. If the initial population densities were set to value other than the equilibrium values, the dynamics of the system would go from being constant to being periodic.

Now we can move forward by running other experiments with predators and prey in the system. The first of these experiments used a constant initial distribution of 9 for the prey animals and 2 for the predators. The expected outcome for this particular experiment is for the population density functions should cycle up and down as the predators and prey without reaching an equilibrium, as was alluded to earlier.

Next, the initial distribution of predators and prey animals is altered so that the population density is higher in specific areas of x; the prey animals have three high density locations ($x = (.25, .5, .75)$) and the predators have one at $x = .25$. Figure 3.5 shows a the total number of predators and prey present in the system at all times during the experiment, measured out to a final time of 80 hypothetical months. The resulting cycling population densities in x are reflective of a real world predator-prey system in which the predator population shrinks and grows as prey become more scarce of plentiful. Figure 3.6 shows the resulting dynamics of the experiment.

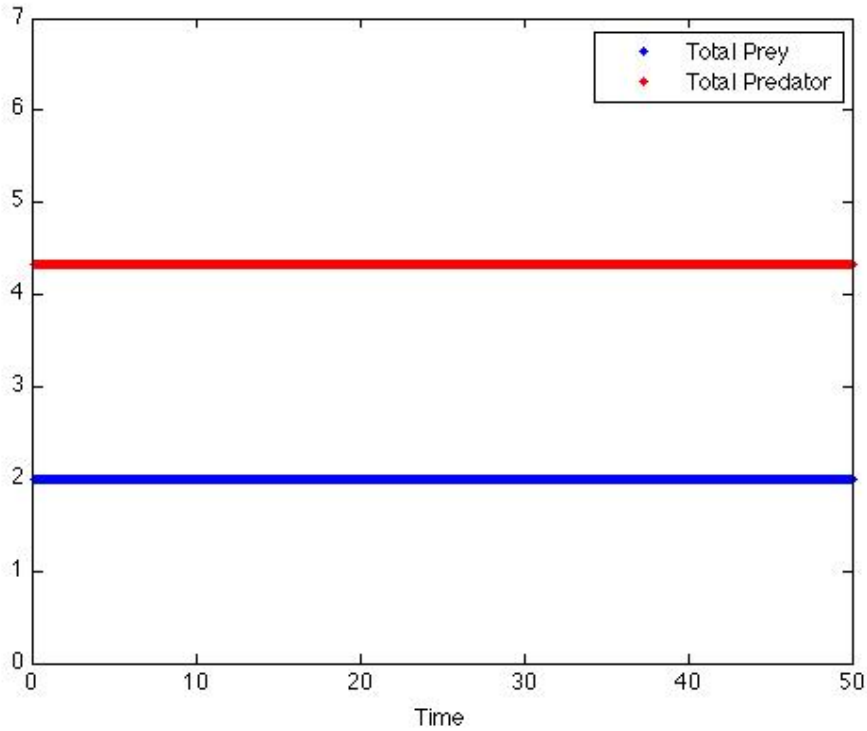


Figure 3.3: Using the trapezoid rule, we can integrate $u(x,t)$ and $v(x,t)$ on the interval of $(0,1)$ to find the total number of predators and prey in the system at all times. Here the integral indicate that when $u(x,0)$ and $v(x,0)$ are set to 2 and 13/3 respectively, the population densities of both predators and prey do not change and the system remains at equilibrium. Time is measured in months.

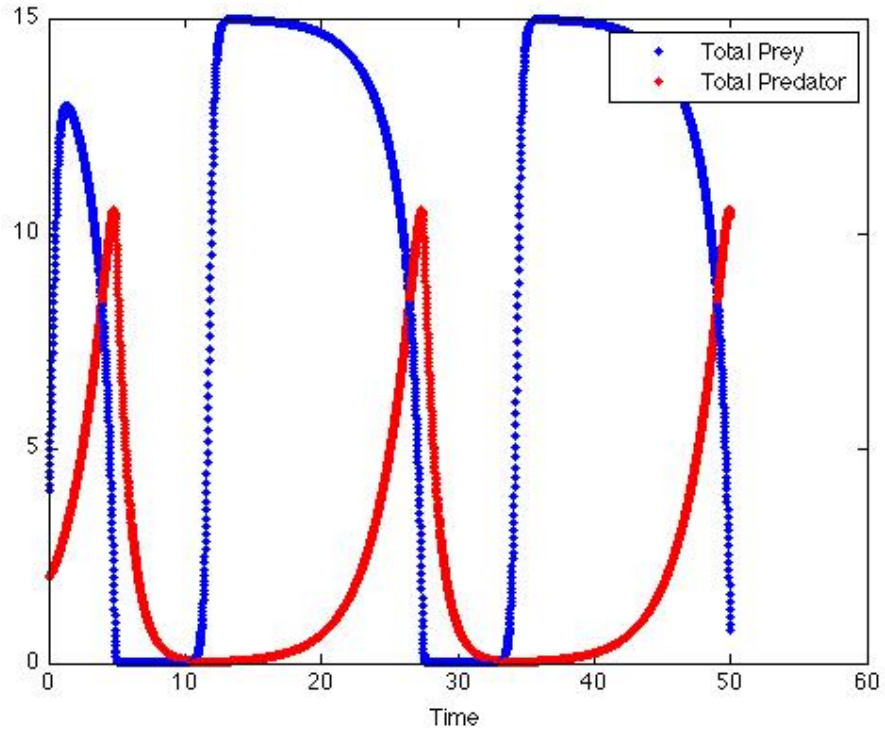


Figure 3.4: When the initial conditions of u and v are set to constant distributions of 9 and 2 respectively, a periodic dynamic emerges in the system. The cycle continues without reaching an equilibrium.

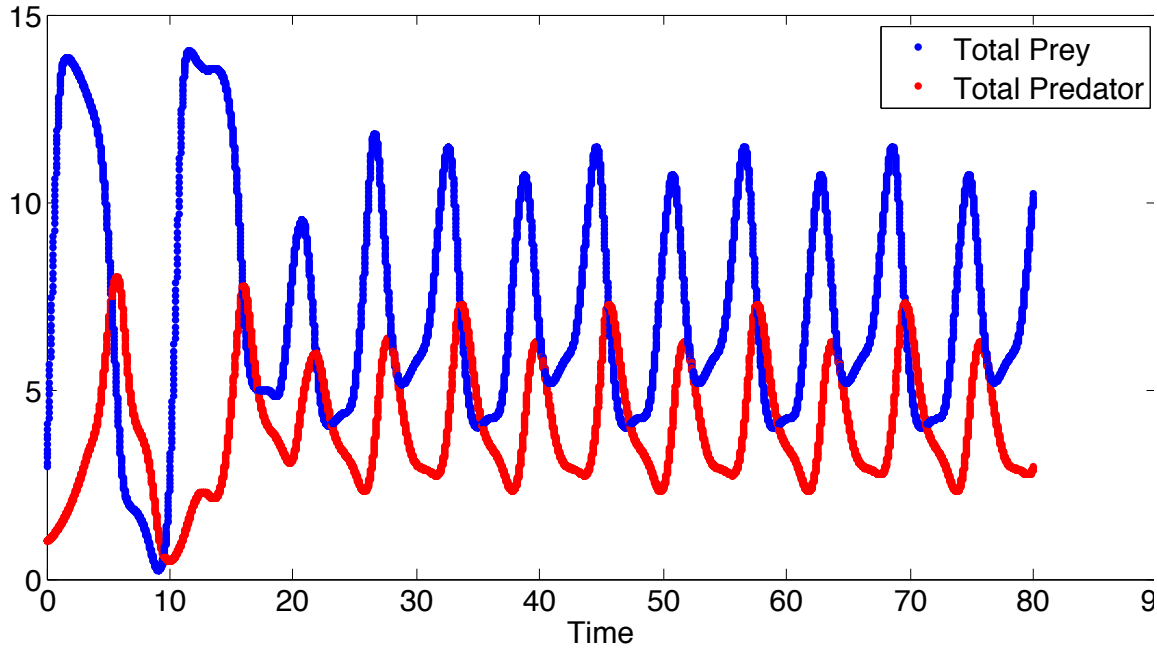


Figure 3.5: Solving for the integral of the differential equations allows us to see the total number of predators and prey in the system at all times. Here we see that with the harvesting functions set to zero at all locations, the predators and prey exhibit a population cycle that is expected in a predator-prey system. The cycle does not appears to settle into a periodic pattern every two peaks.

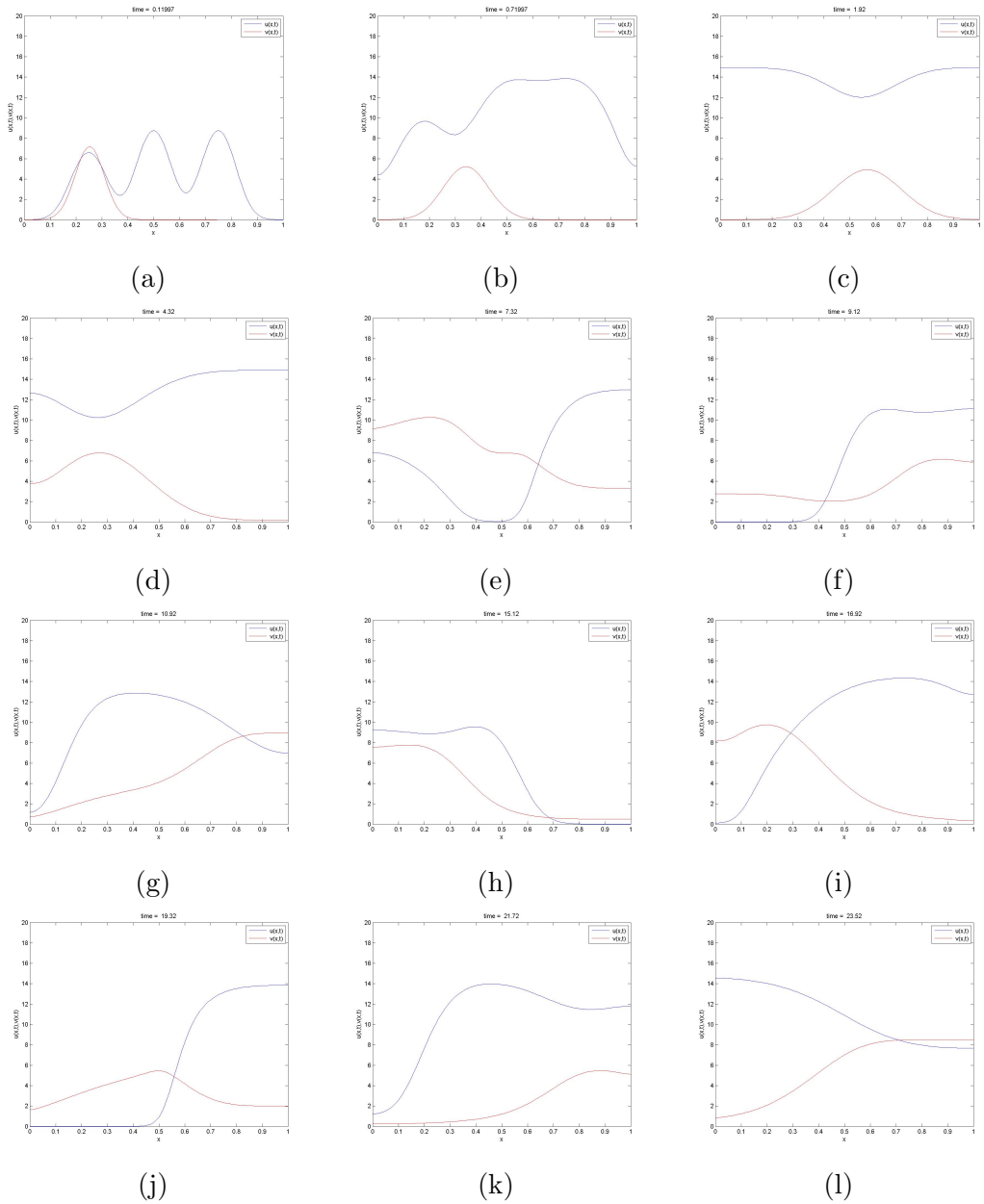


Figure 3.6: A-L show the behavior of the model from over a hypothetical 23 month time period with the harvesting rates set to 0 at all locations on x. Prey population density is represented in blue and predator population density is represented in red. From their initial distributions (identical to those used in the previous two experiments) both predators and prey diffuse across x and the predators migrate over time. The prey animals reproduce quickly and their population density rises quickly early in the experiment. As the prey become more plentiful, the predators begin to consume more of them until the predators outnumber the prey in some locations. As this occurs, the predators have trouble finding prey and their population slowly decreases. This allows the prey animals to replenish their population. The cycle continues as time is added to the experiment.

Migration of Predators

In previous experiments, the predator migration function $m(t)$ was initially set to $-.25\sin(t\frac{\pi}{2})$ (i.e. $b = .25$ and $a = \pi/2$). Other parameters were tried for this function to see if any other set would allow for more healthy predator-prey interactions. However, experimentation with the parameter values of $m(t)$ showed that these parameter values were optimal because they allowed for the most healthy recovery of prey populations after hunting by predators reduced the population density to 0 at certain locations.

3.3 Harvesting Predator or Prey

Now that we know how the model behaves without harvesting of predator and prey, it is time to conduct experiments in the system where the harvesting of rate of prey and/or predator animals is some positive constant. Predictably, when harvesting is added to the model in which either the predator or prey have initial population densities of zero at all locations on x the results are the same as in the preliminary tests for predators and prey alone. The only difference is that the carrying capacity for the prey animals is slightly reduced and predator population density reaches 0 at all locations more quickly. The next series of experiments will examine the behavior of the model when both predators and prey are present in the system, but one species is harvested at a "sustainable" or "unsustainable" rate.

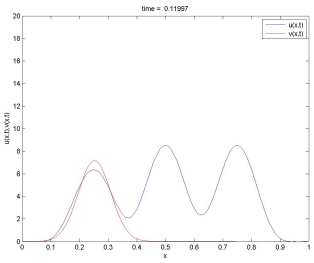
Prey Harvesting

The next experiment examines the system when the prey are harvested at sustainable and unsustainable rates and predators are not harvested at all. An important characteristic of this experiment is that the harvesting rate of prey is constant at all locations on x. x therefore, can be thought of as a region of ocean the entirety of which is fished for the prey animals.

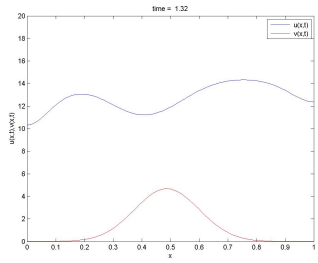
When H_u is set to a sustainable rate of $\alpha_1 = 2$ at all locations, the expected result is a predator-prey population cycle that looks similar to the system without harvesting implemented. The behavior of the system with a sustainable harvest rate of the prey animals is shown in Figure 3.7. The expected result for an unsustainable harvesting rate ($\alpha_1 = 3$) of the prey animals is that the population densities of both predators and prey eventually reaches 0 at all points on x. Figure 3.8 shows the behavior of our model under such conditions.

Predator Harvesting

Next we examine how the system behaves with only harvesting of predators implemented. Once again, the entire region of x will be fished at a constant rate at all locations. When H_v is set to a sustainable rate of $\alpha_2 = 0.47$ at all locations, the expected result is, once again, a predator prey population density cycle similar to that found in section 3.2 with no harvesting implemented. Figure 3.9 show the behavior of the model under such conditions. The expected result for unsustainable H_v of $\alpha_2 = 0.5$ is similar to the result found in section 3.1 when the prey animals grow to their carrying capacity. The results of the system behavior under these conditions are show in Figure 3.10.



(a)



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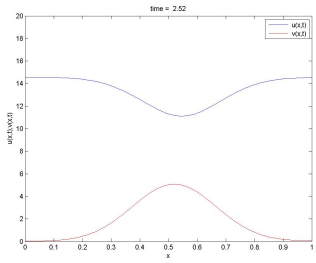

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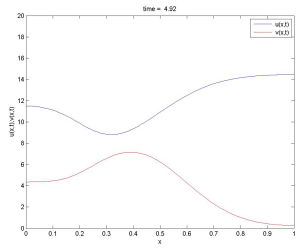

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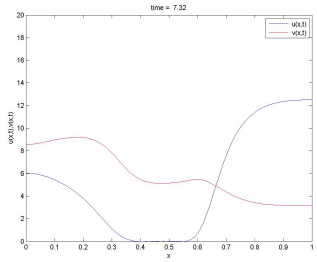

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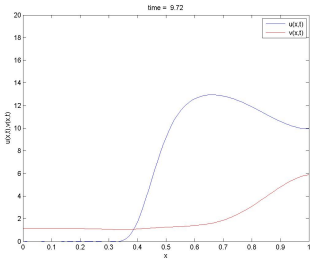

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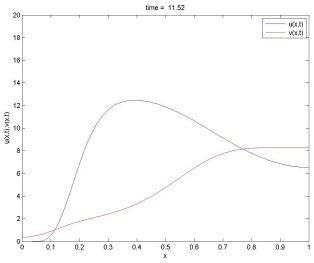

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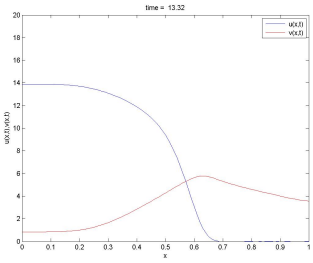

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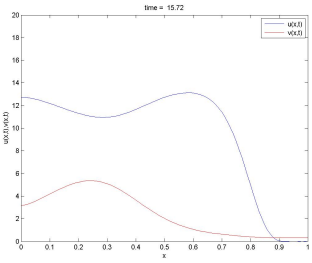

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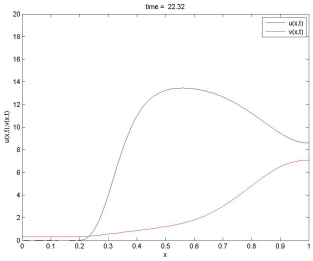

(j)

Figure 3.7: A-J show the behavior the system of equations over a hypothetical 23 month time period with a sustainable fishing rate for H_u at all locations on x. A population density cycle similar to the one shown in Figure 3.6 is achieved, as the fishing rate is not so aggressive that it prevents predators from finding prey, or kill off the population of prey fish.

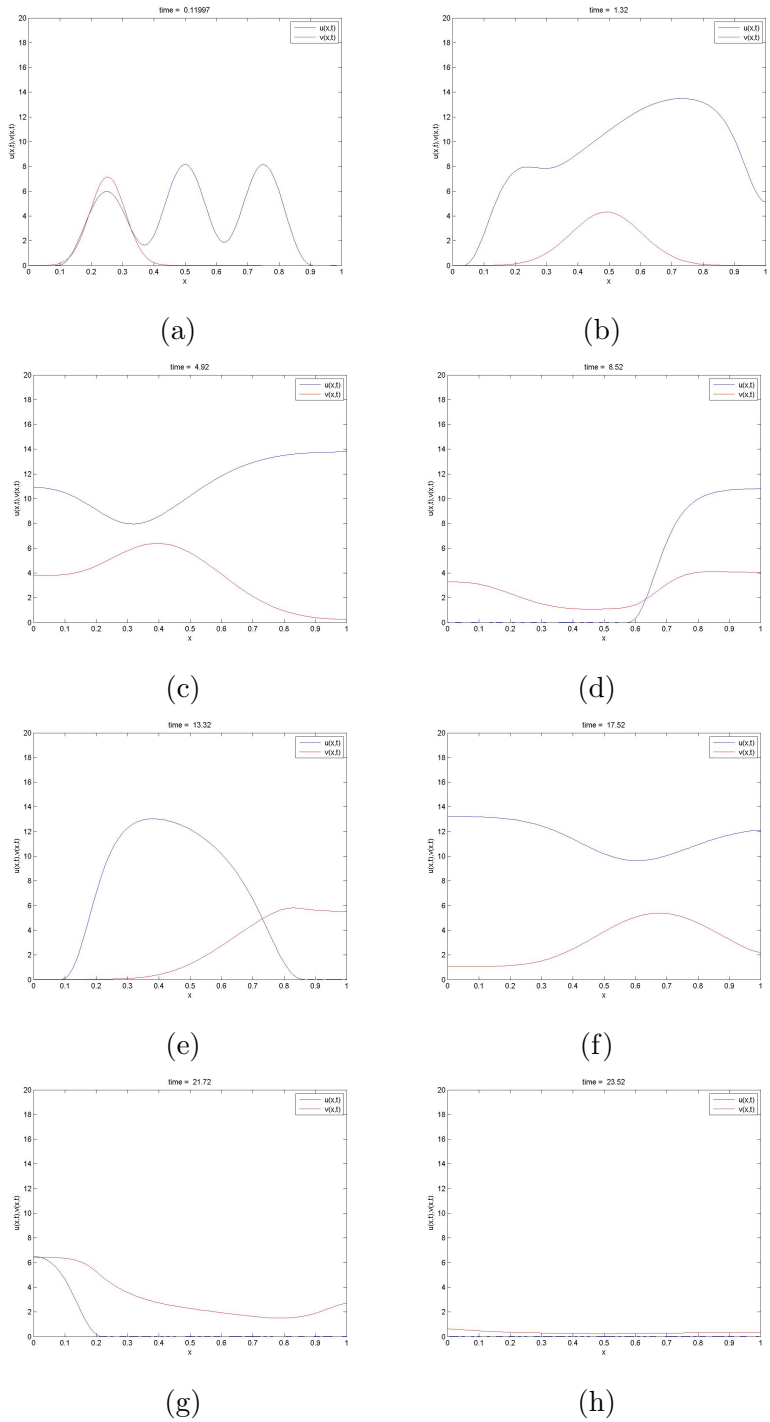

Figure 3.8: A-H show the behavior the system of equations the same hypothetical 23 month period with a unsustainable fishing rate for H_u at all locations on x. The overfishing of the prey population causes the population density of the prey to eventually shrink to 0 at all locations. As a result, the predator population soon suffers the same fate.

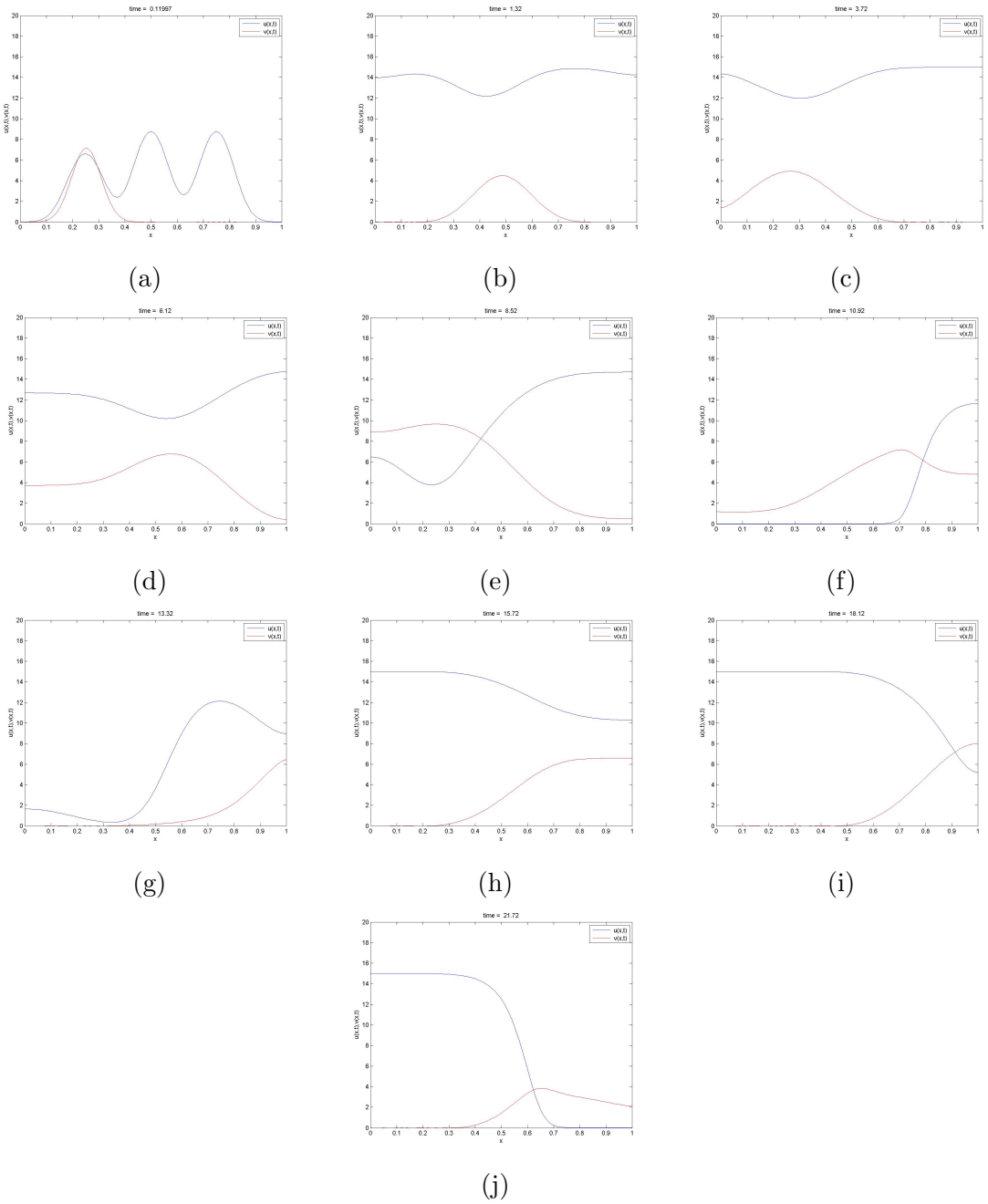

Figure 3.9: A-J show the behavior the system of equations over the hypothetical 23 months with a sustainable fishing rate for H_v at all locations on x. A population density cycle similar to the one shown in Figure 3.6 is achieved as the fishing rate is not so aggressive that it kills off the population of predators faster than the predators can consume prey and rebuild their population.

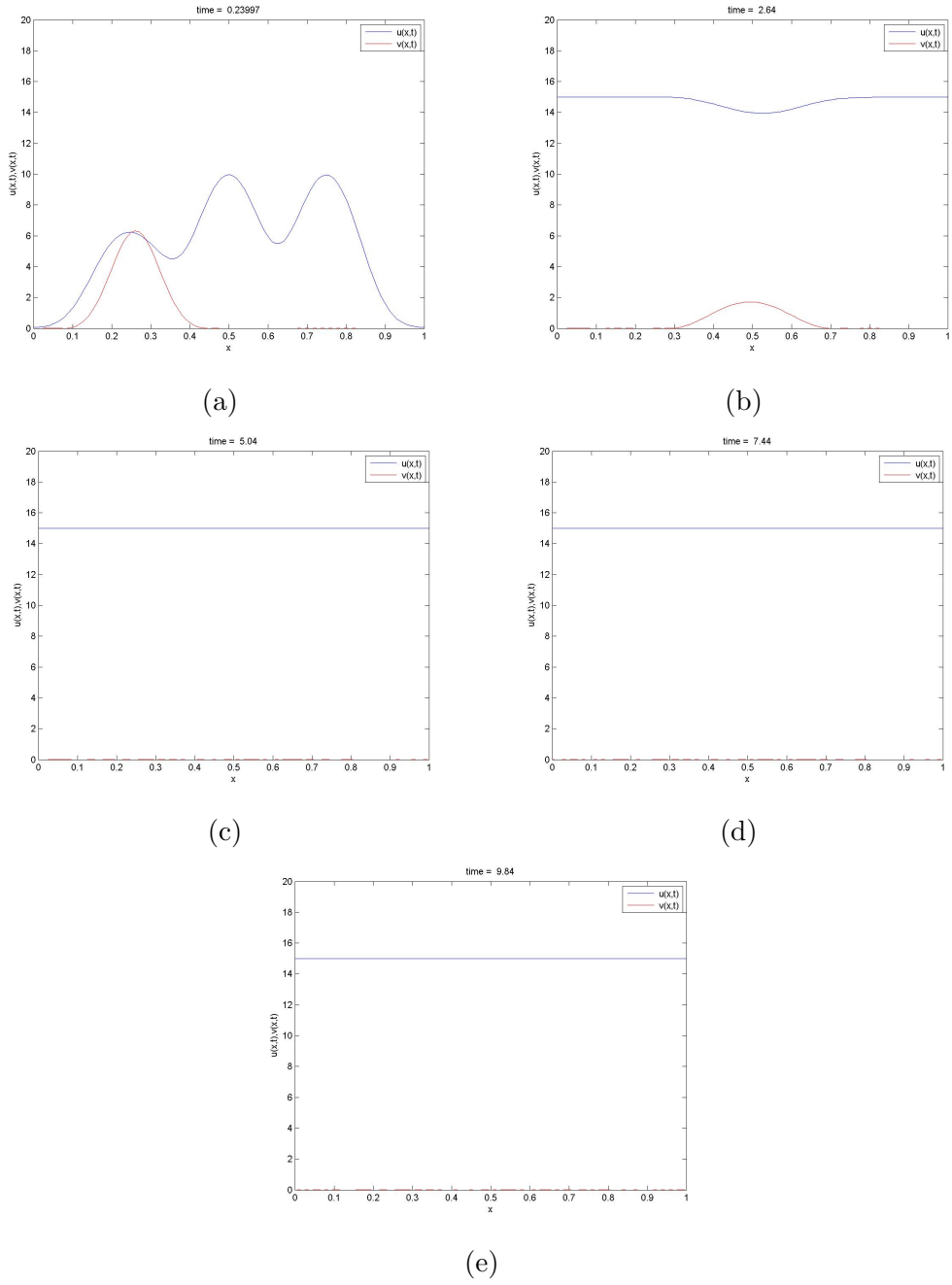

Figure 3.10: A-E show the behavior the system of equations with a unsustainable fishing rate for H_v at all locations on x over 23 months. The result is similar to that of the behavior of the model with only prey in the system and no harvesting. Once the predator animals are hunted to extinction in the region, the prey animals are free to reproduce until they reach their carrying capacity.

3.4 Harvesting Predator and Prey

The next series of experiments examines how the model behaves when both the predator and prey animals are harvested at a constant rate at all locations on x. While this step may seem somewhat trivial since the expected results are more or less the same as those of the experiments conducted in the last section, it allows for us to see whether or not the rate for H_v and H_u that were considered "sustainable" rates remain so when harvesting of both predators and prey is implemented.

Sustainable and Unsustainable Fishing Rates

Experiments showed that the fishing rates that were previously considered sustainable and unsustainable for the prey animals from the previous series of experiments more or less remained sustainable and unsustainable when harvesting of predators was added to the model. However, a harvesting rate of $\alpha_2 = 0.47$ was no longer sustainable. The fishing rate of the predators had to be reduced to $\alpha_2 = 0.4$ order to keep generate a predator-prey population density cycle that indicated sustainable harvesting. Figures 3.11 and 3.12 show the results of the experiments.

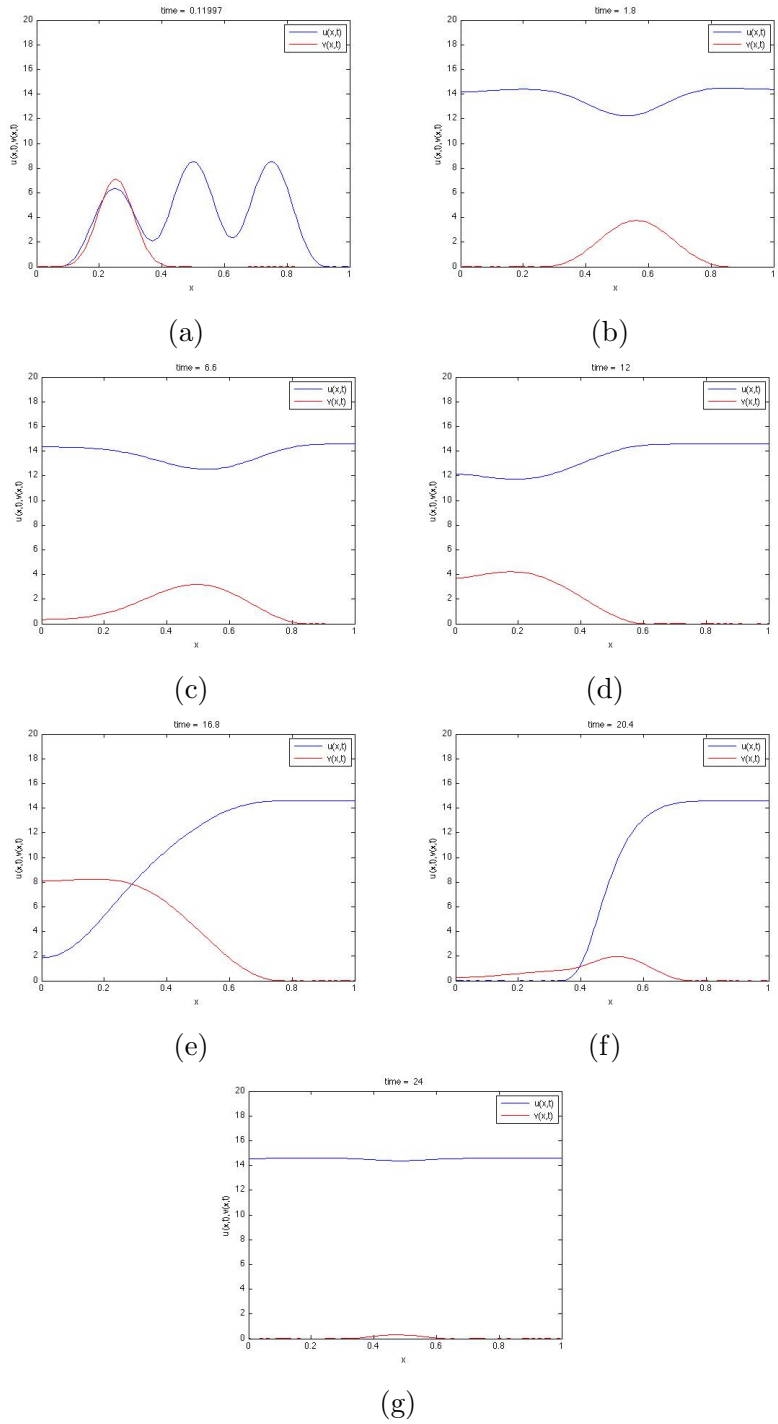

Figure 3.11: A-G show the behavior the system of equations with sustainable fishing for H_u and an unsustainable fishing rate for H_v at all locations on x. An important aspect of this particular experiment is that the fishing rate for H_v was set to a value that was shown to be sustainable if the prey animals were not harvested. This is an indication of how sensitive a predator-prey system in general can be to outside influences like harvesting. While extinction of predators does not occur immediately, the fact that it occurs at all is significant.

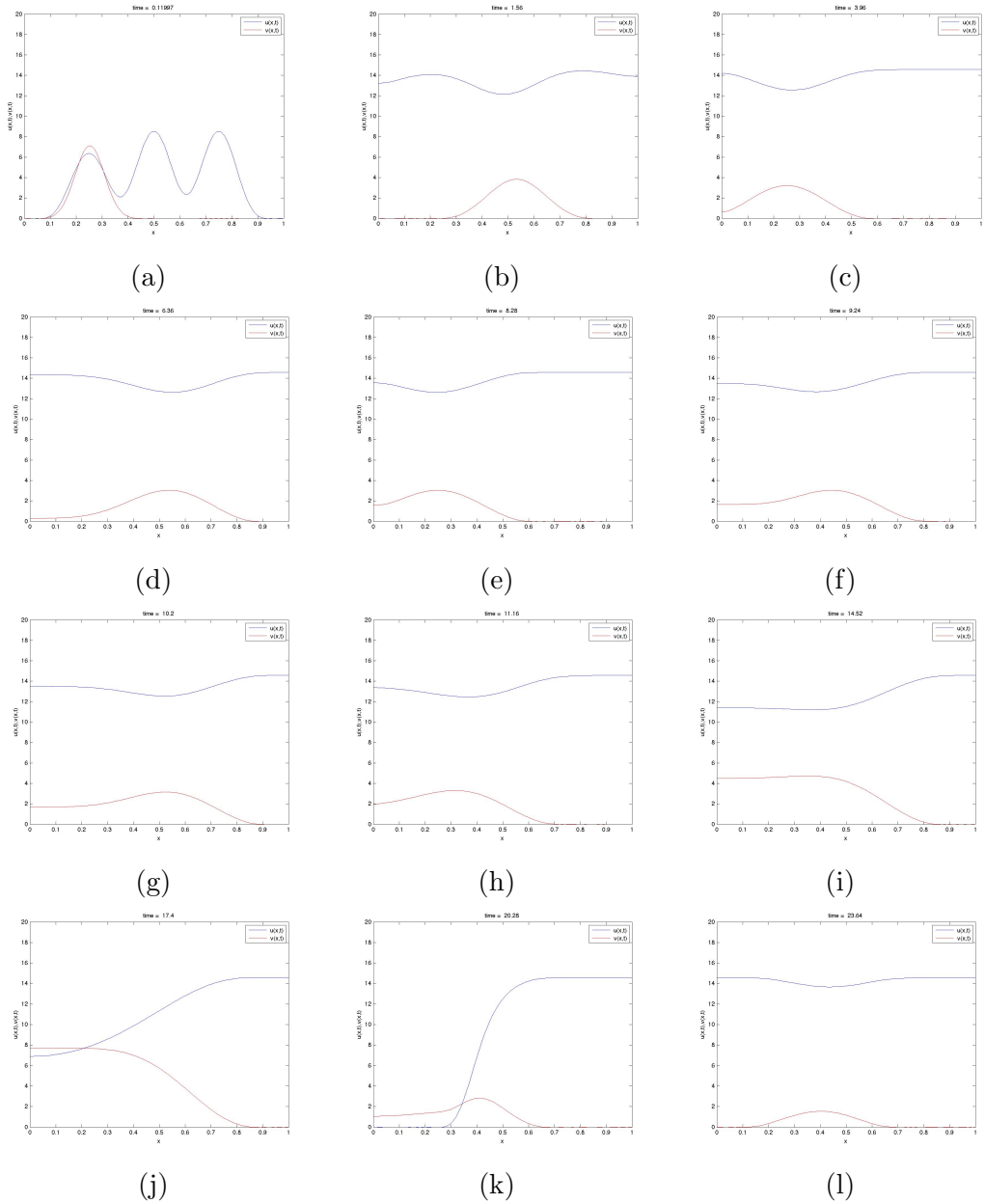

Figure 3.12: A-L show the behavior the system of equations from with sustainable fishing for H_u and H_v at all locations on x. It should be noted that the value of H_v in this experiment was reduced from 0.47 which was shown to be sustainable when prey harvesting was not implemented to .4. Reducing the fishing rate of the predators causes a population density cycle indicative of a healthy system.

Experiment	Max Sustainable Harvesting Rate ($H_v(x)$)	Harvesting Domain
No Prey Harvesting, No Sanctuary	.47	[0,1]
Prey Harvesting, No Sanctuary	.4	[0,1]
Prey Harvesting, Sanctuary	.47	$x \notin [\frac{3}{8}, \frac{5}{8}]$

Table 3.2: According to the behavior of the model, the sustainable and unsustainable fishing rates exist for predators regardless of whether their prey are harvested or not. However, it can also be seen that the maximum sustainable fishing rate of the predators is higher when either their prey are not fished or they are protected by sanctuaries.

Sanctuaries

Since shark fishing is prohibited in some areas of water like marine wildlife sanctuaries, the next series of experiments will deal with examining how the model behaves when the predator animals are not harvested in a small area of x. This measure was not implemented for the prey animals for the sake of simplicity. The results of these experiments should provide an indication of the importance of wildlife sanctuaries in any predator-prey system.

Implementation

For this experiment, H_v is initially set to the same rate that was found to be sustainable in the experiments in the last section, with a small portion of H_v set to zero (i.e. $H_v(x) = 0 \ \forall x \in [\frac{3}{8}N, \frac{5}{8}N]$). The expected result of adding this sanctuary to the model is that the maximum sustainable fishing rate will increase from that found in the last section, though it will not be as high as if there were no predator harvesting at all. A summary of the maximum sustainable fishing rates through all experiments presented in this chapter can be found in Table 3.2. The results of this experiment can be seen in Figure 3.13.

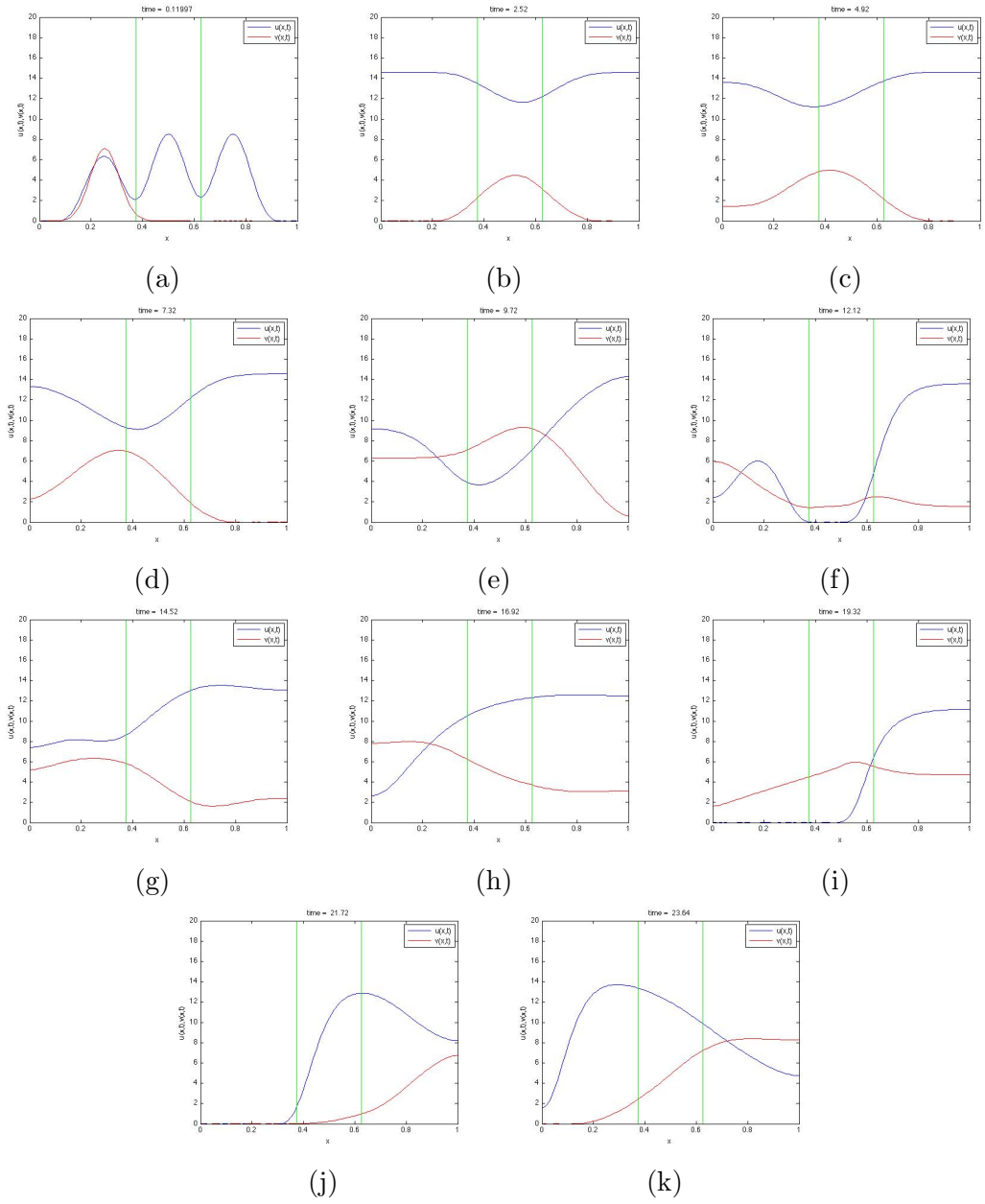

Figure 3.13: A-M show the behavior the system of equations from timestep with sustainable fishing for H_u and H_v at all locations on x after the addition of a sanctuary in which the shark population is not harvested (marked off by the green dashed lines). The predator-prey population density cycle indicative of a healthy system is present after the implementation of the sanctuary at the same value for H_v ($\alpha_2 = 0.47$) shown to be unsustainable in the last experiment.

Chapter 4

Results and Discussion

4.1 Summary of Results

The first two sets of experiments examined the behavior of the system of equations with the harvesting rates for both predators and prey set at 0. The first series of experiments showed how the system behaved as expected when initial population density of either the predator or prey animals was set to 0 at all locations in x. If prey were alone in the system, they reproduced freely until they reached carrying capacity. If predators were alone in the system, they would eventually die out in all locations. In the second series of experiments, the interactions of predators and prey without harvesting in the system were examined. A stable equilibrium could be reached in the system of the initial predator and prey population densities were set to the equilibrium values solved for in section 3.2 and evenly distributed. If the initial conditions of u and v were set to any other value, a population density cycle would result. Changing the distributions of predator and prey population densities form constant to gaussian caused the dynamics of the system to become more varied, but a periodic cycle still emerged at every two peaks. All of the results were as expected and indicate that the model functions as a predator-prey model should without additional factors, such as harvesting.

The next series of experiments examined the behavior of the system with a constant harvesting rate was added to the system at all locations. When both predator and prey were harvested alone in the system at sustainable rates, the behavior of the system remained healthy, indicated by the same kind of continuous population density cycle over time as was observed in the system without harvesting. When both groups of animals were harvested together however, the maximum sustainable harvesting rate for the predator animals decreased from 0.47 to 0.4. This result is an indication of how sensitive the predator-prey system is to harvesting in general.

The last set of experiments examined how the addition of a predator sanctuary to the model affected its behavior. As was expected, blocking off an area in x within predator animals could not be harvested raised the maximum sustainable harvesting rate for the predators, though not to the same value as when they were harvested alone. The simple fact that the sanctuary allowed more sharks to remain in the system allowed humans to harvest more of them and have the system remain ecologically healthy with predators and prey undergoing the same type of population density cycle as the previous the system without harvesting.

4.2 Discussion

The results of all experiments indicate that the numerical model developed in this thesis behaves as expected for all tested conditions. The population density cycles that are observed are indicative of how an ecologically healthy predator-prey system behaves over time. Sustainable harvesting rates for both predator and prey animals exist in all situations tested. It appears that the next step in improving this model's accuracy is to match the parameters in the model to observed parameters in real-world systems.

There are plenty of other improvements that could be implemented in this

particular model. For instance, the model could be modified so that there were more than one type of prey animal present in the system; perhaps one prey type that could be harvested and another that could not. Another improvement would be to add juvenile sharks to the model, as it is the harvesting of these immature sharks that causes many of ecological related to shark populations in the real world. Finally, the harvesting functions in the model could be modified to not only vary with respect to location, but with respect time as well. Most fisheries only harvest certain animals during the time in which they are in season and as of yet, this model does not account for this mathematically.

While theoretical in nature, it is hoped that the results achieved by the experiments in which predator sanctuaries were implemented will illustrate the important role that marine sanctuaries can play in the sustainability of harvesting a predator in a predator-prey system. The fact that simply adding an area where the predator animals could not be harvested made a less the ideal situation better for the predator population in general shows that sustainable fishing would undoubtedly become more realistic if more such sanctuaries existed. It is also hoped that this model could eventually be applied to other predator-prey system in which the predators face the threat of extinction.

References

[1] Marcus R. Garviea and C. Trencheab. Spatiotemporal dynamics of two generic predator–prey models. *Journal of Biological Dynamics*, 00(0):1–12, November 2009.

[2] International Union for Conservation of Nature and Natural Resources Iucnredlist.org. Galeorhinus galeus.

[3] Vincent F. Gallucci, Ian G. Taylor, and Karim Erzini. Conservation and management of exploited shark populations based on reproductive value. *Can. J. Fish. Aquat.*, 63:931–942, 2006.

[4] Christine A. Ward-Paige, Camilo Mora, Heike K. Lotze, Christy Pattengill-Semmens, Loren McClenachan, Ery Arias-Castro, and Ransom A. Myers. Large-scale absence of sharks on reefs in the greater- caribbean: A footprint of human pressures. *PLoS ONE*, 5(8), August 2010.

[5] Julia K. Baum and Wade Blanchard. Inferring shark population trends from generalized linear mixed models of pelagic longline catch and effort data. *Fisheries Research*, 102(1):229–239, November 2009.

[6] Richard Haberman. *Predator-Prey Models*. Society for Industrial and Applied Mathematics, 1977.

[7] Jicai Huang and Yijun Gong. Bifurcation analysis in a predator-prey model with

constant-yield predator harvesting. *Discrete and Continuous Dynamical Systems Series B*, 18(8):2101–2121, October 2013.

[8] Pascal Frey. *The Finite Difference Method*, March 2008.