SHELLABILITY OF THE BRUHAT ORDER ON BOREL ORBIT CLOSURES

AN ABSTRACT
SUBMITTED ON THE SIXTEENTH DAY OF APRIL, 2013
TO THE DEPARTMENT OF MATHEMATICS
OF THE GRADUATE SCHOOL OF
TULANE UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
BY

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Abstract

Involutions and fixed-point-free involutions arise naturally as representatives for certain Borel orbits in invertible matrices. Similarly, partial involutions and partial fixed-point-free involutions represent certain Borel orbits in matrices which are not necessarily invertible. Inclusion relations among Borel orbit closures induce a partial order on these discrete parameterizing sets. In this dissertation we investigate the associated order complex of these posets. In particular, we prove that the order complex of the Bruhat poset of Borel orbit closures is shellable in symmetric as well as skew-symmetric matrices.

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Contents

Chapter 1

Introduction

Simplicial complexes are the building blocks of algebraic topology and play an important role in geometry. Subcomplexes of a simplicial complex are called faces, and the faces that are maximal under inclusion are called facets. A simple example is the three dimensional cube where the faces are the vertices, edges, sides and the empty set. The sides of the cube are the faces maximal under inclusion, so they are the facets. The idea of "shellability" is to build the complex successively from its facets. For this, the facets are ordered in a way such that the intersection of a facet with all the preceding facets is a simplicial subcomplex of codimension 1. If such an ordering on the facets exists, then we say that the complex is shellable. The exact definition of shellability is in general very difficult to verify, but it implies a number of important properties. For example, it is known that a shellable simplicial complex has the homotopy type of a wedge of spheres. Also, a shellable simplicial complex is Cohen-Macaulay.

The concept of shellability naturally applies to partially ordered sets, since every partially ordered set can be uniquely identified with its order complex. In detail, one identifies the elements of the poset with vertices of the complex and the chains with the faces. We call a poset shellable if its order complex is shellable. Since maximal chains correspond to facets, shellability requires an ordering on the maximal chains.

However, it is desirable to find a condition on posets that implies shellability without the need to consider the corresponding order complex. Such a condition was found by Björner in [1] and is termed "lexicographic shellability." For this, every edge of the Hasse diagram is labeled and hence every chain in the poset can be identified with a sequence of labels. A poset is called lexicographically shellable if there exists a labeling such that the lexicographically first chain between any two elements is increasing, and there is no other increasing chain between these elements. For a precise definition see Section 2.3. Ordering the maximal chains lexicographically using these labels gives us the desired order on maximal chains required for shellability. Hence, lexicographic shellability implies shellability. However, we should point out that lexicographic shellability of the poset is a stronger property than the associated order complex being shellable. In fact, Vince and Wachs found an example of a poset that is shellable, but not lexicographically shellable [2].

The posets we are investigating in this thesis are orbits of Borel group actions related to Schubert varieties ordered by inclusion. A Borel subgroup B of an algebraic group G is a maximal connected solvable algebraic subgroup. In GL_n, Borel subgroups are the subgroups that are conjugate to the subgroup of invertible upper triangular matrices B_n. GL_n/B_n forms an algebraic variety called the flag variety. The Borel orbit of each flag under left multiplication by B_n is called a Schubert cell. Schubert cells are open sets and their Zariski closures are called Schubert varieties. Schubert varieties were first introduced by Hermann Schubert in 1879 in his celebrated treatise "Kalkül der abzählenden Geometrie" (Calculus of Enumerative Geometry [3]). They form one of the best studied classes of algebraic varieties and play an important role in representation theory. The Bruhat-Chevalley order on Schubert varieties is defined by set inclusion. The study of this partial order is equivalent to the study of the cell decomposition of these varieties in the topological sense.

In order to prove the lexicographic shellability of posets of the kind studied in this thesis, two things have to be done. First, we have to show that the set of orbits is finite, and we have to find a way to parameterize them. Second, we have to investigate the covering relations in the poset and prove that there is a labeling satisfying the properties for lexicographic shellability.

It is well-known that every Schubert variety can be parameterized by a permutation, and Edelman proves in [4] that this poset is indeed lexicographically shellable.

We focus on the space of symmetric and the space of skew-symmetric invertible matrices on which B_n acts by $g \cdot A = (g^{-1})^\top A g^{-1}$. Richardson and Springer show in [5] that the Borel orbits in the case of symmetric matrices can be parameterized by symmetric permutation matrices, and in the skew-symmetric case they show that the Borel orbits can be parameterized by symmetric fixed-point-free permutation matrices. It is shown in [6] that the Borel orbits in the case of symmetric matrices form a shellable simplicial complex under inclusion. Our first main result in this thesis is that the poset of fixed-point-free permutation matrices is also lexicographically shellable. This was proved in joint work with Can and Cherniavsky [7].

One can extend these results by dropping the requirement of invertibility and look at the monoid of $n \times n$ matrices M_n. Note that GL_n is Zariski dense in M_n. First, we look at the classic example of Schubert varieties again. Given the action of B_n on GL_n we can investigate the same action on M_n. It was shown by Renner in [8] that the Borel orbits in this case can be identified with elements of the rook monoid. Recall that the rook monoid is the finite monoid of 0/1 matrices with at most one 1 in each row and column. Can extended the results of Edelman and proved that rook monoid is shellable too [9].

We are interested in symmetric and skew-symmetric matrices in M_n. These spaces are the Zariski closures of the invertible symmetric and skew-symmetric matrices. We again investigate the action $g \cdot A = (g^{-1})^\top A g^{-1}$ of B_n. Szechtman proved that in the case of symmetric matrices the closure of these Borel orbits in M_n can be identified with symmetric rook matrices [10]. Our second main result in this thesis is the lexicographic shellability of this poset. This was proved in joint work with Can [11]. In the case of skew-symmetric matrices it is possible to identify the orbits with fixed-point-free symmetric rook matrices as shown by Cherniavsky in [12]. Our third main result is that this poset is lexicographically shellable as well.

The organization of this thesis is as follows. In Chapter 2 we rigorously define the terms introduced above and provide first examples. Chapters 3, 4 and 5 each have two subsections. In the first of these subsections we treat the invertible cases; in the following we handle the non-invertible cases. In detail, Chapter 3 recalls that the posets of permutations and rooks is lexicographically shellable. In Chapter 4 we recall Incitti's results about the poset of involutions [6] and establish the shellability of partial involutions. Chapter 5 analyzes the posets of fixed-point-free and partial fixed-point-free involutions. In Chapters 6, 7 and 8 we look at related results for the investigated posets. In particular, Chapter 6 explores Eulerian intervals in the rook monoid and partial involutions. In Chapter 7 we investigate the difference between the Bruhat order on fixed-point-free involutions and a similar order discovered by Deodhar and Srinivasan in [13]. Chapter 8 analyzes the order complexes of fixed-point-free and partial fixed-point-free involutions.

Chapter 2

Preliminaries

In this chapter we define all the terms needed and provide first examples. Our main references are [14], [15], [16] and [17]. For general background on posets we recommend [14]. In this thesis, we consider shellability only for the pure case. We recommend [15] and [16] as references. [17] is a nice summary of all basic definitions needed for this thesis and provides further implications of shellability and lexicographic shellability. In [17] pureness is not assumed.

2.1 Basic Definitions

Definition 1. *A partially ordered set P (poset, for short) is a set, together with a binary relation $\leq$ satisfying*

1. *For all $t \in P, t \leq t$ (reflexivity).*
2. *If $s \leq t$ and $t \leq s$, then $s = t$ (antisymmetry).*
3. *If $s \leq t$ and $t \leq u$, then $s \leq u$ (transitivity).*

Definition 2. *A subposet of P is a subset Q of P and a partial ordering of Q such that if $s \leq t$ in Q, then $s \leq t$ in P. By an induced subposet of P, we mean a subset Q of P and a partial ordering of Q such that for $s, t \in Q$ we have $s \leq t$ in Q if and only if $s \leq t$ in P. We then say that the subset Q of P has the induced order.*

As a special type of subposet we define an *interval* of P to be $[s,t] = \{u \in P : s \leq u \leq t\}$.

In a poset P, an element y is said to *cover* another element x, if $x < y$, and if $x \leq z \leq y$ for some $z \in P$, then either $z = x$ or $z = y$. In this case we write $y \to x$. Given P, we denote by $C(P)$ the set of all covering relations of P,

$$C(P) = \{(x,y) \in P \times P : \ y \text{ covers } x\}.$$

An (increasing) *chain* in P is a sequence of distinct elements such that $x = x_1 < x_2 < \cdots < x_{n-1} < x_n = y$. A chain in a poset P is called *saturated*, if it is of the form $x = x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{n-1} \leftarrow x_n = y$. A saturated chain in an interval $[x,y]$ is called *maximal*, if the end points of the chain are x and y. A poset is called *graded* if all maximal chains between any two comparable elements $x \leq y$ have the same length. For a finite poset with a minimal and a maximal element, denoted by $\hat{0}$ and $\hat{1}$, respectively this amounts to the existence of an integer valued function $\ell_P : P \to \mathbb{N}$ satisfying

1. $\ell_P(\hat{0}) = 0$,

2. $\ell_P(y) = \ell_P(x) + 1$ whenever y covers x in P.

ℓ_P is called the *length function* of P. In this case, the length of the interval $[\hat{0}, \hat{1}] = P$ is called the *length* of the poset P.

We recall the definition of an abstract simplicial complex.

Definition 3. *An abstract simplicial complex on a vertex set V is a collection Δ of subsets of V satisfying:*

1. *If $T \in V$ then $\{T\} \in \Delta$,*

2. *if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.*

An element $F \in \Delta$ is called a face of Δ.

The faces which are maximal under inclusion are called *facets*. A complex is *pure* if all its facets are equicardinal.

Now we define the *order complex* of a poset P as follows.

Definition 4. *Let P be any poset, then let $\Delta(P)$ be the simplicial complex where the vertices of $\Delta(P)$ are the elements of P, and the faces of $\Delta(P)$ are the chains of P. The simplicial complex $\Delta(P)$ is called the order complex of P.*

For every poset P let $\widehat{P}$ denote the poset P with $\hat{0}$ and $\hat{1}$ adjoined. We define the notion of a Cohen-Macaulay poset.

Definition 5. *A finite poset P is said to be Cohen-Macaulay over an abelian group A if for every $s < t$ in $\widehat{P}$, the order complex $\Delta(s,t)$ of the open interval (s,t) satisfies*

$$\widetilde{H}_i(\Delta(s,t); A) = 0, \text{ if } i < \dim \Delta(s,t).$$

where $\widetilde{H}_i(\Delta(s,t); A)$ is the i-th reduced homology of the topological realization of $\Delta(s,t)$ with coefficients in A.

Definition 6. *The Möbius function of P is defined recursively by the formula*

$$\mu([x,x]) = 1,$$
$$\mu([x,y]) = -\sum_{x \leq z < y} \mu([x,z])$$

for all $x \leq y$ in P.

The main reason for our interest in the Möbius function is its connection to the Euler characteristic of the associated order complex. Recall that the reduced Euler characteristic $\widetilde{\chi}(\Delta)$ of a simplicial complex Δ is defined to be

$$\widetilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i f_i(\Delta),$$

where $f_i(\Delta)$ is the number of i-dimensional faces of Δ. Here, $f_{-1}(\Delta) = 1$ and corresponds to the empty face.

Proposition 7. *(Philip Hall Theorem)*

For any poset P,

$$\mu(\hat{P}) = \widetilde{\chi}(\Delta(P)).$$

The Euler characteristic is a topological invariant. Hence by Proposition 7, $\mu_P(x, y)$ depends only on the topology of the open interval (x, y) of P.

Definition 8. *A finite graded poset P with $\hat{0}$ and $\hat{1}$ is called Eulerian if $\mu_p(s, t) = (-1)^{\ell(t)-\ell(s)}$ for all $s \leq t$ in P.*

2.2 Shellablility

For each face F of a simplicial complex Δ, let $\langle F \rangle$ denote the subcomplex generated by F, i.e., $\langle F \rangle = \{G : G \subseteq F\}$.

Definition 9. *A pure simplicial complex Δ is said to be shellable if its facets can be given a linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $\left(\cup_{i=1}^{k-1} \langle F_i \rangle\right) \cap \langle F_k \rangle$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$. A linear order of the facets which satisfies this requirement is called a shelling.*

Example 10. *[18], [16]*

1. *Every 0-dimensional complex, that is, every set of points, is shellable, by definition.*

2. *Every simplex is shellable. In fact, any ordering of its facets yields a shelling. This is easily shown by induction on the dimension, since the intersection of any two facets F_i and F_j is a facet of both F_i and F_j.*

3. *The d-cubes are shellable. By induction on the dimension, it can be shown that every ordering of the 2d facets $F_1, \ldots, F_{2d}$ such that F_1 and F_{2d} are opposite (that is, $F_{2d} = F_1$) yields a shelling.*

However, already for 2-complexes, problems arise. For example, in Figure 2.1, the left complex is not shellable but the right complex is shellable.

The problem with the left complex is that cells 1 and 2 intersect at a vertex, which is not 1-dimensional. In contrast, the ordering of the right complex is a shelling.

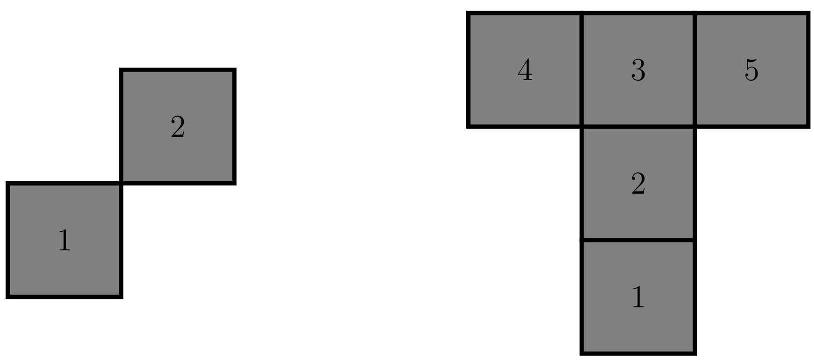


Figure 2.1: Non shellable and shellable 2-complexes

One of the most famous results about convex polytopes is the *Euler-Poincaré formula*:

$$-f_{-1} + f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} + (-1)^d f_d = 0,$$

where f_i denotes the number of i-dimensional faces of a d-polytope P. Grünbaum [19] observed that all classical inductive proofs of the Euler-Poincaré formula starting with Schläfli's proof in the middle of the nineteenth century [20] assumed that the boundary of every polytope can be built up inductively in a nice way. That this is in fact possible was proved by Bruggesser and Mani in 1971 [21]. In their paper shellability was formally introduced. The theory of shellability was extended by Björner and Wachs [22] where Δ is no longer assumed to be pure.

Although the definition of shellability is not very illuminating, it implies a number of strong topological and algebraic properties.

Theorem 11. *If Δ is a shellable simplicial complex then Δ is Cohen-Macaulay.*

Theorem 12. *[23] A shellable simplicial complex has the homotopy type of a wedge of spheres.*

Corollary 13. *If Δ is shellable, then for all i,*

$$\widetilde{H}_i(\Delta;\mathbb{Z}) \cong \widetilde{H}^i(\Delta;\mathbb{Z}) \cong \mathbb{Z}^{r_i}.$$

It should be mentioned that shellability depends on the triangulation of a complex.

Theorem 14. *[24] The tetrahedron can be triangulated in a nonshellable way.*

2.3 Lexicographic shellability

In 1980 Björner found a condition for the edge labeling of a poset which implies the shellability of the poset [1]. From this the theory of lexicographic shellability emerged. Björner and Wachs further developed the notion to the nonpure case [25]. It should be mentioned, that there are two versions of lexicographic shellability, EL-shellability and CL-shellability. We work with EL-shellability which is known to imply CL-shellability.

Let P be a finite poset with a maximum and a minimum element, denoted by $\hat{1}$ and $\hat{0}$, respectively. We assume that P is graded of rank n. Recall that by

$C(P)$ we denote the set of covering relations. An *edge-labeling* on P is a map $f = f_{P,\Gamma} : C(P) \to \Gamma$ into some totally ordered set Γ. The *Jordan-Hölder sequence* (with respect to f) of a chain $\mathfrak{c} : x_0 < x_1 < \cdots < x_{n-1} < x_n$ of P is the n-tuple

$$f(\mathfrak{c}) := (f((x_0, x_1)), f((x_1, x_2)), \ldots, f((x_{n-1}, x_n))) \in \Gamma^n.$$

Fix an edge labeling f, and a chain $\mathfrak{c} : \; x_0 < x_1 < \cdots < x_n$. We call both the chain $\mathfrak{c}$ and its image $f(\mathfrak{c})$ *increasing*, if

$$f((x_0, x_1)) \leq f((x_1, x_2)) \leq \cdots \leq f((x_{n-1}, x_n))$$

holds in Γ.

Let $k > 0$ be a positive integer and let Γ^k denote the $k-$fold cartesian product $\Gamma^k = \Gamma \times \cdots \times \Gamma$, totally ordered with respect to the lexicographic ordering. An edge labeling $f : C(P) \to \Gamma$ is called an *$EL-$labeling*, if

1. in every interval $[x, y] \subseteq P$ of rank $k > 0$ there exists a unique maximal chain $\mathfrak{c}$ such that $f(\mathfrak{c}) \in \Gamma^k$ is increasing,

2. the Jordan-Hölder sequence $f(\mathfrak{c}) \in \Gamma^k$ of the unique chain $\mathfrak{c}$ from (1) is the smallest among the Jordan-Hölder sequences of maximal chains $x = x_0 < x_1 < \cdots < x_k = y$.

A poset P is called *EL-shellable*, if it has an $EL-$labeling.

The following result of Björner justifies the term lexicographic shellability.

Theorem 15. *([1]) Suppose P is a bounded poset with an EL-labeling. Then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$.*

Lexicographic shellability is not equivalent to shellability. In 1985, Vince and

Wachs found an example of a shellable poset that is not lexicographically shellable [2].

Example 16. *We depict an example of an EL-shellable poset with an EL-labeling as well as a non EL-shellable poset.*

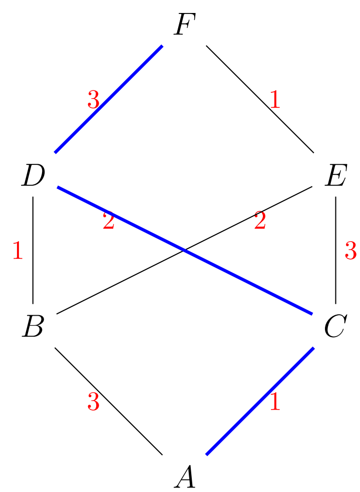


Figure 2.2: EL-shellable

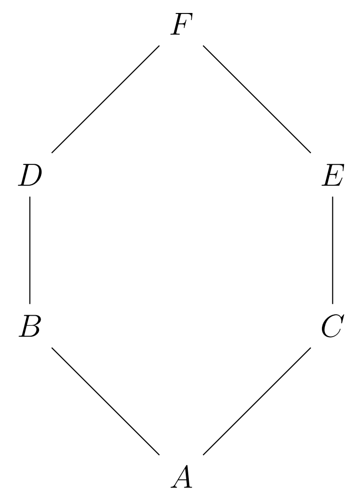


Figure 2.3: not EL-shellable

2.4 Posets of Borel orbit closures

For the remainder of our thesis we fix an algebraically closed field k and consider all groups and semigroups over k. For notational ease we do not use k as long as it is clear from the context.

We denote by GL_n the general linear group of $n \times n$ invertible matrices. Let Y be a variety on which a subgroup of GL_n acts. We denote by $B(Y;G)$ the set of G-orbits in Y. We focus on the following examples:

- $Y = GL_n$ and $G = B_n \times B_n$ where B_n is the Borel subgroup of invertible upper triangular matrices acting on Y via

$$(x, y) \cdot A = xAy^{-1}, \tag{2.1}$$

where $x, y \in B_n$ and $A \in GL_n$.

- $Y = Sym_n^0$, the space of symmetric matrices in GL_n, and $G = B_n$ acting on Y via

$$x \cdot A = (x^{-1})^\top A x^{-1}, \tag{2.2}$$

where $x^\top$ denotes the transpose of the matrix $x \in B_n$ and $A \in Sym_n^0$.

- $Y = Skew_{2n}^0$, the space of skew-symmetric matrices in GL_n, and $G = B_n$ acting on Y via

$$x \cdot A = (x^{-1})^\top A x^{-1}, \tag{2.3}$$

where $x^\top$ denotes the transpose of the matrix $x \in B_n$ and $A \in Skew_{2n}^0$.

In all of these examples, $B(Y; G)$ is a finite poset with respect to set inclusion. It is well known that the symmetric group of permutation matrices, S_n, parametrizes the orbits of (2.1). For $u \in S_n$, let $\dot{u}$ denote the right coset in GL_n/B represented by u. The classical *Bruhat-Chevalley ordering* is defined by $u \leq_{S_n} v \iff B \cdot \dot{u} \subseteq \overline{B \cdot \dot{v}}$ for $u, v \in S_n$. In [4] Edelman proves the lexicographic shellability of this poset.

A permutation $u \in S_n$ called an *involution* if $u^2 = id$, or equivalently, if its permutation matrix is a symmetric matrix. We denote by I_n the set of all involutions in S_n, and consider it as a subposet of the Bruhat-Chevalley poset $(S_n, \leq_{S_n})$. In [5], Richardson and Springer show that I_n parametrizes the elements of $B(Y; G)$ of (2.2). In [6] Incitti proves that $(I_n, \leq_{S_n})$ is a lexicographically shellable poset.

An involution $x \in I_{2n}$ is called *fixed-point-free*, if the matrix of x has no non-zero diagonal entries. In [5], Richardson and Springer show that there exists a poset isomorphism between F_{2n}, the poset of (2.3) and the subposet of fixed-point-free involutions in I_{2n}. Unfortunately, F_{2n} does not form an interval in I_{2n}, hence it does not immediately inherit nice properties therein. In fact, this is easily seen for $n = 2$ from the Hasse diagram of I_4 in Figure 2.4, in which the fixed point free involutions are boxed.

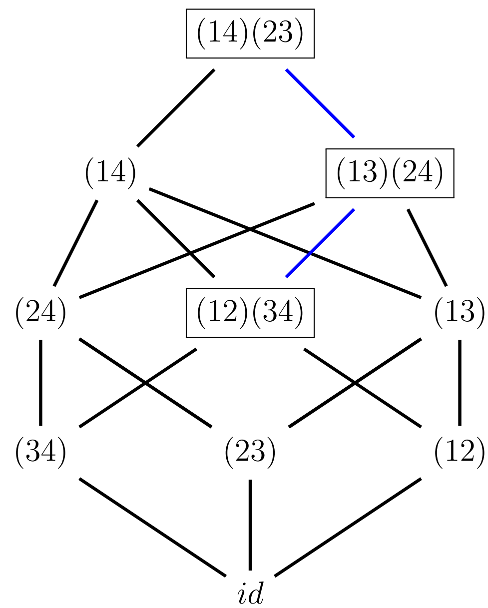


Figure 2.4: F_4 in I_4

These classical cases have a straight-forward but important generalization. The closure of GL_n is M_n, the monoid of $n \times n$ matrices. Starting with M_n we get to the following spaces:

- $Y = M_n$ and $G = B_n \times B_n$ where B_n is the group of invertible upper triangular matrices acting on Y via

$$(x, y) \cdot A = xAy^{-1}, \tag{2.4}$$

  where $x, y \in B_n$ and $A \in M_n$.

- $Y = Sym_n$, the space of symmetric matrices in M, and $G = B_n$ acting on Y

via

$$x \cdot A = (x^{-1})^\top A x^{-1}, \tag{2.5}$$

where $x^\top$ denotes the transpose of the matrix $x \in B_n$ and $A \in Sym_n$.

- $Y = Skew_n$, the space of skew-symmetric matrices in M_n, and $G = B_n$ acting on Y via

$$x \cdot A = (x^{-1})^\top A x^{-1}, \tag{2.6}$$

where $x^\top$ denotes the transpose of the matrix $x \in B_n$ and $A \in Skew_n$.

Also in these examples, $B(Y; G)$ is finite and partially ordered with respect to set inclusion. The *rook monoid* R_n is the finite monoid of 0/1 matrices with at most one 1 in each row and each column. The elements of R_n parametrize the orbits of the action (2.4) of $B_n \times B_n$ on M [8]. The elements of R_n are called *rooks*, or *rook matrices*.

It is shown by Szechtman in [10] that each orbit closure in (2.5) has a unique corresponding symmetric rook in R_n. Following [26], we call these rooks *partial involutions* as they satisfy the quadratic equation

$$x^2 = e,$$

where $e \in R_n$ is a diagonal matrix. We denote the set of all partial involutions in R_n by P_n.

In [12], Cherniavsky shows that the Borel orbits in (2.6) are parametrized by those elements $x \in \mathrm{Skew}_n$ such that

1. the entries of x are either 0,1 or -1,

2. any non-zero entry of x that is above the main diagonal is a $+1$,

3. in every row and column of x there exists at most one non-zero entry.

Note that when the -1's in x are replaced by +1's, the resulting matrix $\tilde{x}$ is a partial involution with no diagonal entries. In other words, $\tilde{x}$ is a partial fixed-point-free involution. It is easy to check that this correspondence is a bijection, hence PF_n parametrizes the Borel orbits in $Skew_n$.

The *Bruhat-Chevalley-Renner ordering* on rooks is defined by

$$r \leq t \iff B_n r B_n \subseteq \overline{B_n t B_n},\ r, t \in R_n.$$

Here, the bar in our notation stands for the Zariski closure in M. The corresponding partial order on P_n and PF_n, denoted by $\preceq$ is defined similarly; if A and A' are two B_n-orbit closures in $B(Q; B_n)$, and, r and r' are two partial involutions or partial fixed-point-free involutions representing A and A', respectively, then $r \preceq r' \iff A' \subseteq A$.

It is desirable to obtain a combinatorial description of these partial orderings. For R_n a useful characterization is found by Can and Renner in [27]. Related combinatorial descriptions for the partial involutions and for the partial fixed point free involutions are given by Bagno and Cherniavsky in [26] and by Cherniavsky in [12], respectively. For the sake of space, we review the latter two descriptions, only. In fact, the Can-Renner description can be translated into a form similar to that found by Bagno and Cherniavsky (see final remarks of [26]).

Let $X = (x_{ij})$ be an $n \times m$ matrix. For each $1 \leq k \leq n$ and $1 \leq l \leq m$, denote by X_{kl} the upper-left $k \times l$ submatrix of X. The *rank-control matrix* of X is the $n \times m$ matrix $Rk(X) = (r_{kl})$ with entries given by

$$r_{kl} = \text{rank}(X_{kl}),$$

for $1 \leq k \leq n$ and $1 \leq l \leq m$. For example, the rank-control matrix of the partial involution $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is

$$Rk(x) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \tag{2.7}$$

For two matrices $A = (a_{kl})$ and $B = (b_{kl})$ of the same size with integer entries, we write $A \leq_R B$, if $a_{kl} \leq b_{kl}$ for all k and l. Then

$$x \preceq y \text{ if and only if } Rk(x) \leq_R Rk(y). \tag{2.8}$$

Although $\preceq$ is more natural from a geometric point of view, we prefer to work with its opposite, which we denote, by abuse of notation, by $\leq$, also. The same criterion holds for the posets $\leq_{Sym}$ and $\leq_{Skew}$.

2.5 Relations between the posets

We recall some fundamental facts about the covering relations of $\leq_{Sym}$ and $\leq_{Skew}$. Our references are [26] and [12].

Lemma 17. *The intersection $PF_{2n} \cap I_{2n}$ is equal to F_{2n}, and furthermore, $(F_{2n}, \leq_{Sym})$ and $(F_{2n}, \leq_{Skew})$ are isomorphic.*

Proof. The first claim is straightforward. For the second it is enough to observe that the partial orders $\leq_{Skew}$ and $\leq_{Sym}$ are both given by the same rank-control matrix comparison. Therefore, they restrict to give the same poset structure on F_{2n}. $\square$

Whenever it is clear from the context, we write $(F_{2n}, \leq)$ instead of $(F_{2n}, \leq_{Sym})$

or $(F_{2n}, \leq_{Skew})$.

Remark 18. *It is easy to see that the sets PF_{2n} and I_{2n} have the same cardinality. Indeed, let $x \in PF_{2n}$ be a partial fixed-point-free involution with determinant 0. We denote by $\tilde{x}$ the completion of x to an involution in I_{2n} by adding the missing diagonal entries. For example,*

$$x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \tilde{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define $\phi : PF_{2n} \to I_{2n}$ by setting

$$\phi(x) = \begin{cases} \tilde{x} & \text{if } x \in PF_{2n} - F_{2n}, \\ x & \text{otherwise.} \end{cases} \tag{2.9}$$

It is not difficult to check that ϕ is a bijection between PF_{2n} and I_{2n} such that $\phi(x) = x$ for all $x \in F_{2n}$. However, as pointed out in [12] the posets $(PF_{2n}, \leq_{Skew})$ and $(I_{2n}, \leq_{Sym})$ are not isomorphic. (Compare the Hasse diagram of PF_4 as depicted in Example 5.1 in [12] and the Hasse diagram of the opposite of I_4 as depicted in Figure 2 of [6].)

Lemma 19. *Let $w_0 \in P_{2n}$ denote the "longest permutation," namely, the $2n \times 2n$ anti-diagonal permutation matrix, and let $j_{2n} \in F_{2n}$ denote the $2n \times 2n$ fixed-point-free involution having non-zero entries at the positions $(1,2), (2,1), (3,4), (4,3), \dots, (2n-1, 2n), (2n, 2n-1)$, only. In other words, j_{2n} is the fixed-point-free involution with the only non-zero entries along its super-diagonals. Then*

1. *I_{2n} is an interval in P_{2n} with smallest element id_{2n} and largest element w_0.*

2. F_{2n} is an interval in PF_{2n} with smallest element j_{2n} and largest element w_0.

Proof. Let $x \in I_{2n}$. The $(2n, 2n)$-th entry of $Rk(x)$ is equal to $2n$ because x is invertible. On the other hand, if $x \in P_{2n}$ is an element not contained in I_{2n}, then its rank is less than $2n$. In other words, its $(2n, 2n)$-th entry cannot be $2n$, and therefore, it cannot be greater than or equal to any element of I_{2n}. Hence, I_{2n} is an interval. It follows from (2.8) that w_0 is the smallest, id_{2n} is the largest element of I_{2n}.

To prove the second claim, it is enough to prove that j_{2n} is the smallest element of F_{2n} because we already know that $F_{2n} = I_{2n} \cap PF_{2n}$. Now, the minimality of j_{2n} follows from induction by using the combinatorial criterion (2.8).

□

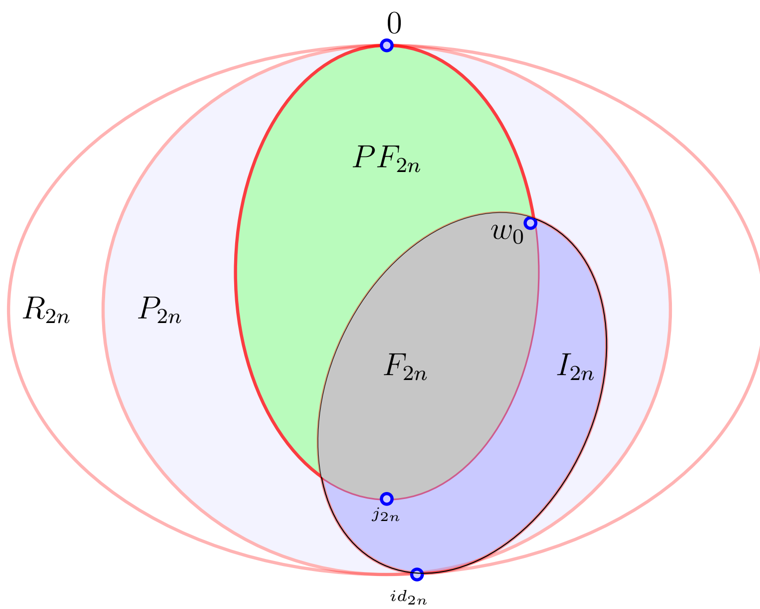


Figure 2.5: The schematic diagrams of $R_{2n}, P_{2n}, PF_{2n}, I_{2n}$ and F_{2n}.

2.6 Rooks and their enumeration.

This section is joint work with Can [11].

We set up our notation for rook matrices and establish a preliminary enumerative result.

Let $x = (x_{ij}) \in R_n$ be a rook matrix of size n. Define the sequence $(a_1, \dots, a_n)$ by

$$a_j = \begin{cases} 0 & \text{if the } j\text{-th column consists of zeros,} \\ i & \text{if } x_{ij} = 1. \end{cases} \tag{2.10}$$

By abuse of notation, we denote both the matrix and the sequence $(a_1, \dots, a_n)$ by x. For example, the associated sequence of the partial permutation matrix

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is $x = (3, 0, 4, 0)$.

Once n is fixed, a rook matrix $x \in R_n$ with k-nonzero entries is called a *k-rook*. Observe that the number of k-rooks is given by the formula

$$|R_{n,k}| = k! \cdot \binom{n}{k}^2. \tag{2.11}$$

Indeed, to determine a k-rook, we first choose $n-k$ 0-zero rows and $n-k$ 0-columns. This is done in $\binom{n}{n-k}^2$ ways. Next we decide for the non-zero entires of the k-rook. Since deleting the zero rows and columns results in a permutation matrix of size k, there are $k!$ possibilities. Hence, the formula follows.

Let τ_n denote the number of invertible partial involutions. By default, we set

$\tau_0 = 1$.

There is no closed formula for τ_n, however, there is a simple recurrence that it satisfies;

$$\tau_{n+1} = \tau_n + (n-1)\tau_{n-1} \ (n \geq 1). \tag{2.12}$$

There is a similar recurrence satisfied by the number of invertible n-rooks (permutations);

$$(n+1)! = n! + n^2 \cdot (n-1)! \ (n \geq 1). \tag{2.13}$$

It follows that

Lemma 20. *For all $n \geq 1$,*

1. $|R_{n,n-1} \cup R_{n,n}| = (n+1)!$,

2. $|P_{n,n-1} \cup P_{n,n}| = \tau_{n+1}$.

Proof. The first assertion follows from equations (2.13) and (2.11). The second assertion follows from equation (2.12) and the fact that $|P_{n,n-1}| = (n-1)\tau_{n-1}$. $\square$

Chapter 3

The posets of permutations S_n and rooks R_n

3.1 The poset of permutations S_n

3.1.1 An EL-labeling of permutations

We follow the summary of Can in [9].

The symmetric group S_n is the set of all permutations of $[n]$. We represent the elements of S_n in one line notation $w = (w_1, ..., w_n) \in S_n$ so that $w(i) = w_i$. It is well known that the S_n is a graded poset with respect to Bruhat-Chevalley ordering. Let B be the Borel subgroup of invertible upper triangular matrices in SL_n. The grading on S_n is given by the length function

$$\ell(w) = \dim(BwB) - \dim(B) = inv(w), \tag{3.1}$$

where

$$inv(w) = |\{(i,j) : \ 1 \leq i < j \leq n, \ w_i > w_j\}|. \tag{3.2}$$

Note that $\dim B = \binom{n+1}{2}$.

The Bruhat-Chevalley ordering on S_n is the smallest partial order generated by the transitive closure of the following (covering) relations. The permutation $x = (a_1, ..., a_n)$ is covered by the permutation $y = (b_1, ..., b_n)$, if $\ell(y) = \ell(x) + 1$ and

1. $a_k = b_k$ for $k \in \{1, ..., \widehat{i}, ..., \widehat{j}, ..., n\}$ (hat means omit those numbers),

2. $a_i = b_j$, $a_j = b_i$, and $a_i < a_j$.

An $EL-$labeling for S_n is constructed by Edelman [4] as follows. Let $\Gamma = [n] \times [n]$ be the poset of pairs, ordered lexicographically: $(i, j) \leq (r, s)$ if $i < r$, or $i = r$ and $j < s$. Define $f((x, y)) = (a_i, a_j)$, if $y = (b_1, ..., b_n)$ covers $x = (a_1, ..., a_n)$ such that

1. $a_k = b_k$ for $k \in \{1, ..., \widehat{i}, ..., \widehat{j}, ..., n\}$,

2. $a_i = b_j$, $a_j = b_i$, and $a_i < a_j$.

Theorem 21. *([4]) The symmetric group S_n with Bruhat-Chevalley ordering is lexicographically shellable.*

We depict the the EL-labeling of S_3:

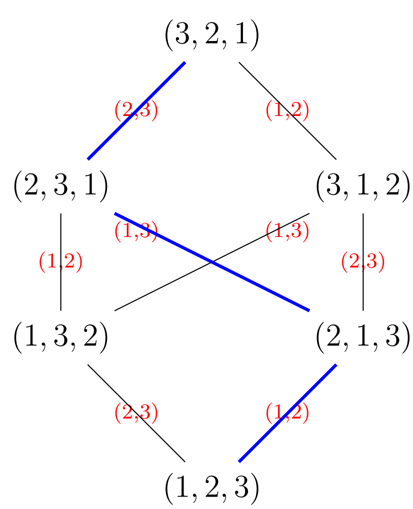


Figure 3.1: The EL-labeling of S_3.

3.2 The rook monoid R_n

3.2.1 Lexicographic shellability of the rook monoid

We revisit the results of Can in [9]. Recall that the rook monoid R_n is the monoid of all 0-1 matrices with at most one 1 in each line and each column. We use the one-line notation as defined in Section 2.6.

There are two different types of covering relations in R_n as layed out in the following lemmas.

Lemma 22. *([9]) Let $x = (a_1, ..., a_n)$ and $y = (b_1, ..., b_n)$ be elements of R_n. Suppose that $a_k = b_k$ for all $k = \{1, ..., \widehat{i}, ..., n\}$ and $a_i < b_i$. Then, y covers x if and only if either*

1. *$b_i = a_i + 1$, or*
2. *there exists a sequence of indices $1 \leq j_1 < \cdots < j_s < i$ such that the set $\{a_{j_1}, ..., a_{j_s}\}$ is equal to $\{a_i + 1, ..., a_i + s\}$, and $b_i = a_i + s + 1$.*

Lemma 23. *([9]) Let $x = (a_1, ..., a_n)$ and $y = (b_1, ..., b_n)$ be two elements of R_n. Suppose that $a_j = b_i$, $a_i = b_j$ and $a_j < a_i$ where $i < j$. Furthermore, suppose that for all $k \in \{1, ...\widehat{i}, ..., \widehat{j}, ..., n\}$, $a_k = b_k$. Then, y covers x if and only if for $s = i + 1, ..., j - 1$, either $a_j < a_s$, or $a_s < a_i$.*

If a covering relation is as in Lemma 22 we call it *type 1* and it is called *type 2* if it is as in Lemma 23.

Using these two lemmas, we define an $EL-$labeling on R_n

$$F : C(R_n) \longrightarrow \mathit{\Gamma},$$

where $\mathit{\Gamma}$ is the poset $\mathit{\Gamma} = \{0, 1, ..., n\} \times \{0, 1, ..., n\}$ with respect to lexicographic ordering.

Let $(x, y) \in C(R_n)$. We define

$$F((x,y)) = \begin{cases} (a_i, b_i), & \text{if } y \text{ covers } x \text{ by type 1} \\ (a_i, a_j), & \text{if } y \text{ covers } x \text{ by type 2.} \end{cases} \tag{3.3}$$

Theorem 24. *[9] Let $\mathit{\Gamma} = \{0, 1, ..., n\} \times \{0, 1, ..., n\}$, and let $F : C(R_n) \longrightarrow \mathit{\Gamma}$ be the edge-labeling, defined as in (3.3). Then F is an $EL-$labeling for R_n.*

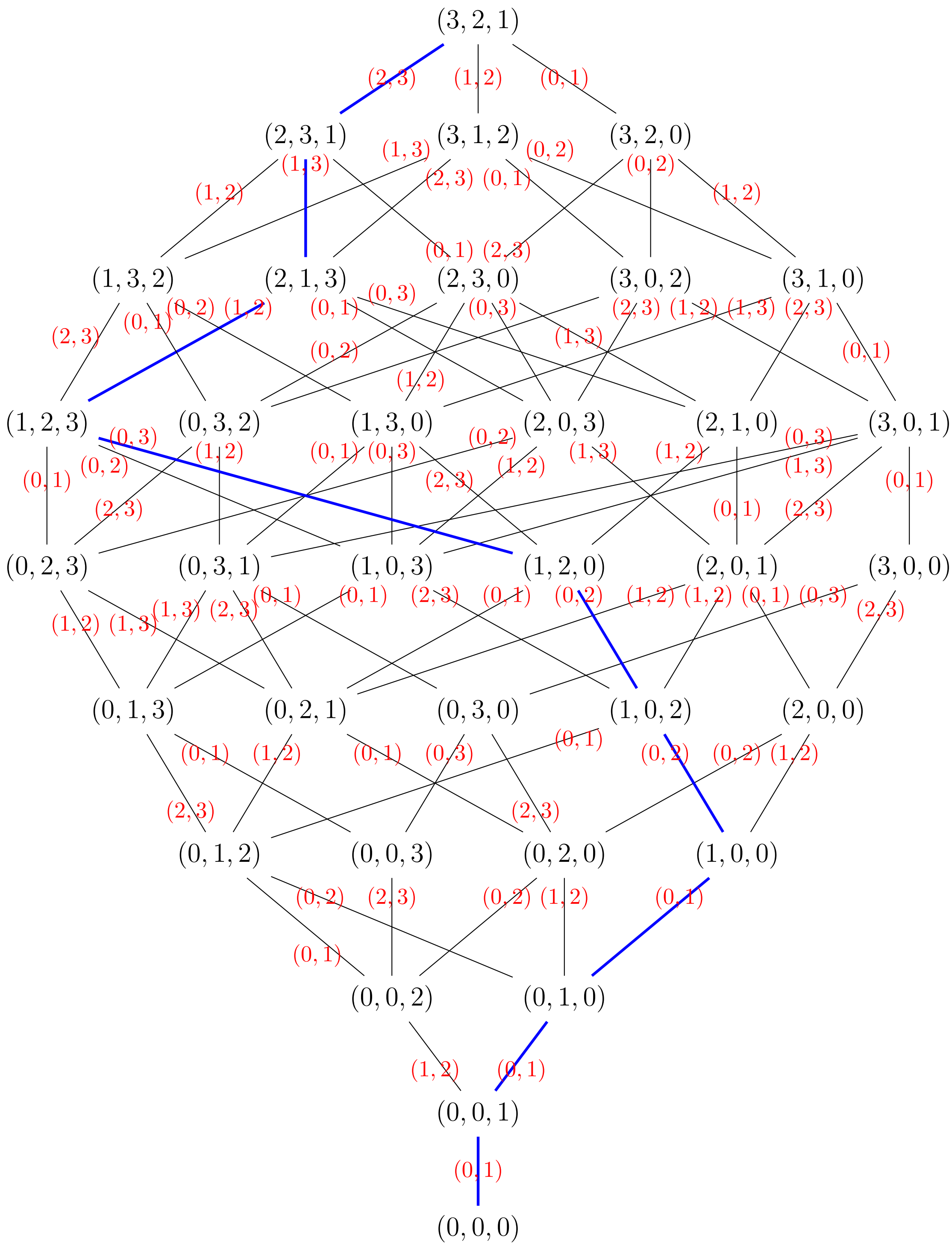


Figure 3.2: $EL-$labeling of the rook monoid R_3.

Chapter 4

The posets of involutions I_n and partial involutions P_n

4.1 The poset of involutions I_n

4.1.1 An EL-labeling of invertible involutions.

In [6], Incitti shows that the poset of invertible involutions is EL-shellable. Let us briefly recall his arguments.

For a permutation $\sigma \in S_n$, a *rise* of σ is a pair $(i, j) \in [n] \times [n]$ such that

$$i < j \text{ and } \sigma(i) < \sigma(j).$$

A rise (i, j) is called *free*, if there is no $k \in [n]$ such that

$$i < k < j \text{ and } \sigma(i) < \sigma(k) < \sigma(j).$$

For $\sigma \in S_n$, define its *fixed point set, its exceedance set* and its *defect set* to be

$$I_f(\sigma) = Fix(\sigma) = \{i \in [n] : \sigma(i) = i\},$$

$$I_e(\sigma) = Exc(\sigma) = \{i \in [n] : \sigma(i) > i\},$$

$$I_d(\sigma) = Def(\sigma) = \{i \in [n] : \sigma(i) < i\},$$

respectively.

The *type* of a rise (i, j) is defined to be the pair (a, b), if $i \in I_a(\sigma)$ and $j \in I_b(\sigma)$, for some $a, b \in \{f, e, d\}$. We call a rise of type (a, b) an *ab-rise*. Two kinds of ee-rises have to be distinguished from each other; an ee-rise is called *crossing*, if $i < \sigma(i) < j < \sigma(j)$, and it is called *non-crossing*, if $i < j < \sigma(i) < \sigma(j)$. The rise (i, j) of an involution $\sigma \in I_n$ is called *suitable*, if it is free and if its type is one of the following: $(f, f), (f, e), (e, f), (e, e), (e, d)$. We depict these possibilities in the first two columns in Figure 4.2, below. It is easy to check that each involution τ in the right column in Figure 4.2 covers the corresponding σ in the left column. In this case, the covering relation is called a *covering transformation* of type (i, j), and τ is denoted by $ct_{(i,j)}(\sigma)$. In [6], Incitti shows that these covering transformations exhaust all possible covering relations in I_n, and moreover, he shows that the labeling

$$F((\sigma, ct_{(i,j)}(\sigma))) := (i, j) \in [n] \times [n]$$

is an EL-labeling for I_n.

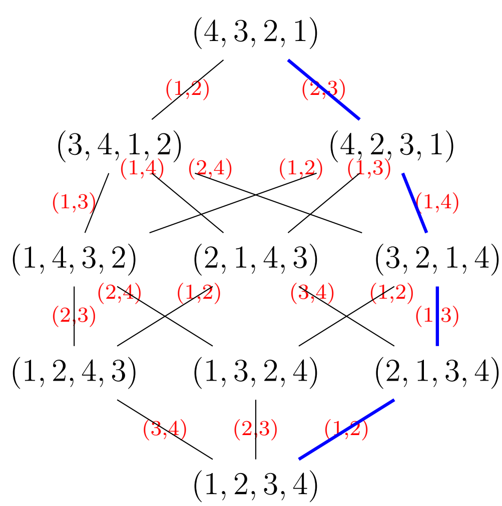


Figure 4.1: The EL-labeling of I_4.

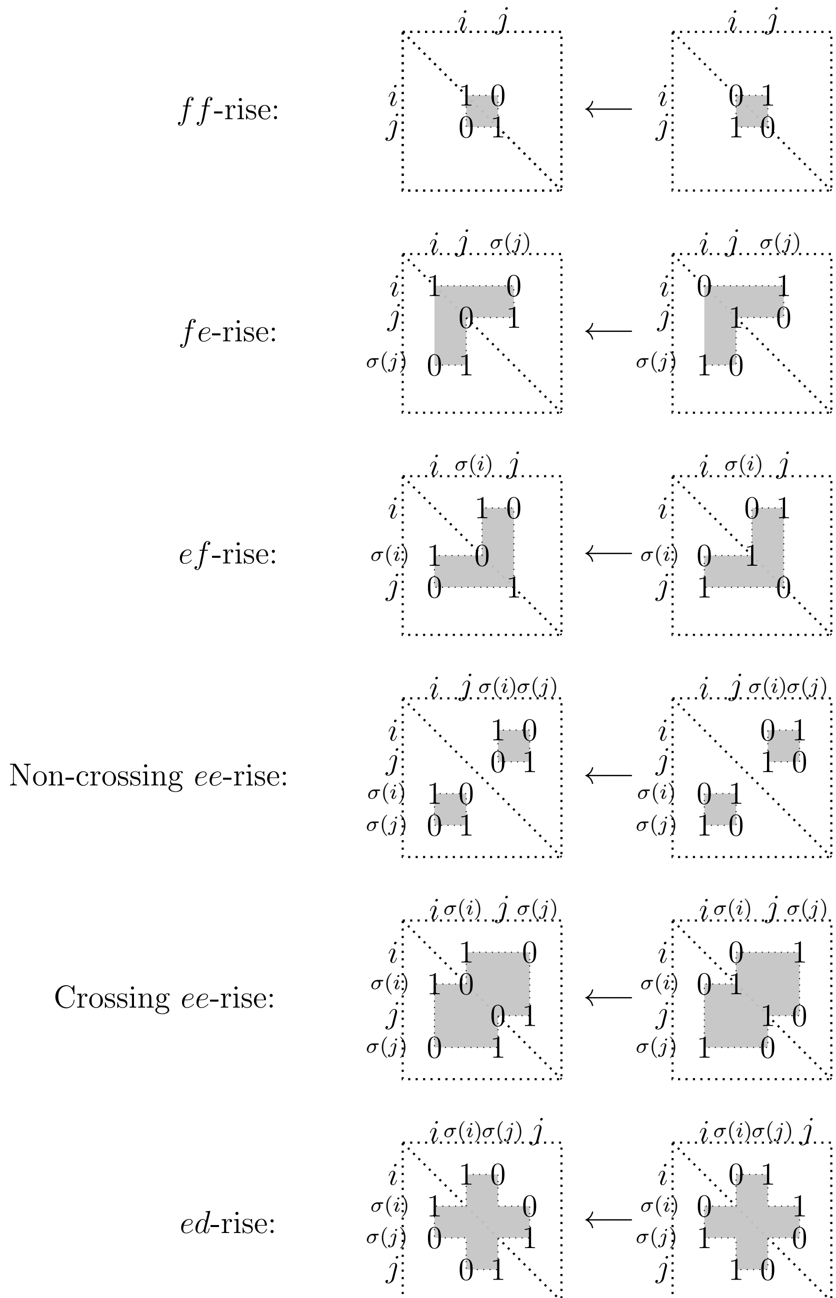


Figure 4.2: Covering transformations $\sigma \leftarrow \tau = ct_{(i,j)}(\sigma)$ of I_n.

4.2 The poset of partial involutions P_n

The results of this section were obtained in joint work with Can [11].

4.2.1 Covering relations of partial involutions P_n.

Covering relations in P_n depend on a numerical invariant associated with the rank-control matrices. For any non-negative integer k, define $r_{0,k}$ to be 0. For a rank-control matrix $Rk(x) = (r_{ij})$, define

$$D(x) = \#\{(i,j) | 1 \leq i \leq j \leq n \text{ and } r_{ij} = r_{i-1,j-1}\}.$$

For example, let $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $Rk(x) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Then $D(x) = \#\{(2,2),(2,3)\} = 2$.

In [26], Bagno and Cherniavsky prove that, in $(P_n, \leq)$,

$$x \text{ covers } y \iff R(x) \leq_R R(y) \text{ and } D(x) = D(y) + 1.$$

However, we need a finer classification of the covering types. The notion of a suitable rise on involutions extends to the partial involutions $(P_n, \leq)$, verbatim. Of course, there are additional covering relations. In this section we exhibit all of them.

Lemma 25. *Let x and y be two partial involutions. Then x covers y if and only if one of the following is true:*

1. *x and y have the same zero-rows and columns. Let $\widetilde{x}$ and $\widetilde{y}$ denote the full rank involutions obtained from x and y, respectively, by deleting common zero rows and columns. Then x covers y if and only if $\widetilde{x}$ covers $\widetilde{y}$. For example,*

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

2. *Without removing a suitable rise, x is obtained from y by one of the following moves:*

   (a) *a 1 on the diagonal is moved down diagonally to the first available diagonal entry. It is possible for a 1 to be pushed out of the matrix. For example,*

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (b) *Two off-diagonal symmetric 1's are pushed right/down or down/right to the first available entries at symmetric positions. There are two cases which we demonstrate by examples:*

      i. $y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *is covered by* $x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, *ii.* $y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *is covered by* $x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

As a special case of ii., if there are no available entries at symmetric positions to push down and right, then the two 1's at positions (i, j) *and* (j, i) *with* $i > j$ *are pushed to* (i, i)*, and to the first available diagonal entry below* (i, i)*. For example,*

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, a single 1 is allowed to be pushed out of the matrix. For example, $y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *is covered by* $x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Before we start the proof, let us illustrate by an example, what it means to remove a suitable rise:

Example 26. *Let* $y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *and let* $x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$*. Then* x *is obtained from* y *by a move as in 2.(b)i., however, it removes the suitable rise* $(1, 3)$*. Therefore, it is not a covering relation.*

Proof. Comparing the rank-control matrices $Rk(\cdot)$ as well as the invariants $D(\cdot)$ of x and y, the "if" direction of the claim is straightforward to verify.

We prove the "only if" direction by contraposition. To this end let x denote a partial involution that covers $y \in P_n$, and x is not obtained by one the moves as in 1., 2.(a), or 2.(b).

Since 1. does not hold, x has a smallest row consisting of zeros such that the corresponding row of y contains a non-zero entry. Let i denote the index of this zero row of x. Notice that, if there is a zero row for both x and y with the same index, then removing this row and the corresponding column does not have any effect on the remaining entries of the rank-control matrices. Therefore, we assume that neither x nor y has a zero row before the i-th row.

There are two subcases;

I) the nonzero entry in row i of y does not occur on the i-th column,

II) it occurs on the i-th column.

Realize that if the nonzero entry in row i of y occurs after the i-th column, then the 1's are in the symmetric positions (i, j) and (j, i). Without loss of generality we can therefore assume that $i > j$ in case I), i.e. the nonzero entry in row i of y occurs before the i-th column.

We proceed with I). Then y and x are as in

$$y = \begin{pmatrix} & & & 0 & & \\ & A & & \vdots & B & \\ & & & 1 & & \\ & & & \vdots & & \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ & C & & \vdots & D & \\ & & & 0 & & \end{pmatrix} \text{ and } x = \begin{pmatrix} & & & 0 & & \\ & A' & & \vdots & B' & \\ & & & 0 & & \\ & & & \vdots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & C' & & \vdots & D' & \\ & & & 0 & & \end{pmatrix},$$

where $A, A'B, B', \ldots$ stand for appropriate size matrices. Let $1 \leq k < i$ denote the index of the row of y with a 1 on its i-th entry.

Let Γ denote the set of coordinates of non-zero entries (r, s) of x satisfying $k \leq r \leq n$ and $i < s \leq n$. Since the upper $k \times n$ portions of both of y and x are of rank k, $\Gamma \neq \emptyset$.

Let $(r, s) \in \Gamma$ denote the entry with smallest column index. Unless $r = s$, we define x_1 to be the matrix obtained from x by moving the non-zero entries at the positions (r, s) and (s, r) (which exists, by symmetry) to the positions (r, i) and (i, r). If $r = s$, then x_1 is defined by moving the non-zero entry to the (i, i)-th position.

We claim that $y \leq x_1 < x$. Indeed, since x_1 is obtained from x by reverse of the one of the moves 2.(a) or 2.(b), the second inequality is clear. The first inequality follows immediately from checking the corresponding rank-control matrices of x, x_1 and of y. Let us illustrate the procedure by two possible scenarios:

Example 27. *Let*

$$y = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $i = 5$, $k = 1$, *and the rank-control matrices of* y *and* x *are*

$$Rk(y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\ 1 & 2 & 3 & 3 & 4 & 4 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 & 5 & 5 & 6 \\ 1 & 2 & 3 & 3 & 4 & 5 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 6 & 7 \end{pmatrix} \text{ and } Rk(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 0 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 1 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\ 1 & 2 & 3 & 3 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

In this case,

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } Rk(x_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 0 & 1 & 2 & 2 & 3 & 4 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 & 5 & 5 & 6 \\ 1 & 2 & 3 & 3 & 4 & 5 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 6 & 7 \end{pmatrix}.$$

If

$$y = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \textit{ and } x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then $i = 4$, $k = 1$ *and the rank-control matrices are*

$$Rk(y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 4 & 5 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \textit{ and } Rk(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 2 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 3 & 4 & 5 & 6 & 6 \\ 1 & 2 & 3 & 3 & 4 & 5 & 6 & 6 \end{pmatrix}.$$

In this case,

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } Rk(x_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 2 & 2 & 3 & 4 & 4 \\ 0 & 1 & 2 & 2 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 4 & 5 & 6 & 6 \\ 1 & 2 & 3 & 4 & 4 & 5 & 6 & 6 \end{pmatrix}.$$

We proceed with case II) that the non-zero entry of y in its i-th column occurs at the k-th row, where $k \geq i$.

First of all, without loss of generality, we may assume that x has a non-zero entry in its $(i+1)$-st row, whose column index we denote by j_x.

Denote by y_1 the partial involution obtained from y by interchanging its i-th and $(i+1)$-st rows as well as interchanging its i-th and $(i+1)$-st columns. If it exists, let j_y denote the column index of the non-zero entry of y_1 in its i-th row. If $j_y < k$, then, $y < y_1$. Furthermore, in this case, because the i-th row of x consists of 0's, $y_1 < x$. In other words, we have $y < y_1 < x$.

Therefore, we assume that $k < j_y$. In this case, if $k < j_x$, then let x_1 denote the partial involution obtained from x by interchanging its i-th and $(i+1)$-st rows as well as interchanging its i-th and $(i+1)$-st columns. Then we have $y < x_1 < x$ and we are done. Therefore, we assume that $k > j_x$. But in this case $y < y_1 < x$ holds. This finishes the proof of the case 2), and we conclude the result.

□

4.2.2 EL-labeling of P_n.

We define an edge labeling of P_n and prove that it is an EL-labeling.

1. If the covering relation is derived from a regular covering of an involution, namely from a move that is as in Lemma 25, Part 1., then we use the labeling as defined in [6].

2. If the covering relation results from a move as in Lemma 25 Part 2.(a), namely from a diagonal push where the element that is pushed from is at the position (i, i), then we label it by (i, i).

3. Suppose that a covering relation is as in Lemma 25 (b). Observe that, in all of these covering relations, one of the 1's is pushed down and the other is pushed right. Let i denote the column index of the first 1 that is pushed to the right, and let j denote the index of the resulting column. Then we label the move by (i, j).

To illustrate the third labeling let us present a few examples. Also, see Figure 4.3 on page 47 for the labeling of P_3 which is depicted in one-line notation.

Example 28.

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \textit{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding labeling here is $(3, 5)$.

Example 29.

$$y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \textit{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding labeling here is $(1, 3)$.

Example 30.

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \textit{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding labeling here is $(2, 3)$.

If x covers y with label (i, j), then we refer to it as an $(i, j) - covering$ and say that y is obtained from x by an (i, j)-move. Alternatively, we call a covering relation a *c-cover*, if it is derived from an involution; a *d-cover*, if it is obtained by a shift of a diagonal element; an *r-cover*, if it is derived from a right/down or a down/right move. We will refer to the corresponding moves as *c*-, *d*- and *r*-moves.

Let Γ denote the lexicographic order on the product $[n] \times [n]$. Then, for any $k > 0$, $\Gamma^k = \Gamma \times \cdots \times \Gamma$ is totally ordered with respect to lexicographic ordering. Finally, let $F : C(P_n) \to \Gamma$ denote the labeling function defined above.

For an interval $[x, y] \subseteq P_n$ and a maximal chain $\mathfrak{c} : x = x_0 < \cdots < x_k = y$, we denote by $F(\mathfrak{c})$ the Jordan-Hölder sequence of labels of $\mathfrak{c}$:

$$F(\mathfrak{c}) = (F((x_0, x_1)), \ldots, F((x_{k-1}, x_k))) \in \mathit{\Gamma}^k.$$

Proposition 31. *Let $\mathfrak{c} : x = x_0 < \cdots < x_k = y$ be a maximal chain in $[x, y]$ such that its Jordan-Hölder sequence $F(\mathfrak{c})$ is lexicographically smallest among all Jordan-Hölder sequences (of chains in $[x, y]$) in $\mathit{\Gamma}^k$. Then,*

$$F((x_0, x_1)) \leq F((x_1, x_2)) \leq \cdots \leq F((x_{k-1}, x_k)). \tag{4.1}$$

Proof. Assume that (4.1) is not true. Then, there exist three consecutive terms

$$x_{t-1} < x_t < x_{t+1}$$

in $\mathfrak{c}$, such that $F((x_{t-1}, x_t)) > F((x_t, x_{t+1}))$. We have 9 cases to consider.

Case 1: $\mathit{type}(x_{t-1}, x_t) = c$, and $\mathit{type}(x_t, x_{t+1}) = c$.

Case 2: $\mathit{type}(x_{t-1}, x_t) = d$, and $\mathit{type}(x_t, x_{t+1}) = d$.

Case 3: $\mathit{type}(x_{t-1}, x_t) = d$, and $\mathit{type}(x_t, x_{t+1}) = c$.

Case 4: $\mathit{type}(x_{t-1}, x_t) = c$, and $\mathit{type}(x_t, x_{t+1}) = d$.

Case 5: $\mathit{type}(x_{t-1}, x_t) = r$, and $\mathit{type}(x_t, x_{t+1}) = r$.

Case 6: $\mathit{type}(x_{t-1}, x_t) = d$, and $\mathit{type}(x_t, x_{t+1}) = r$.

Case 7: $\mathit{type}(x_{t-1}, x_t) = r$, and $\mathit{type}(x_t, x_{t+1}) = d$.

Case 8: $\mathit{type}(x_{t-1}, x_t) = r$, and $\mathit{type}(x_t, x_{t+1}) = c$.

Case 9: $\mathit{type}(x_{t-1}, x_t) = c$, and $\mathit{type}(x_t, x_{t+1}) = r$.

In each of these 9 cases, we either produce an immediate contradiction by showing that we can interchange the two moves, or we construct an element $z \in [x, y]$

which covers x_{t-1}, and such that $F((x_{t-1}, z)) < F((x_{t-1}, x_t))$. Since we assume that $F(\mathfrak{c})$ is the lexicographically first Jordan-Hölder sequence, the existence of z is a contradiction, too.

Case 1: Straightforward from the fact that type c covering relations have identical labelings with Incitti's [6].

Case 2: Suppose that the first move is labeled (i, i) and the second one (j, j) with $j < i$. If the two moves are not interchangeable then (j, i) is a legal c-move in x_{t-1}. Since (j, i) is lexicographically smaller than (i, i), we derive a contradiction.

Case 3: Let (i, i) be moved to (j, j) in the first step (type d move), hence $i < j$. If the following c-move does not involve the entry at (j, j), then either the c- and the d-move commute with each other, or the rise for the c-move is not free in x_{t-1}. In that case there has to be an ef-rise involving the entry at the position (i, i). This ef-rise has a smaller label than (i, i), which is a contradiction.

Thus we may assume that the c-move involves the entry at the (j, j)-th position. Then the c-move has to come from either an ff-, an fe-, or an ef-rise.

Type ff is not possible: Let (a, b) the corresponding label. The move involves the entry at the (j, j)-th position if either $a = j$ or $b = j$. If $a = j$ then $(a, b) > (i, i)$ and the labels are increasing. If $b = j$, then we must have $a < i$ for $(a, b) < (i, i)$. Therefore, there is a legal c-move (a, i) in x_{t-1} has a smaller label than (i, i).

Type fe is not possible since (j, b) is greater than (i, i).

Finally, ef is not possible: Let (k, j) be the label of the c-move. If (k, i) is a suitable rise in x_{t-1}, then $(k, i) < (i, i)$. If (k, i) is not a suitable rise in x_{t-1}, let $(j, j), (k, l)$, and (l, k) denote the entries involved in the c-move where $l < k$. Then

$l < i < k$ and $(l, j) < (i, i)$. (l, j) is a legal r-move in x_{t-1} with a smaller label than (i, i). This concludes case 3.

Case 4: This is not possible since no c-move places a 1 on the diagonal such that moving this 1 gives rise to a smaller labeling than the c-move. Note that if there is a 1 on the diagonal before the c-move takes place, then moving this 1 first creates an element z with covering label lexicographically smaller that of the c-move. Thus we are done with this case.

Case 5: Let the first move be labeled (i, j) and the second (k, l). Then it is clear that if $k = j$ we obtain an increasing sequence. If $k = i$, then we can switch the order of the moves. If $k \neq \{i, j\}$, then, if possible, we perform the second move first. If it is not possible to interchange the order of the r-moves, then by the moving the element on column k in x_{t-1} to the right a suitable rise is removed. But then the corresponding c-cover has a smaller label in x_{t-1} than (i, j).

Case 6: We either perform the r-move first if possible, or perform the c-cover corresponding to the suitable rise removed by d-move which has a smaller label than the d-move in x_{t-1}.

Case 7: Similar to *Case 6* so we omit the proof.

Case 8: The c-move has to include the elements moved by the previous r-move since otherwise the c-move can be performed first.

If the suitable rise is created by the r-move then the label of the r-move is smaller than the label of the c-move. Otherwise, there is a suitable rise in x_{t-1} involving the elements moved by the r-move. But the c-move corresponding to this suitable rise has a smaller label than the r-move.

Case 9: If the r-move does not involve an element moved by the c-move then perform the r-move first. If this is not possible then a suitable rise is removed by moving it. The c-move corresponding to this suitable rise has a smaller label than the other c-move.

If the r-move involves an element that is placed at this position by the preceding c-move, then we proceed to exhibit every c-move to exclude all of them:

ff: The label of c-move is (i, j). The smaller r-move involving a new element can only be (i, k) with $k < j$. But then (i, i) is possible in x_{t-1} and $(i, i) < (i, j)$.

fe: Similar to *ff* so we omit the proof.

ef: The label of c-move is (i, j). The smaller r-move involving a new element can only be (i, k) with $k < j$. Then (i, k) is possible in x_{t-1} and $(i, k) < (i, j)$.

The cases of non-crossing ee, crossing ee and ed are similar to ef so we omit the proof.

$\square$

Proposition 32. *We continue the notation of Proposition 31. There exists a unique maximal chain* $x = x_0 < \cdots < x_k = y$ *with* $F((x_0, x_1)) \leq \cdots \leq F((x_{k-1}, x_k))$.

Proof. We already know that the lexicographically first chain is increasing. Therefore, it is enough to show that there is no other increasing chain. We prove this by induction on the length of the interval $[x, y]$. Clearly, if y covers x, there is nothing to prove. So, we assume that for any interval of length k there exists a unique increasing maximal chain.

Let $[x, y] \subseteq P_n$ be an interval of length $k+1$, and let

$$\mathfrak{c} : x = x_0 < x_1 < \cdots < x_k < x_{k+1} = y$$

be the maximal chain such that $F(\mathfrak{c})$ is the lexicographically first Jordan-Hölder sequence in Γ^{k+1}.

Assume that there exists another increasing chain

$$\mathfrak{c}' : x = x_0 < x_1' < \cdots < x_k' < x_{k+1} = y.$$

Since the length of the chain

$$x_1' < \cdots < x_k' < x_{k+1} = y$$

is k, by the induction hypotheses, it is the lexicographically first chain between x_1' and y.

We are going to find contradictions to each of the following possibilities.

Case 1: $type(x_0, x_1) = c$, and $type(x_0, x_1') = c$,

Case 2: $type(x_0, x_1) = d$, and $type(x_0, x_1') = d$,

Case 3: $type(x_0, x_1) = d$, and $type(x_0, x_1') = c$,

Case 4: $type(x_0, x_1) = c$, and $type(x_0, x_1') = d$,

Case 5: $type(x_0, x_1) = r$, and $type(x_0, x_1') = r$,

Case 6: $type(x_0, x_1) = d$, and $type(x_0, x_1') = r$,

Case 7: $type(x_0, x_1) = r$, and $type(x_0, x_1') = d$,

Case 8: $type(x_0, x_1) = r$, and $type(x_0, x_1') = c$,

Case 9: $type(x_0, x_1) = c$, and $type(x_0, x_1') = r$,

In each of these cases we will construct a partial involution z such that z covers

x_1' and $F((x_1', z)) < F((x_1', x_2'))$. Contradiction to the induction hypothesis.

Case 1: Proved in [6].

Case 2: $F(x_0, x_1) = (i, i) < F(x_0, x_1') = (j, j)$ with $i < j$. In x_1' (i, i) is a legal covering move. Hence we have our desired contradiction: $(j, j) \leq F((x_1', x_2')) \leq (i, i)$.

Case 3: $F(x_0, x_1) = (i, i) < F(x_0, x_1') = (j, k)$. There are two cases to consider: $i = j$ and $i < j$. If $i < j$ then we can reverse the order of the d and c move and get to the same contradiction as in *Case 2*. If $i = j$ then $k \neq i + 1$ since otherwise (i, i) is not a possible move x_0. The d-cover moves (i, i) to (l, l) where $l < k$. But then (i, l) is a legal move of x_1' and $(i, l) < (i, k)$ which is a contradiction.

Case 4: $F(x_0, x_1) = (i, j) < F(x_0, x_1') = (k, k)$. There are two cases to be considered: $j = k$ and $j \neq k$. If $j \neq k$ then $k \notin [i, j]$ since otherwise (i, j) is not a suitable rise, hence $k > j$. But this means the two covering moves are interchangeable. We get to the same contradiction as in the preceding cases.

If $j = k$ then (i, j) is a legal r-move of x_1' with $(i, j) < (k, k)$.

Case 5: $F(x_0, x_1) = (i, j) < F(x_0, x_1') = (k, l)$. $k > i$ since there is at most one legal r-move of each element. We also have $j < l$ since otherwise either (i, k) or (i, l) is a suitable rise with a label less than (i, j). We have two cases to consider:

(a) $i < j < k < l$

(b) $i < k < j < l$

In case (a), the two moves are interchangeable.

In case (b), (i,k) is a suitable rise of x_0 with $(i,k) < (i,j)$.

Case 6: $F(x_0, x_1) = (i,i) < F(x_0, x_1') = (j,k)$. By construction $j \neq i$. Hence $i < j < k$ and therefore (j,k) does not influence the move (i,i) and we derive a contradiction.

Case 7: $F(x_0, x_1) = (i,j) < F(x_0, x_1') = (k,k)$. We have $k > j$ since otherwise a suitable rise is removed by (i,j). But then (i,j) is a legal move of x_1'.

Case 8: $F(x_0, x_1) = (i,j) < F(x_0, x_1') = (k,l)$. Two cases need to be considered: $i < k$ and $i = k$, $i < j < l$. If $i < k$ then $j < k$ since otherwise (i,j) removes a suitable rise. But this means that (i,j) is a legal move of x_1'.

If $i = k$ then the c-move corresponds to an ef, non-crossing ee, crossing ee or a ed rise. In each of these cases (i,j) is a legal move of x_1'.

Case 9: $F(x_0, x_1) = (i,j) < F(x_0, x_1') = (k,l)$. We have two cases to consider:

(a) $i = k$, $i < j < l$

(b) $i < k$

(a) does not occur because then the r-move removes a suitable rise, hence, it is not a covering relation.

(b) Since we have $i < k < l$ and $i < j$, we consider $i < j < k < l$, $i < k < j < l$ and $i < k < l < j$. In all these cases the c- and the r-moves are interchangeable. This concludes our proof.

$\square$

Combining previous two propositions we obtain our first main result.

Theorem 33. *The poset of partial involutions is lexicographically shellable.*

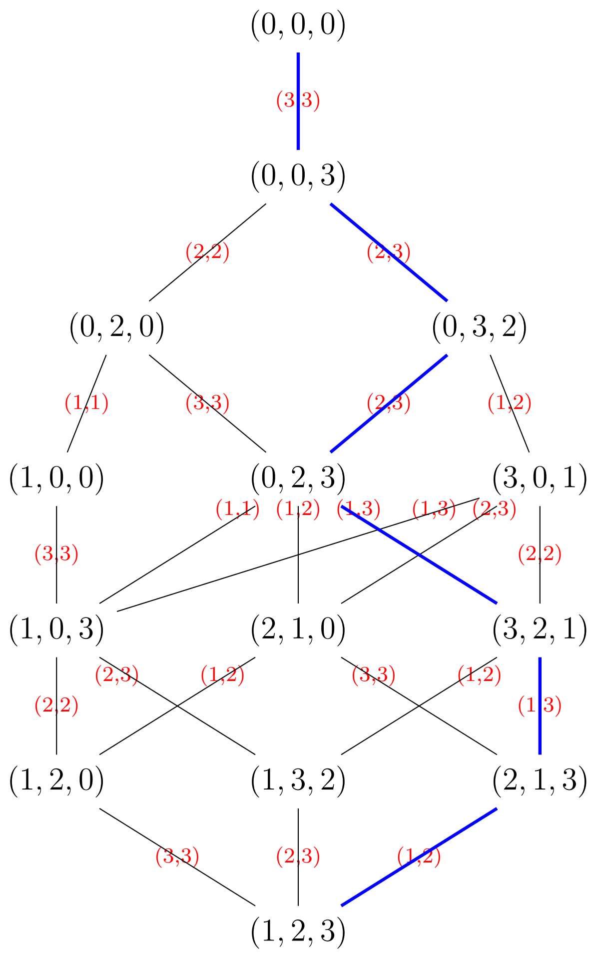


Figure 4.3: The EL-labeling of P_3.

Chapter 5

Fixed-Point-Free Involutions F_{2n} and Partial Fixed-Point-Free Involutions PF_n

5.1 Fixed-point-free involutions F_{2n}

This section is joint work with Can and Cherniavsky [7].

5.1.1 Saturated chains of fixed-point-free involutions

Lemma 34. *Let $x = (x_{ij})_{i,j=1}^{2n}$ be an element in F_{2n}. Then the number of equalities along the main diagonal of $Rk(x)$ is equal to n.*

Proof. Since all diagonal entries of x are zero, as we move down along the main diagonal of $Rk(x)$, in each step there are exactly two possibilities: (1) $r_{i+1,i+1} = r_{ii}$, or (2) $r_{i+1,i+1} = r_{ii}+2$. Indeed, as we move from the (i,i)-th position to the $(i+1,i+1)$-th entry, the new minor gains either two new non-zero entries, say $x_{i+1,k} = 1$ and $x_{k,i+1} = 1$, or $x_{i+1,k} = x_{k,i+1} = 0$ for all $1 \leqslant k \leqslant i+1$. The matrix x is invertible, so $r_{2n,2n} = 2n$. Since each step can increase the value r_{ii} only by 2 or leave it alone, r_{ii} has to increase exactly n times and has to stay the same exactly n times.

□

Proposition 35. *Consider F_{2n} as a subposet of I_{2n} and let $x, y \in F_{2n}$ be two elements such that $x \leq y$. Then there exits a saturated chain in I_{2n} from x to y consisting of* *fixed-point-free involutions only.*

Proof. First observe that there exists a saturated chain from x to y in PF_{2n} and since F_{2n} is an interval in PF_{2n} this chain consists of fixed-point-free involutions only. Let $\ell_{P_n}(x)$ and $\ell_{PF_n}(x)$ denote the length functions of P_n and PF_n, respectively. (For a concrete description of $\ell_{PF_{2n}}(x)$ see Section 5.2.3.) For two fixed-point-free involutions u and w of this chain such that w covers u in PF_{2n} we have $Rk(u) < Rk(w)$ and $\ell_{PF_{2n}}(w) - \ell_{PF_{2n}}(u) = 1$. Since $Rk(u) < Rk(w)$, we have $u < w$ in I_{2n} also. By Lemma 34 we have

$$\ell_{P_{2n}}(w) - \ell_{P_{2n}}(u) = \ell_{PF_{2n}}(w) + n - (\ell_{PF_{2n}}(u) + n) = 1$$

Therefore w covers u also in I_{2n}, and hence this chain is saturated in I_{2n}, also.

□

5.1.2 Covering transformations in F_{2n}

Since F_{2n} is a connected graded subposet of I_{2n} its covering relations are among the covering relations of I_{2n}. On the other hand, within F_{2n} we use only two types of covering transformations of Figure 4.2 in Section 4.1. For convenience of the reader, we depict these moves in Figure 5.1 and Figure 5.2.

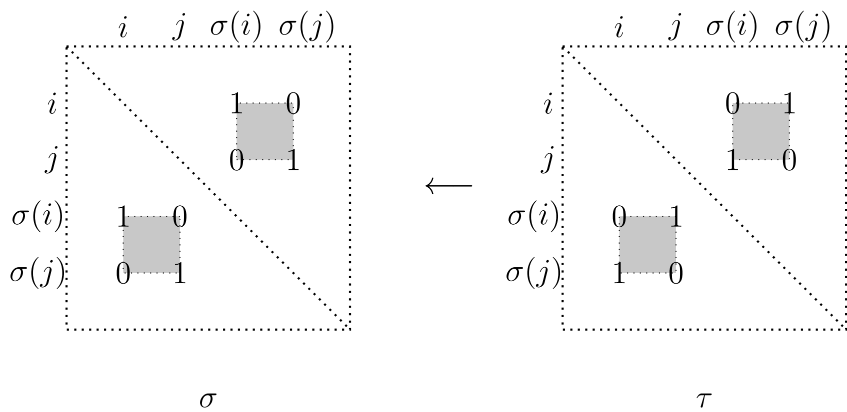


Figure 5.1: (non-crossing) ee-rise for the covering $\tau \to \sigma$.

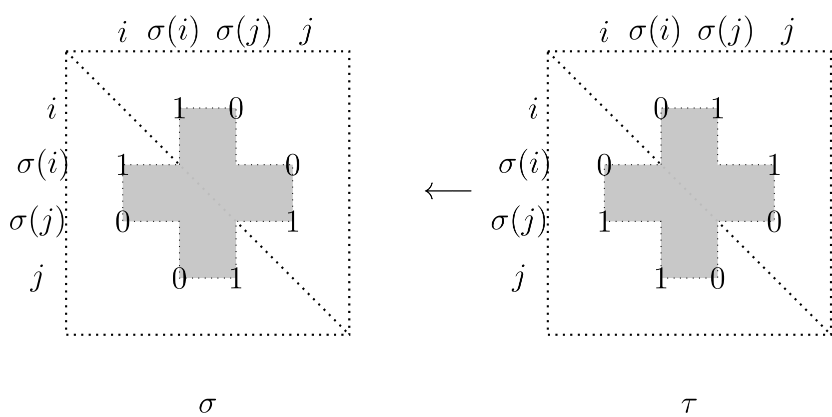


Figure 5.2: ed-rise for the covering $\tau \to \sigma$.

5.1.3 EL-shellability of F_{2n}

Theorem 36. *F_{2n} is an EL-shellable poset.*

Proof. Let x and y be two fixed-point-free involutions. By Proposition 35 we know that there exists a saturated chain between x and y that is entirely contained in F_{2n}. Since lexicographic ordering is a total order on maximal chains, there exists a unique largest such chain. We denote it by

$$\mathfrak{c} : x = x_1 < x_2 < \cdots < x_s = y.$$

The idea of the proof is showing that $\mathfrak{c}$ is the unique decreasing chain and therefore by switching the order of our totally ordered set $\mathbb{Z}^2$ obtaining the lexicographically smallest chain which is the unique increasing chain. See Figure 5.3 on page 62 for an illustration.

Towards a contradiction assume that $\mathfrak{c}$ is not decreasing. Then, there exist three consecutive terms

$$x_{t-1} < x_t < x_{t+1}$$

in $\mathfrak{c}$, such that $f((x_{t-1}, x_t)) < f((x_t, x_{t+1}))$. We have 4 cases to consider.

Case 1: $type(x_{t-1}, x_t) = ee$, and $type(x_t, x_{t+1}) = ee$.

Case 2: $type(x_{t-1}, x_t) = ed$, and $type(x_t, x_{t+1}) = ed$.

Case 3: $type(x_{t-1}, x_t) = ee$, and $type(x_t, x_{t+1}) = ed$.

Case 4: $type(x_{t-1}, x_t) = ed$, and $type(x_t, x_{t+1}) = ee$.

In each of these 4 cases, we either produce an immediate contradiction by showing that the two moves are interchangeable (hence $\mathfrak{c}$ is not the largest chain), or we construct an element $z \in [x, y] \cap F_{2n}$ which covers x_{t-1}, and such that $f((x_{t-1}, z)) > f((x_{t-1}, x_t))$. Since we assume that $f(\mathfrak{c})$ is the lexicographically largest Jordan-Hölder sequence, the existence of z is a contradiction, too.

To this end, suppose that the label of the first move is (i, j), and the second move is labeled by (k, l).

Case 1:

We begin by assuming $\{i, j, x_{t-1}(i), x_{t-1}(j)\} \cap \{k, l\} = \emptyset$.

Assume for the moment that $k > j$. Then the covering transformations (k, l) and (i, j) are independent of each other. Therefore, we assume that $i < k < j$. There are four cases; $x_{t-1}(k) < j$, $j < x_{t-1}(k) < x_{t-1}(i)$, $x_{t-1}(i) < x_{t-1}(k) < x_{t-1}(j)$ or $x_{t-1}(k) > x_{t-1}(j)$. In the first case $(k, x_{t-1}(j))$ is an *ed*-rise for x_{t-1} with a label bigger than (i, j). This is a contradiction. Similarly, in the second case, (k, j) is an *ee*-rise for x_{t-1} with a bigger label than (i, j). The third case leads to a contradiction, because in that case (i, j) is not a suitable rise in x_{t-1}. Finally, in the fourth case the two covering relations (k, l) and (i, j) are independent of each other.

Next we assume that $\{i, j, x_{t-1}(i), x_{t-1}(j)\} \cap \{k, l\} \neq \emptyset$.

We observe that if $k = x_{t-1}(i)$, then we have

$$x_t(k) = x_{t-1} \cdot (i, j) \cdot (x_{t-1}(i), x_{t-1}(j))(k) = j$$

. We obtain $j < x_{t-1}(i) = k < x_t(k) = j$, which is absurd. Similarly, if $k = x_{t-1}(j)$, then we have $x_t(k) = i$, and from $i < x_{t-1}(j) = k < x_t(k) = i$ we obtain another contradiction.

Next observe that if $l = x_{t-1}(i)$, then we have

$$x_t(l) = x_{t-1} \cdot (i, j) \cdot (x_{t-1}(i), x_{t-1}(j))(l) = j$$

, and from $j < x_{t-1}(i) < x_{t-1}(j) = l < x_t(l) = j$ we obtain a contradiction. Likewise,

$l = x_{t-1}(j)$ is impossible.

If $i = k$, then, of course we must have $j < l$. In this case we must also have that $x_t(k) = x_{t-1}(j)$. In this case, it is easy to check that $x_t(l) = x_{t-1}(l)$; therefore, (j,l) is an *ee*-rise for x_{t-1} which is bigger than (i,j), a contradiction. If $j = k$, then we have $x_t(k) = x_{t-1}(i)$. Just as in the previous case, (i,l) is an *ee*-rise for x_{t-1}. Furthermore, $(i,l) > (i,j)$ gives the contradiction. Finally, if $j = l$, then it is easy to check that (k,j) is an *ee*-rise for x_{t-1}; therefore, we have another contradiction, and this finishes the proof of the first case.

Case 2:

We begin with the assumption that $\{i, j, x_{t-1}(i), x_{t-1}(j)\} \cap \{k, l\} = \emptyset$.

Then $k > i$. If $k > x_{t-1}(j)$, then observe that $l > x_t(l) = x_{t-1}(l) > x_t(k) = x_{t-1}(k) > k > x_{t-1}(j)$. It follows that (k,l) is an *ed*-rise for x_{t-1} with a bigger label than (i,j), a contradiction.

We proceed with the assumption that $i < k < x_{t-1}(j)$. If $x_{t-1}(k) > j$, then the two moves are interchangeable. If $x_{t-1}(k)$ is in between i and $x_{t-1}(j)$, then $(k, x_{t-1}(j))$ is an *ee*-rise for x_{t-1}.

We proceed with the assumption that $\{i, j, x_{t-1}(i), x_{t-1}(j)\} \cap \{k, l\} \neq \emptyset$.

If $k = i$, then we have $j < l$. Since x_t is obtained from x_{t-1} by applying the covering transformation (i,j), in this case we see that $x_t(k) = x_{t-1}(j)$. Note also that $x_t(l) = x_{t-1}(l)$. Therefore, $x_{t-1}(j) < x_{t-1}(l) < l$. If $x_{t-1}(l) < j$, then $(x_{t-1}(j), x_{t-1}(l))$ is an *ee*-rise for x_{t-1} with a label bigger than (i,j), which is a contradiction. Otherwise, $(x_{t-1}(j), l)$ is an *ed*-rise for x_{t-1} with a label bigger than (i,j), which is another contradiction.

If $k = j$, then since x_t is obtained from x_{t-1} by the covering transformation of (i,j), $x_t(j) = x_{t-1}(i)$. But this is impossible, because (k,l) is an *ed*-rise for x_t, and hence $k < x_t(k)$ which implies that $j = k < x_{t-1}(i)$.

If $k = x_{t-1}(i)$, then $x_t(k) = j$ hence $x_{t-1}(j) < j < x_t(l) = x_{t-1}(l)$. Therefore, $(x_{t-1}(j), l)$ is an *ed*-rise in x_{t-1}.

If $k = x_{t-1}(j)$, then $k < x_t(k) = i$, which is absurd.

If $l = j$, then we see that (k, l) is an *ed*-rise for x_{t-1}, which is a contradiction.

If $l = x_{t-1}(i)$, then $l > x_t(l) = j > x_{t-1}(i) = l$, which is absurd.

Finally, if $l = x_{t-1}(j)$, then $x_t(k) = x_{t-1}(k)$ and furthermore $x_{t-1}(k) = x_t(k) < x_t(l) = i$. Therefore, $(k, x_{t-1}(i))$ is an *ed*-rise for x_{t-1}, which is a contradiction.

Case 3:

We begin with the assumption that $\{i, j, x_{t-1}(i), x_{t-1}(j)\} \cap \{k, l\} = \emptyset$; hence $k > i$.

If $k > j$, then the order of the covering transformations are interchangeable leading to a contradiction. Therefore we assume that $i < k < j$. If $x_{t-1}(k) > x_{t-1}(j)$, then once again in this case the two moves are interchangeable. On the other hand, if $x_{t-1}(i) < x_{t-1}(k) < x_{t-1}(j)$, then (i, j) is not a suitable rise for x_{t-1}, which is a contradiction.

If $x_{t-1}(k) < x_{t-1}(i)$, then we consider two cases: $x_{t-1}(k) > j$ and $x_{t-1}(k) < j$. In the former case, either the two moves are interchangeable, or (k, j) is an *ee*-rise for x_{t-1} with a bigger label than (i, j), hence a contradiction.

In the latter case, we have $i < x_{t-1}(k) < j$. In this case, if $l < x_{t-1}(i)$, then the two moves are interchangeable. If $x_{t-1}(i) < l < x_{t-1}(j)$, then either $x_{t-1}(l)$ is in between i and j or $x_{t-1}(l)$ is greater than j. In the former case, (i, j) is not a suitable rise. If $x_{t-1}(l) > j$, then (k, l) is not a suitable rise for x_t, because in this case $x_{t-1}(k) < x_t(j) = x_{t-1}(i) < x_{t-1}(l)$. Now, if $x_{t-1}(j) < l$, then we have two possibilities again; either $x_{t-1}(l) > j$ or $x_{t-1}(l) < j$. In the former case, (k, l) is not a suitable rise for x_t. In the latter case, the two moves are interchangeable.

We proceed with the assumption that $\{i, j, x_{t-1}(i), x_{t-1}(j)\} \cap \{k, l\} \neq \emptyset$.

If $k = i$ then (j, l) is an *ed*-rise for x_{t-1}. Indeed, in this case, $x_t(l) = x_{t-1}(l)$ and we have the inequalities $j < x_{t-1}(j) = x_t(k) < x_t(l) = x_{t-1}(l) < l$.

If $k = j$ then either $x_{t-1}(l) < x_{t-1}(j)$, or $x_{t-1}(l) > x_{t-1}(j)$. In the former case, we see that (i, l) is an *ed*-rise for x_{t-1}. In the latter case (j, l) is an *ed*-rise for x_{t-1}.

If $k = x_{t-1}(i)$, then $k < x_t(k) = x_{t-1} \cdot (i, j) \cdot (x_{t-1}(i), x_{t-1}(j))(k) = j$. Since $j < x_{t-1}(i)$, this is a contradiction. Similarly, if $k = x_{t-1}(j)$, then $k < x_t(k) = x_{t-1} \cdot (i, j) \cdot (x_{t-1}(i), x_{t-1}(j))(k) = i$. Since $i < x_{t-1}(i)$, this is a contradiction, also.

If $l = j$, then we obtain a contradiction to the facts that (k, l) is an *ed*-rise, and (i, j) is an *ee*-rise.

If $l = x_{t-1}(i)$, then $(k, x_{t-1}(j))$ is an *ed*-rise for x_{t-1}, because $k < x_{t-1}(k) = x_t(k) < x_t(l) = j < x_{t-1}(j)$.

If $l = x_{t-1}(j)$, then $(k, x_{t-1}(i))$ is an *ed*-rise for x_{t-1}, because $k < x_{t-1}(k) = x_t(k) < x_t(l) = i < x_{t-1}(i)$.

Case 4:

We begin with the assumption that $\{i, j, x_{t-1}(i), x_{t-1}(j)\} \cap \{k, l\} = \emptyset$.

Once again, $k > i$. If $k > x_{t-1}(j)$ then the two moves are interchangeable. Therefore we assume that $i < k < x_{t-1}(j)$.

If $x_{t-1}(k) < x_{t-1}(j)$, then (k, j) is an *ed*-rise for x_{t-1}.

If $x_{t-1}(j) < x_{t-1}(k) < j$ then $(k, x_{t-1}(j))$ is an *ee*-rise for x_{t-1}.

If $j < x_{t-1}(k)$, then it is easy to check that the two moves are interchangeable.

We proceed with the case that $\{i, j, x_{t-1}(i), x_{t-1}(j)\} \cap \{k, l\} \neq \emptyset$.

If $k = i$, then $j < l$. Since (k, l) is an *ee*-rise for x_t, we see that $l < x_t(k) = x_{t-1}(j)$, hence $j < x_{t-1}(j)$. But (i, j) is an *ee*-rise for x_{t-1}, hence $j > x_{t-1}(j)$, a contradiction.

If $k = j$, then $i < k < x_t(k) = x_{t-1} \cdot (i,j) \cdot (x_{t-1}(i), x_{t-1}(j))(k) = x_{t-1}(i)$, a contradiction.

If $k = x_{t-1}(i)$ then either $l > x_{t-1}(j)$, which implies that $(x_{t-1}(j), l)$ is an *ee*-rise for x_{t-1}, or $k < l < x_{t-1}(j)$, which implies that $(i, x_{t-1}(l))$ is an *ed*-rise for x_{t-1} and because $x_{t-1}(l) = x_t(l)$, the label $(i, x_{t-1}(l))$ is bigger than the label (i,j), hence a contradiction.

If $k = x_{t-1}(j)$, then $i < k < x_t(k) = i$, a contradiction.

If $l = j$, then $i < l < x_t(l) = i$, a contradiction.

Similarly, the case $l = i$ is impossible.

If $l = x_{t-1}(i)$, then we have either $x_{t-1}(k) < x_{t-1}(j)$, or $x_{t-1}(j) < x_{t-1}(k)$.

In the first case, if $x_{t-1}(i) < x_{t-1}(k) < x_{t-1}(j)$, then it is easy to check that $i < k < j$, hence (i,j) is not a suitable rise. On the other hand, if $x_{t-1}(k) < x_{t-1}(i)$, we have a contradiction to $x_{t-1}(i) = l < x_t(k) = x_{t-1}(k)$. We proceed with the case $x_{t-1}(k) > x_{t-1}(j)$, then $(k, x_{t-1}(j))$ is an *ee*-rise for x_{t-1}.

If $l = x_{t-1}(j)$, then $x_t(l) = i$ and $i < x_{t-1}(j)$ which is a contradiction.

Our next step is to prove that no other chain is lexicographically decreasing. We prove this by induction on the length of the interval $[x,y]$. Clearly, if y covers x, there is nothing to prove. So, we assume that for any interval of length k there exists a unique decreasing maximal chain.

Let $[x,y] \subseteq F_{2n}$ be an interval of length $k+1$, and let

$$\mathfrak{c} : x = x_0 < x_1 < \cdots < x_k < x_{k+1} = y$$

be the maximal chain such that $f(\mathfrak{c})$ is the lexicographically largest Jordan-Hölder sequence in Γ^{k+1}.

Assume that there exists another decreasing chain

$$\mathfrak{c}' : x = x_0 < x'_1 < \cdots < x'_k < x_{k+1} = y.$$

Since the length of the chain

$$x'_1 < \cdots < x'_k < x_{k+1} = y$$

is k, by the induction hypotheses, it is the lexicographically largest chain between x'_1 and y.

We are going to find contradictions to each of the following possibilities.

Case 1: $type(x_0, x_1) = ee$, and $type(x_0, x'_1) = ee$,

Case 2: $type(x_0, x_1) = ed$, and $type(x_0, x'_1) = ed$,

Case 3: $type(x_0, x_1) = ee$, and $type(x_0, x'_1) = ed$,

Case 4: $type(x_0, x_1) = ed$, and $type(x_0, x'_1) = ee$,

In each of these cases we construct a fixed-point-free involution z such that z covers x'_1 and $f((x'_1, z)) > f((x'_1, x'_2))$, contradicting the induction hypothesis. To this end, let $f((x_0, x_1)) = (i, j)$, $f((x_0, x'_1)) = (k, l)$ and assume that $(k, l) < (i, j)$.

Case 1:

If $k = i$ and hence $l < j$, then (l, j) is an ee-rise for x'_1. Obviously this label is greater than (k, l) hence it is greater than the label of $x'_1 \leftarrow x'_2$, a contradiction.

Let $k < i$, $l \leq i$. Then (i, j) is a suitable rise for x'_1, which is greater than (k, l). This is a contradiction to the maximality of the chain $\mathfrak{c}'$.

Let $k < i$ and $i < l < j$. If either $x_0(l) < x_0(i)$, or $x_0(k) > x_0(j)$ hold, then (i, j) is a suitable rise for x'_1. If $x_0(i) < x_0(l) < x_0(j)$, then (i, j) is not a free rise, which is absurd. If $x_0(i) < x_0(k) < x_0(j)$, then (i,l) is a suitable rise in x_1' with $(i,l) > (k,l)$. If $x_0(k) < x_0(i)$, then (k,l) is not a free rise in x_0.

Let $k < i$ and $l > j$. If $x_0(l) < x_0(i)$, then (i,j) a suitable rise for x_1'. If $x_0(i) < x_0(l) < x_0(j)$ then (i,l) is a suitable rise for x_0 with $(i,l) > (i,j)$. If $x_0(l) > x_0(j)$ then (j,l) is a suitable rise for x_0 with $(j,l) > (i,j)$. This contradicts with our assumption that (i,l) is the lexicographically largest covering label for x_0.

Case 2:

Suppose that $k = i$. Then we have $l < j$. If $x_0(l) < x_0(j)$, then (i,j) is not a suitable rise, a contradiction. On the other hand, if $x_0(l) > x_0(j)$, then the *ee*-rise $(x_0(i), x_0(j))$ is an *ee*-rise for x_1', another contradiction.

Next, suppose that $k < i$. If $x_0(k) > j$, then (i,j) and (k,l) are interchangeable rises.

Therefore, we look at the case when $x_0(j) < x_0(k) < j$. In this case, if $x_0(l) < j$, then (k,l) is not a free rise. On the other hand, if $x_0(l) > j$ then (i,j) and (k,l) are interchangeable rises.

If $x_0(i) < x_0(k) < x_0(j)$, then because (i,j) and (k,l) are free rises for x_0, we must have either $l > j, x_0(l) < x_0(j)$, or $l < j, x_0(l) < x_0(j)$. In both of these cases, (i,j) gives a suitable rise for x_1' with a larger label than (k,l).

If $i < x_0(k) < x_0(i)$ then (i,j) is interchangeable with (k,l) in x_1'.

If $x_0(k) < i$, then either $x_0(l) < i$ or $l < x_0(i)$. In both cases (i,j) and (k,l) are interchangeable rises for x_1', which leads to the contradiction that we seek.

Case 3:

In this case, $k = i$ with $l < j$ is not possible because then (k,l) is not a free rise in x_0.

If $k < i$ and $x_0(k) > x_0(j)$, then (i, j) and (k, l) are interchangeable for x_1'.

If $k < i$ and $x_0(i) < x_0(k) < x_0(j)$, then for (k, l) to be a free rise in x_0 we have to have that $x_0(k) < x_0(l) < x_0(j)$. In this case, (i, j) is a suitable rise for x_1'.

Suppose now that $k < i$ and $x_0(k) < x_0(i)$.

If $l = x_0(j)$, then $i < x_0(k) < j$ and therefore, $(i, x_0(j))$ is a suitable rise for x_1' with a label greater than (k, l).

If $l = x_0(i)$, then $x_0(k) < i$. In this case, let m be the largest integer less than i such that $x_0(i) \le x_1'(m) < x_0(j)$. (Note that such an m exists because $x_0(k)$ satisfies these conditions.) Then (m, j) is a suitable rise for x_1'.

If $x_0(l) < i$ or if $l < x_0(i)$ then (i, j) is a suitable rise for x_1' and the contradiction is as before. If $x_0(i) < l < x_0(j)$, since (i, j) is a suitable rise, we have either $x_0(l) < i$ and we are done by the previous case, or $x_0(l) > j$. In the latter case, we have $x_0(l) < x_0(i)$; otherwise (k, l) is not a free rise. Then, $(i, x_0(l))$ is a suitable *ee*-rise for x_0 with a label greater than (i, j).

If $l > x_0(j)$, then we have three cases: $x_0(l) < i$, $i < x_0(l) < j$, or $x_0(l) > j$. The first case is already taken care of. For the second possibility, because (k, l) is a free rise for x_0, it follows that $i < x_0(k) < x_0(l)$. Then (i, j) and (k, l) are interchangeable. Finally, if $x_0(l) > j$, then (i, j) and (k, l) are interchangeable.

Case 4:

Let $k = i, l < j$. Since (k, l) is a non-crossing *ed*-rise we have to have $l < x_0(k) = x_0(i)$. Then (l, j) is a suitable *ed*-rise for x_1' with a label greater than (k, l), a contradiction.

Suppose now that $k < i$.

If $x_0(k) > j$, then (i, j) is a suitable rise for x_1' with a greater label than (k, l), a contradiction.

If $x_0(i) < x_0(k) < j$, then we look at the location of l.

If $l < i$, then (i, j) and (k, l) are interchangeable.

If $x_0(i) < l < x_0(j)$, then $x_0(l) > j$ since otherwise (i, j) is not a free rise for x_0. It follows that $(l, x_0(j))$ is a suitable rise in x_1', a contradiction.

If $l = x_0(j)$ then $(i, x_0(k))$ is possible in x_1'. If $i < l < x_0(j)$, then $x_0(l) > j$ since otherwise (i, j) is not a free rise for x_0. But then $(l, x_0(j))$ is a suitable rise for x_1', another contradiction.

If $x_0(i) < l < i$ and also if $x_0(l) < x_0(i)$, then (i, j) and (k, l) are interchangeable.

If $x_0(i) < l < i$ and $x_0(i) > x_0(l) > x_0(j)$, then (i, j) is not a suitable rise.

If $x_0(i) < l < i$ and $x_0(j) < x_0(l)$ then either (i, j) and (k, l) are interchangeable, or (l, j) is a suitable rise.

If $x_0(j) < l$, then because (k, l) is a free rise we must have that $x_0(l) < j$.

If $l > x_0(j)$, then once again (i, j) and (k, l) are interchangeable. This finishes the checking of the cases for $x_0(i) < x_0(k) < j$.

Our final case is when $x_0(k) < x_0(i)$. In this case, we look at the location of l.

If $l = i$, then let m denote the largest integer less than i such that $x_0(i) \leq x_1'(m) < x_0(j)$. Such an m exists because k satisfies these conditions. Then (m, j) is a suitable *ed*-rise in x_1' with a label greater than (k, l). If $l \neq i$, then we have either $l < i$, or $x_0(l) < x_0(i)$. In both of these cases (i, j) and (k, l) are interchangeable, hence a contradiction. This finishes our proof of the uniqueness, and hence the theorem follows.

$\square$

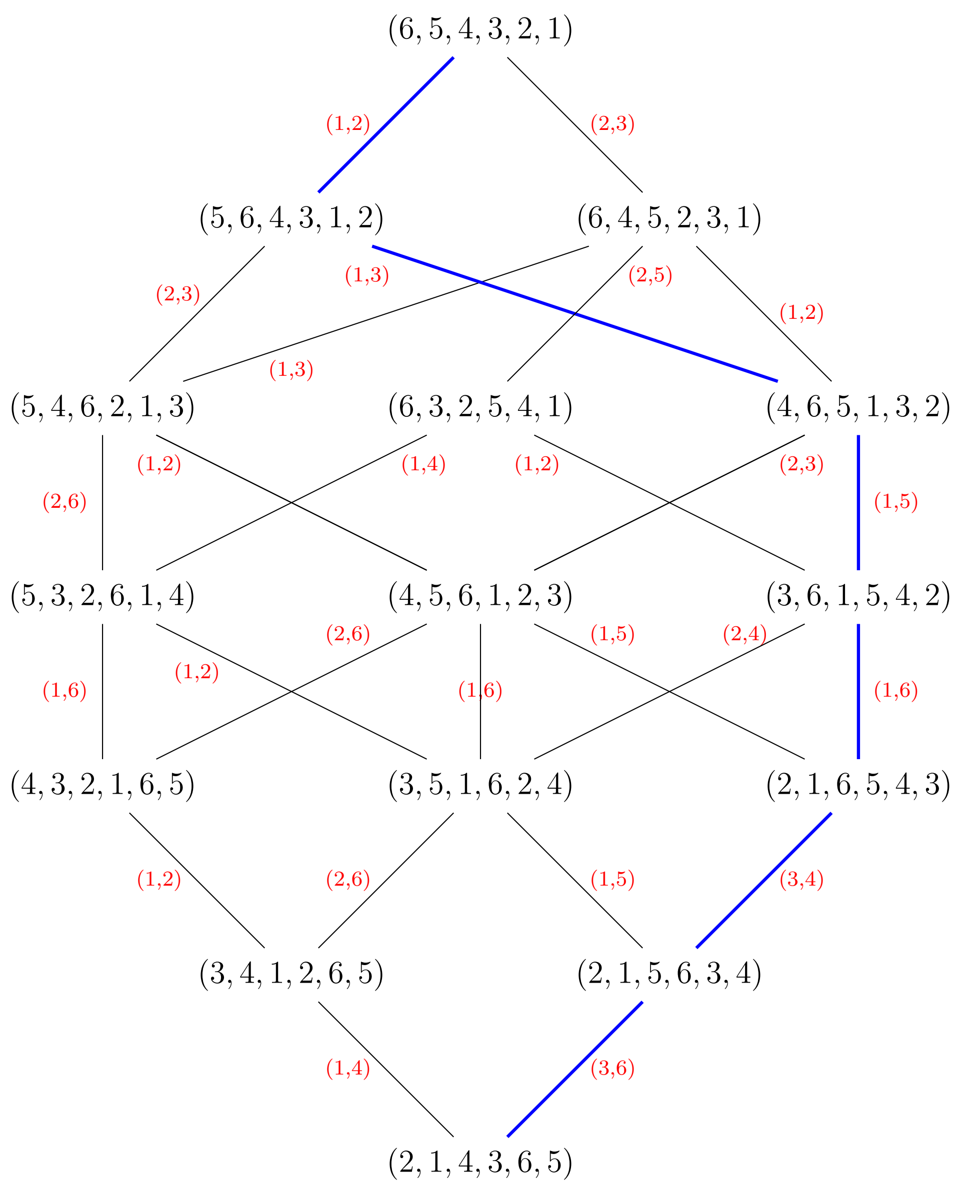


Figure 5.3: Bruhat-Chevalley order on F_6.

5.2 Partial Fixed-Point-Free Involutions PF_n

5.2.1 The covering relations of PF_n

We investigate the poset of 0/1 partial fixed-point-free involution matrices PF_n. Unfortunately, this poset is not a connected subposet of the partial involutions. This can easily be seen by looking at the example of 2×2 matrices.

Example 37. *There are only two partial fixed-point-free involutions when $n = 2$.*

$$x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and therefore x covers y as a partial fixed-point-free involution.

However, viewed as a partial involution x does not cover y because $y < z < x$ where

$$z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

It is clear from the rank-control description of $\leq$ that if y covers x in P_n and if both x and y are members of PF_n, then y covers x in PF_n too.

These constellations can only arise in the following cases:

1. $y \to x$ is an r-cover in P_n, and $x, y \in PF_n$

2. $y \to x$ corresponds to a non-crossing *ee* or an *ed*-rise

There are additional covering relations though. These covering relations arise if two fixed-point-free involutions x and y form an interval of length three in P_n and all elements $z \in P_n$ such that $x < z < y$ are not in PF_n. Such a constellation can happen only if y is obtained from x by an r-move followed by a d-move where at every move the rank of the matrix drops by 1. It is clear that these situations lead to coverings. (For an illustration see Example 37.) To see that these are all the covering relations, suppose there are additional covering relations. Then there exist $x, y \in PF_n$ such that y covers x in PF_n, but y does not cover x in P_n and y is not obtained from x by an r-move followed by a d-move as described above. This means that $[x, y] \subset P_n$ contains elements in $P_n \setminus PF_n$ only, except for x and y. Let $z \in [x, y]$ be an element that covers x in P_n. Then z is not obtained from x by a c-cover of type non-crossing ee or ed. But this means z can only be obtained from x be a c-cover of type ff, fe, ef, or crossing ee, or by an r-move where two symmetric entries are moved to the diagonal. Assume that z is obtained from x by a c-move. Since $x \in PF_n$ we know that x does not contain any rises of type ff, fe or ef. Therefore, the only possibility is that z is obtained from x by a transformation corresponding to a crossing ee-rise. We already realized, though, that every crossing ee-rise can also be viewed as an ed-rise. We claim that the element z_1 corresponding to this rise lies in $[x, y]$ contradicting the assumption. Let the crossing ee-rise in x that z corresponds to be labeled (i, j) (recall the label from 4.2). We assume $z_1 \notin [x, y]$. Comparing the rank-control matrices of z_1 and z we see that for $y \not> z_1$ there must be a 1 in y on position (k, l) where $k, l \in [x(i), j-1]$. Since y is a partial fixed-point-free involution we may assume that $k < l$. The 1 on position (k, l) has to arise from z through a series of covering relations. There are two ways for this to happen: it is placed there either by an ff-move or by an r-move. If it arises from an ff-move then it has to involve the 1's on positions (i, i) and (j, j) or these 1's pushed further down on the diagonal. But this is not possible because $j \not< j$. It is also impossible to involve only the 1 on position (i, i) and another diagonal entry (m, m) where $m < i$, because then the same argument can be used for the first 1 on the diagonal. It is also not possible for the 1 to be placed at (k, l) by an r-move because in this case the 1 being moved to (k, l) is to the upper left of (k, l) in z_1 and thus $Rk(z_1) > Rk(y)$.

Assume that z is obtained from x by an r-move which places two symmetric entries on the diagonal. In this case, another r-move is possible in x involving the same 1's. Let the resulting partial fixed-point-free involution be z_1. A similar argument shows that also in this case $z_1 \in [x, y]$.

Therefore $[x, y]$ does not contain elements in $P_n \setminus PF_n$ only, which is a contradiction. Hence we identified all possible covering relations.

Summarizing we have the following covering relations in PF_n:

1. $y \to x$ corresponds to a non-crossing ee or an ed-rise;
2. $y \to x$ is an r-cover in P_n, and $x, y \in PF_n$;
3. $y \to x$ is an r-cover followed by a d-cover in P_n where at each step the rank drops by 1.

We define an EL-labeling for PF_n:

1. If the covering relation is derived from a covering of an invertible fixed-point-free involution, then we use the labeling as defined in Section 5.1 and transform this label (i, j) into $(n - i, n - j)$.
2. If the covering relation results from a right move, then we define the label to be $(i + n, j)$ where $y > x$ results from x by moving the 1 in column i to row j. If the 1 is pushed out of the matrix then $j = n + 1$.

Remark 38. *In the case of invertible fixed-point-free involutions we showed that the lexicographically largest chain is the only decreasing chain. Since the label is transformed from (i, j) to $(n - i, n - j)$ now the lexicographically smallest chain is increasing.*

The reason the label of r-moves is shifted by n in the first coordinate is to ensure that every r-cover has a bigger label than any c-cover.

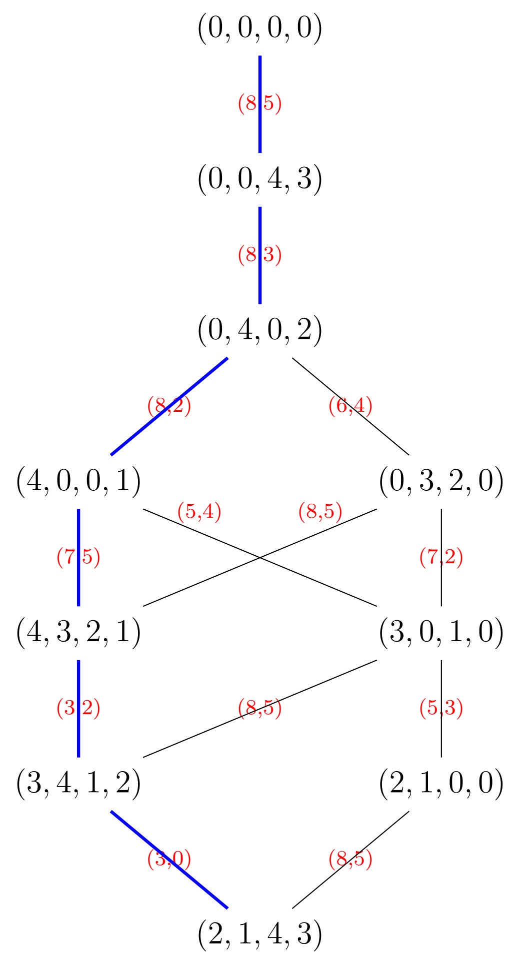


Figure 5.4: The EL-labeling of PF_4.

5.2.2 Lexicographic shellability of PF_n

Theorem 39. *PF_n is a lexicographically shellable poset.*

Proof. Let x and y be two partial fixed-point-free involutions.

Since lexicographic ordering is a total order on maximal chains, there exists a unique smallest such chain. We denote it by

$$\mathfrak{c} : x = x_1 < x_2 < \cdots < x_s = y.$$

The idea of the proof is showing that $\mathfrak{c}$ is the unique increasing chain and therefore the unique lexicographically smallest chain which is increasing.

Towards a contradiction assume that $\mathfrak{c}$ is not increasing. Then, there exist three consecutive terms

$$x_{t-1} < x_t < x_{t+1}$$

in $\mathfrak{c}$, such that $f((x_{t-1}, x_t)) > f((x_t, x_{t+1}))$. We have 4 cases to consider.

Case 1: $type((x_{t-1}, x_t)) = c$, and $type((x_t, x_{t+1})) = c$.

Case 2: $type((x_{t-1}, x_t)) = r$, and $type((x_t, x_{t+1})) = r$.

Case 3: $type((x_{t-1}, x_t)) = c$, and $type((x_t, x_{t+1})) = r$.

Case 4: $type((x_{t-1}, x_t)) = r$, and $type((x_t, x_{t+1})) = c$.

In each of these 4 cases, we either produce an immediate contradiction by showing that either the two moves are interchangeable (hence $\mathfrak{c}$ is not the smallest chain), or we construct an element $z \in [x, y] \cap PF_n$ which covers x_{t-1}, and such that $f((x_{t-1}, z)) < f((x_{t-1}, x_t))$. Since we assume that $f(\mathfrak{c})$ is the lexicographically smallest Jordan-Hölder sequence, the existence of z is a contradiction, too.

To this end, suppose that the label of the first move is (i, j), and the second move is labeled by (k, l).

Case 1:

Done in the proof for invertible fixed-point-free involutions.

Case 2:

If $i = k$ then $l > j$. Therefore we assume that $k < i$.

If $k - n = j$ then $j < i - n$ and $(m + n, l)$ is possible in x_{t-1} with $m < j < i$ where $(m, i - n)$ is the position of the 1 in x_{t-1}.

If $k - n \neq j$ then either the two moves are interchangeable or (k, l) removes a suitable rise in x_{t-1} which corresponds to a move with a smaller label than (i, j).

Case 3:

This case is impossible since every c-move has a smaller label than any r-move.

Case 4:

If the r-cover labeled (i, j) is the covering relation with the lexicographically smallest label then there is no suitable rise in x_{t-1}. The c-move has to involve one of the moved 1's since otherwise there is a suitable rise in x_{t-1}. For this, one of the moved 1's has to have a 1 to the upper left or the lower right in x_t that was not to the upper left or lower right of it in x_{t-1}. Since the 1's are moved right and down respectively, it is impossible that there is a 1 to the lower right in x_t that is not to the lower right in x_{t-1}. If the c-cover corresponds to the suitable rise $(m, i-n)$ (with label $(n - m, i)$) then (i, j) is not the r-move with the smallest label in x_{t-1} since in this case $(m + n, j)$ is possible in x_{t-1} with $(n + m, j) < (i, j)$. If the c-cover corresponds to the rise (m, j), then the r-move $(m + n, i - n)$ is possible in x_{t-1} which again has a smaller label than (i, j).

We will now use induction to prove that no other chain is lexicographically increasing. We prove this by induction on the length of the interval $[x, y]$. Clearly, if y covers x, there is nothing to prove. So, we assume that for any interval of length k there exists a unique increasing maximal chain.

Let $[x, y] \subseteq PF_{2n}$ be an interval of length $s + 1$, and let

$$\mathfrak{c} : x = x_0 < x_1 < \cdots < x_s < x_{s+1} = y$$

be the maximal chain such that $f(\mathfrak{c})$ is the lexicographically smallest Jordan-Hölder sequence in Γ^{k+1}.

Assume that there exists another increasing chain

$$\mathfrak{c}' : x = x_0 < x_1' < \cdots < x_s' < x_{s+1} = y.$$

Since the length of the chain

$$x_1' < \cdots < x_s' < x_{s+1} = y$$

is s, by the induction hypotheses, it is the lexicographically smallest chain between x_1' and y.

We are going to find contradictions to each of the following possibilities.

Case 1: $type(x_0, x_1) = c$, and $type(x_0, x_1') = c$,

Case 2: $type(x_0, x_1) = r$, and $type(x_0, x_1') = r$,

Case 3: $type(x_0, x_1) = c$, and $type(x_0, x_1') = r$,

Case 4: $type(x_0, x_1) = r$, and $type(x_0, x_1') = c$,

In each of these cases we will construct a partial fixed-point-free involution $z \in [x, y]$ such that z covers x_1' and $F((x_1', z)) < F((x_1', x_2'))$, contradicting the induction hypothesis. To this end, let $f((x_0, x_1)) = (i, j)$, $f((x_0, x_1')) = (k, l)$ and assume that $(k, l) < (i, j)$.

Case 1:

Done in the proof for the invertible case.

Case 2:

It is impossible for $i = k$ since there is only one r-move for each 1.

Therefore assume that $i < k$. Let the moved 1's be on the symmetric positions $(i - n, m)$ and $(m, i - n)$ in x_0. If $k = m + n$ then $(l + n, j)$ is possible in x_1' with $(l + n, j) < (k, l)$. If $k \neq m$ then either the two moves are interchangeable or the suitable rise $(n - i, n - k)$ is possible in x_1'.

Case 3:

Since no r-move can remove a suitable rise, there exists a legal c-move in x_1'. But this c-move has a smaller label than (k, l) which is our desired contradiction.

Case 4:

This case is not possible because every c-move has a smaller label than any r-move.

□

5.2.3 The length function of PF_n

For any non-negative integer k we define $r_{0,k}$ to be 0. Let $Rk(x) = (r_{ij})$ be the rank-control matrix of x. Recall from Section 4.2.1 that the length function of the poset of partial involutions is given by $l_{P_n}(x) = D(x)$ where

$$D(x) = \#\{(i,j) | 1 \leq i \leq j \leq n \text{ and } r_{ij} = r_{i-1,j-1}\}.$$

The length function of PF_n only differs from the length function of P_n in two ways: The rank of two matrices in PF_n can only differ by a multiple of 2 and the smallest element in P_n is the identity which is not in PF_n. The minimal element in PF_n is given by the matrix with the largest rank-control matrix.

Example 40.

$$\text{When } n = 6,\ \hat{0} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and when } n = 5,\ \hat{0} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This means that in the case when n is even $l_{PF_n}(x) = l_{P_n}(x) - \frac{n-rk(x)}{2} - \frac{n}{2}$. We subtract $\frac{n-rk(x)}{2}$ so that the length function increases only by 1 if the rank drops by 2 and we subtract $\frac{n}{2}$ because the minimal element has to have length zero. Similarly, when n is odd we have to subtract $\frac{n-1-rk(x)}{2}$ and $\frac{n+1}{2}$.

Summarizing, we see that for all n the length function $l_{PF_n}(x)$ of PF_n is given by

$$l_{PF_n}(x) = l_{P_n}(x) - \frac{n-rk(x)}{2} - \frac{n}{2} = l_{P_n}(x) - \frac{2n-rk(x)}{2} = D(x) - \frac{2n-rk(x)}{2}.$$

Chapter 6

Eulerian intervals in R_n and P_n

The results in this chapter were found in joint work with Can [11].

There is a concrete way to compare two rooks given in one line notation in Bruhat-Chevalley-Renner ordering. For an integer valued vector $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, let $\widetilde{a} = (a_{\alpha_1}, \dots, a_{\alpha_n})$ be the rearrangement of the entries $a_1, \dots, a_n$ of a in a non-increasing fashion;

$$a_{\alpha_1} \geq a_{\alpha_2} \geq \cdots \geq a_{\alpha_n}.$$

The *containment ordering*, "$\leq_c$," on $\mathbb{Z}^n$ is then defined by

$$a = (a_1, \dots, a_n) \leq_c b = (b_1, \dots, b_n) \iff a_{\alpha_j} \leq b_{\alpha_j} \text{ for all } j = 1, \dots, n.$$

where $\widetilde{a} = (a_{\alpha_1}, \dots, a_{\alpha_n})$, and $\widetilde{b} = (b_{\alpha_1}, \dots, b_{\alpha_n})$.

Example 41. *Let $x = (4, 0, 2, 3, 1)$, and let $y = (4, 3, 0, 5, 1)$. Then $x \leq_c y$, because*

$$\widetilde{x} = (4, 3, 2, 1, 0) \quad \textit{and} \quad \widetilde{y} = (5, 4, 3, 1, 0).$$

For $k \in [n]$, the *k-th truncation* $a(k)$ of $a = (a_1, \dots, a_n)$ is defined to be

$$a(k) = (a_1, a_2, \dots, a_k).$$

Let $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ be two rooks in R_n. It is shown in [27] that

$$v \leq w \iff \widetilde{v(k)} \leq_c \widetilde{w(k)} \text{ for all } k = 1, \ldots, n.$$

Example 42. *Let $x = (0, 1, 2, 3, 4)$, and let $y = (4, 3, 2, 5, 1)$. Then $x \leq y$, because*

$$\begin{aligned}
\widetilde{x(1)} = (0) &\leq_c \widetilde{y(1)} = (4), \\
\widetilde{x(2)} = (1, 0) &\leq_c \widetilde{y(2)} = (4, 3), \\
\widetilde{x(3)} = (2, 1, 0) &\leq_c \widetilde{y(3)} = (4, 3, 2), \\
\widetilde{x(4)} = (3, 2, 1, 0) &\leq_c \widetilde{y(4)} = (5, 4, 3, 2), \\
\widetilde{x(5)} = (4, 3, 2, 1, 0) &\leq_c \widetilde{y(5)} = (5, 4, 3, 2, 1).
\end{aligned}$$

The next lemma, whose proof is omitted, shows that for two permutations x and y of S_n, the inequality $x \leq y$ can be decided in $n - 1$ steps.

Lemma 43. *Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be two permutations in S_n. Then $x \leq y$ if and only if*

$$\widetilde{x(k)} \leq_c \widetilde{y(k)} \text{ for } k = 1, \ldots, n-1.$$

Proposition 44. *The union $(R_{n,n-1} \cup R_{n,n}, \leq)$ is isomorphic to the poset $(S_{n+1}, \leq)$.*

We depict the isomorphism between S_4 and $R_{3,3} \cup R_{3,2}$ in Figure 6.1.

Proof. Let u and w denote the rooks $u = (0, 1, 2, \ldots, n)$ and $w = (n, n-1, \ldots, 2, 1)$. Then $R_{n,n-1} \cup R_n = [u, w]$.

We define a map ψ between $[v, w]$ and S_{n+1} as follows. If $x = (a_1, \ldots, a_n) \in [v, w]$, then

$$\psi(x) = (a_1 + 1, a_2 + 1, \ldots, a_n + 1, a_x), \tag{6.1}$$

where a_x is the unique element of the set

$$[n+1] \setminus \{a_1+1, a_2+1, \dots, a_n+1\}.$$

We have two immediate observations.

1. If x is already a permutation (in $R_{n,n}$), then $a_x = 1$.

2. ψ is injective, hence by Lemma 20, it is bijective as well.

Now, let $x = (a_1, \dots, a_n)$ and $y = (b_1, \dots, b_n)$ be two elements in $[v, w]$ such that $x \leq y$. For the sake of brevity, denote the "shifted" sequence $(a_1+1, \dots, a_n+1)$ associated with x by x'. Since increasing each entry of x and y by 1 does not change the relative sizes of the entries of x and y, we have

$$x' \leq y'.$$

Recall that this is equivalent to saying that $\widetilde{x'(k)} \leq_c \widetilde{y'(k)}$ for all $k = 1, \dots, n$. Since, x' is the n-th truncation $\psi(x)(n)$ of the permutation $\psi(x)$, the proof of the theorem is complete by considering Lemma 43. The converse statement "$\psi(x) \leq \psi(y) \implies x \leq y$" follows from the same argument. Therefore, ψ is a poset isomorphism. $\square$

Unfortunately, the map ψ defined in (6.1) does not restrict to partial involutions nicely enough, therefore, we need another order preserving injection in $P_{n,n-1} \cup P_n$ onto I_{n+1}.

Let $u = (0, n, n-1, \dots, 2)$ and let $\iota = (1, 2, \dots, n)$. Observe that the rank-control matrix of u is the smallest, and that the rank-control matrix of ι is the largest among all elements of $P_{n,n-1} \cup P_{n,n}$. Therefore, the union $P_{n,n-1} \cup P_{n,n}$ is the underlying set of the interval $[\iota, u]$ of P_n.

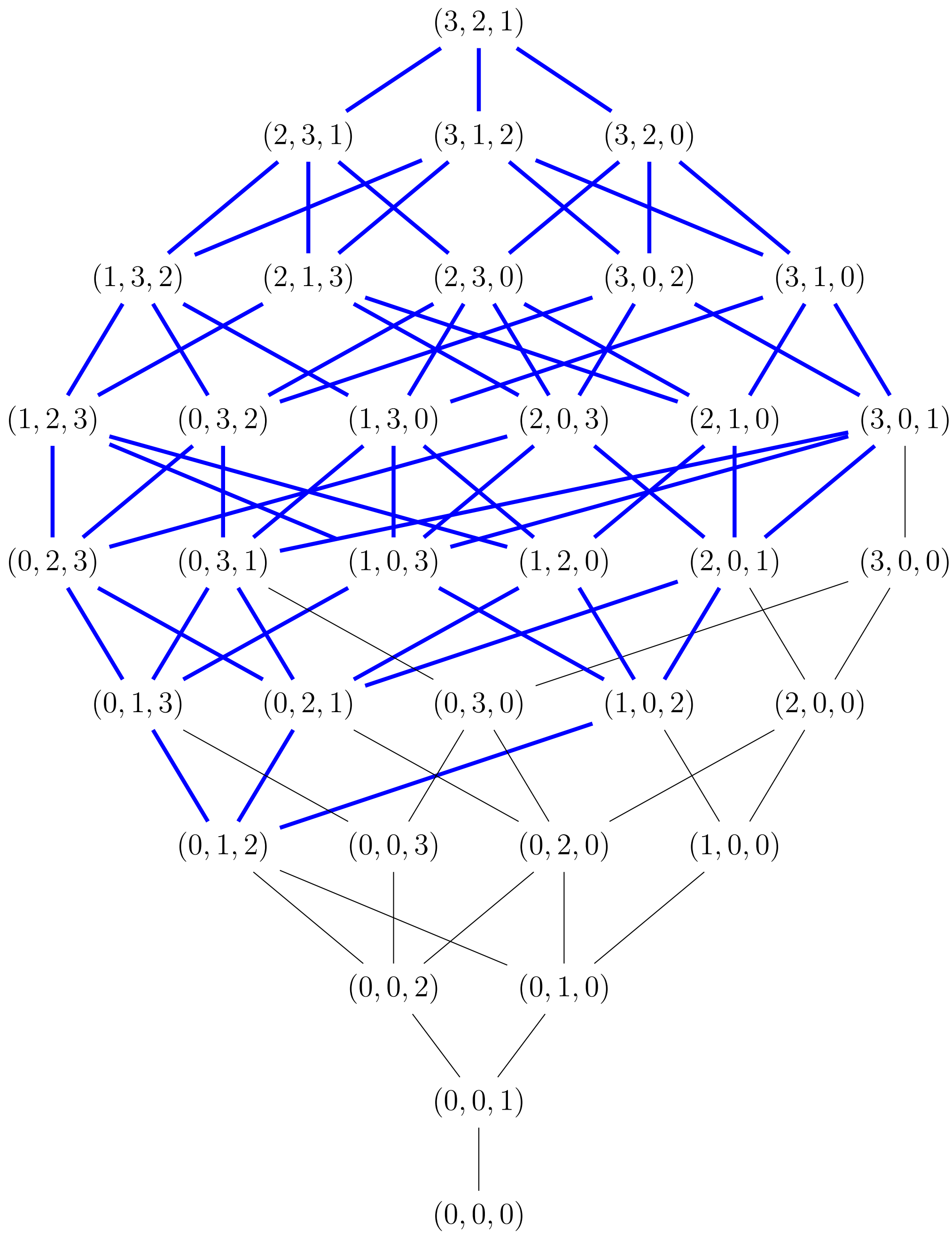


Figure 6.1: S_4 in $(R_3, \leq)$.

Let $x = (a_1, \dots, a_n) \in [\iota, u]$ be given in one-line notation. Then there are two cases:

1. there is an $i \in [n]$ such that $a_i = 0$,
2. x is a permutation.

We start with the first case. If $a_i = 0$ for some $i \in [n]$, then we define $b_i = n + 1$ and for $j \in [n] \setminus \{i\}$ we set $b_j = a_j$. In addition, in this case, we define b_{n+1} to be the unique element of the set $\{0, 1, \dots, n\} - \{a_1, \dots, a_n\}$. In the latter case, we set $b_j = a_j$ for $j = 1, \dots, n$ and define $b_{n+1} = n + 1$. Finally, we define $\phi : [\iota, u] \to I_{n+1}$ by

$$\phi(x) = (b_1, \dots, b_{n+1}). \tag{6.2}$$

For example,

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \phi(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Proposition 45. *The union $(P_{n,n-1} \cup P_{n,n}, \leq)$ is isomorphic to the poset $(I_{n+1}, \leq)$.*

We depict the isomorphism between I_4 and $P_{3,3} \cup P_{3,2}$ in Figure 6.2.

Proof. Let ϕ be defined as in (6.2). By its construction, ϕ is injective. Therefore, by Lemma 20, Part 2., it is enough to show that ϕ is order preserving.

Let x and y be two elements in $[\iota, u]$ such that $x \leq y$. Then $Rk(y) \leq_R Rk(x)$.

Note that the upper-left $n \times n$ portion of the rank-control matrix of $\phi(x)$ is equal to $Rk(x)$. The same is true for $\phi(y)$ and $Rk(y)$.

Let $R^{\phi(x)}_{i,j}$ denote the (i, j)-th entry of $Rk(\phi(x))$. Then, since $\phi(x)$ is a permutation in I_{n+1}, we have

$$R^{\phi(x)}_{n+1,i} = i \text{ and } R^{\phi(x)}_{j,n+1} = j$$

for all $i, j \in [n + 1]$. The same is true for $Rk(\phi(y))$. Therefore,

$$Rk(\phi(y)) \leq_R Rk(\phi(x))$$

and the proof is complete. $\square$

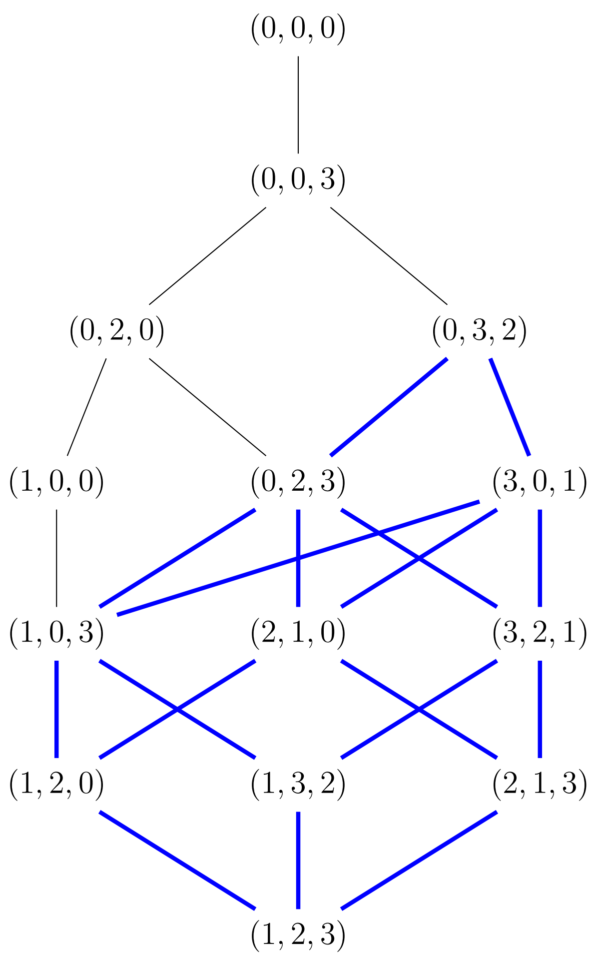


Figure 6.2: I_4 in $(P_3, \leq)$.

Theorem 46. *$R_{n,k}$ and $P_{n,k}$ are Eulerian if and only if $k = n$ or $k = n - 1$.*

Proof. First of all, $R_{n,n} \cong S_n$, and $R_{n,n-1}$ is isomorphic to an interval in S_{n+1}. Thus, both $R_{n,n}$ and $R_{n,n-1}$ are Eulerian. The same argument is true for both of the posets $P_{n,n}$ and $P_{n,n-1}$. Therefore, to finish the proof, it is enough to show that, for $k \neq n, n-1$, $R_{n,k}$ and $P_{n,k}$ are not Eulerian. To this end, for $k \leq n-2$, let v_k, v'_k and v''_k denote the elements

$$v_k = (0, \ldots, 0, 0, 1, 2, \ldots, k),$$
$$v'_k = (0, \ldots, 0, 1, 0, 2, \ldots, k),$$
$$v''_k = (0, \ldots, 1, 0, 0, 2, \ldots, k)$$

in $R_{n,k}$. Then the interval $[v_k, v''_k] \subset R_{n,k}$ has exactly three elements v_k, v'_k, v''_k, hence it cannot be Eulerian.

Similarly, for $k \leq n - 2$, let u_k, u'_k and u''_k denote the elements

$$u_k = (1, 2, \ldots, k, 0, \ldots, 0),$$
$$u'_k = (1, 2, \ldots, k-1, 0, k+1, 0, \ldots, 0),$$
$$u''_k = (1, 2, \ldots, k-1, 0, 0, k+2, 0, \ldots, 0)$$

in $I_{n,k}$. Then the interval $[u_k, u''_k] \subset P_{n,k}$ has exactly three elements u_k, u'_k, u''_k, and therefore, it cannot be Eulerian. This finishes the proof. $\square$

Chapter 7

Deodhar-Srinivasan poset vs. $(F_{2n}, \leq)$

All results in this chapter were obtained in joint work with Can and Cherniavsky [7].

In [13] Deodhar and Srinivasan investigate a poset $(\tilde{F}_{2n}, \leq_{DS})$. $(F_{2n}, \leq)$ and $(\tilde{F}_{2n}, \leq_{DS})$ are different, indeed, for $2n = 6$ the Hasse diagrams of these two posets differ by an edge.

In this section we show that $\tilde{F}_{2n}$ is a subposet of F_{2n}. We proceed by recalling the definition of the length function of $\tilde{F}_{2n}$ as defined in [13].

Let $[i_1, j_1] \cdots [i_n, j_n]$ be an element from $\tilde{F}_{2n}$, and let $x \in F_{2n}$ denote the corresponding fixed-point-free involution. The arc-diagram of $x \in F_{2n}$ is defined as follows. We place the numbers 1 to $2n$ on a horizontal line. We connect the numbers i and j by a concave-down arc, if $j = x(i)$. Let $c(x)$ denote the number of intersection points of all arcs.

The length function $\ell_{\tilde{F}_{2n}}$ of $\tilde{F}_{2n}$ is given by

$$\ell_{\tilde{F}_{2n}}([i_1, j_1] \cdots [i_n, j_n]) = \sum_{t=1}^{n} (j_t - i_t - 1) - c(\pi)\,.$$

See Theorem 1.3 in [13].

Our first observation is that $\ell_{\tilde{F}_{2n}}$ is in fact an inversion number. To this end, for x as above, let us define the *modified inversion number* of x to be the number of inversions in the word $i_1 j_1 i_2 j_2 \cdots i_n j_n$, and denote it by $\widetilde{inv}(x)$. Note that i_1 is always 1 for fixed-point-free involutions.

Proposition 47. *Let $[i_1, j_1] \cdots [i_n, j_n] \in \tilde{F}_{2n}$, and let $x \in F_{2n}$ be the corresponding fixed-point-free involution. Then*

$$\widetilde{inv}(x) = \ell_{\tilde{F}_{2n}}([i_1, j_1] \cdots [i_n, j_n]).$$

Proof. An inversion in the word $i_1 j_1 i_2 j_2 \cdots i_n j_n$ is either the pair (j_p, i_q), or the pair (j_p, j_q), where $p < q$ and $j_p > i_q$, or $j_p > j_q$, respectively.

We count inversions in another way. If (i_t, j_t) is a transposition that appears in $[i_1, j_1] \cdots [i_n, j_n]$ of x, then $j_t - i_t - 1 = \#\{m : m \in \mathbb{N}, i_t < m < j_t\}$. On the other hand, each number $m \in \{i_t + 1, \ldots, j_t - 1\}$ appears as an entry in another transposition of $[i_1, j_1] \cdots [i_n, j_n]$.

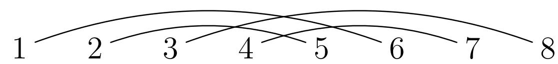
There are three possible cases:

1. the number m is involved in the transposition (a, m), where $a < i_t < m$;
2. the number m is involved in the transposition (a, m), where $i_t < a < m$;
3. the number m is involved in the transposition (m, b), where $m < b$.

In the first case the pair (j_t, m) is not an inversion. Notice that when $a < i_t$, the arc corresponding to the transposition (a, m) crosses the arc corresponding to the transposition (i_t, j_t). In cases 2 and 3, we have the inversion pair (j_t, m) always. For Case 3, whether b is greater than j_t or not is unimportant. So, to get the number of inversion pairs $(j_t, *)$ we have to subtract from $j_t - i_t - 1$ the number of intersections of the arc (i_t, j_t) with the arcs (a, m), where $a < i_t < m < j_t$. Counting the inversions by summing up the contributions of all the transpositions (i_t, j_t) proves our statement. $\square$

Let us illustrate our proof by an example.

Example 48. *Take* $x = (1,6)(2,5)(3,8)(4,7) \in S_8$.



Start with the transposition $(1,6)$. *The numbers between 1 and 6 are 2,3,4,5. All the pairs* $(6,2)$, $(6,3)$, $(6,4)$, $(6,5)$ *are inversions of the word* 16253847*: 2,3,4 are involved in transpositions of the form* $(m,*)$ *which is case (3) in our proof and always gives an inversion, 5 is involved in transposition* $(2,5)$*, it is case (2) since* $1 < 2$*, so it also gives an inversion. Now take the transposition* $(2,5)$. *Both of the numbers 3,4 which are between 2 and 5 are involved in transpositions of case (3),* $(3,8)$ *and* $(4,7)$ *and so both of them give inversions* $(5,3)$ *and* $(5,4)$. *Now consider the transposition* $(3,8)$. *The pair* $(8,4)$ *is an inversion, it is case (3) since 4 is involved in the transposition* $(4,7)$. *The pair* $(8,7)$ *also is an inversion since 7 is involved in the transposition* $(4,7)$ *and* $3 < 4$*, which belongs to case (2). But the pairs* $(8,5)$ *and* $(8,6)$ *are not inversions since 5 and 6 are involved in transpositions* $(2,5)$ *and* $(1,6)$*, where* $1 < 3$ *and* $2 < 3$ *and so both of them are of case (1). By the same reason when we consider the last transposition of* x *which is* $(4,7)$*, the pairs* $(7,5)$ *and* $(7,6)$ *are not inversions, they belong to case (1). So, summing up, we have four inversions of the form* $(6,*)$ *contributed by the transposition* $(1,6)$*, two inversions of the form* $(5,*)$ *contributed by the transposition* $(2,5)$ *and two inversions of the form* $(8,*)$ *contributed by the transposition* $(3,8)$. *Thus,* $\widetilde{inv}(x) = 4+2+2 = 8$. *From the arc diagram depicted above we see that* $c(x) = 4$. *Hence,*

$$\ell_{\tilde{F}_{2n}}(x) = (6-1-1)+(5-2-1)+(8-3-1)+(7-4-1)-4 = 4+2+4+2-4 = 8.$$

So, we see that $\widetilde{inv}(x) = \ell_{\tilde{F}_{2n}}(x)$ *as it is expected.*

Corollary 49. *The length functions of* $(F_{2n}, \leq)$ *and* $(\tilde{F}_{2n}, \leq_{DS})$ *are the same.*

Proof. This follows from Proposition 47 above combined with Proposition 6.2 of [12]. □

Recall that $y \to x = [a_1, b_1] \cdots [a_n, b_n]$ in $\tilde{F}_{2n}$, if $\ell_{\tilde{F}_{2n}}(y) = \ell_{\tilde{F}_{2n}}(x) + 1$ and there exists $1 \leq i < j \leq n$ such that

1. y is obtained from x by interchanging b_i and a_j, where $b_i < a_j$, or
2. y is obtained from x by interchanging b_i and b_j,where $b_i < b_j$.

We call these interchanges *type 1* and *type 2*, respectively. Note that, in a type 1 covering relation we have the inequalities $a_i < b_i < a_j < b_j$. The inequalities of type 2 are $a_i < a_j < b_i < b_j$.

Note also that an arbitrary interchange of the entries in x does not always result in another element of $\tilde{F}_{2n}$. This is because of the ordering of the a_i's. For example, as it is seen from Figure 3 of [13], there is no edge between the elements $[1, 2][3, 6][4, 5]$ and $[1, 4][2, 5][3, 6]$. On the other hand, it is easy to check using rank-control matrices that the corresponding involution $x = (1, 2)(3, 6)(4, 5)$ is covered by $y = (1, 4)(2, 5)(3, 6)$.

Theorem 50. *The covering relations of the poset* $\tilde{F}_{2n}$ *are among the covering relations of* F_{2n}.

Proof. It suffices to prove that a type 1 covering relation of $\tilde{F}_{2n}$ corresponds an *ed*-rise, and a type 2 covering relation of $\tilde{F}_{2n}$ corresponds to an *ee*-rise in F_{2n}.

Let $\tilde{x} = [a_1, b_1][a_2, b_2] \ldots [a_n, b_n]$ be an element from $\tilde{F}_{2n}$ and $x \in F_{2n}$ be the corresponding fixed-point-free element. Suppose we have the inequalities $a_i < a_j < b_i < b_j$, and $\tilde{y} \in \tilde{F}_{2n}$ is obtained from $\tilde{x}$ by a type 2 interchange. Then

$$\tilde{y} = [a_1, b_1][a_2, b_2] \ldots [a_i, b_j] \ldots [a_j, b_i] \ldots [a_n, b_n]$$

It is straightforward to check that the corresponding y is obtained from x by moving the non-zero entries at the positions (a_i, b_i) and (a_j, b_j) (as well as the corresponding symmetric entries) to the positions (a_i, b_j) and (a_j, b_i) (and to the positions of the corresponding symmetric entries). This is an *ee*-rise for x.

Similarly, suppose we have $a_i < b_i < a_j < b_j$ and $\tilde{y} \in F_{2n}$ is obtained from $\tilde{x}$ by a type 1 move. In the matrix of x, there are non-zero entries at the positions $(a_i, b_i), (a_j, b_j)$, as well as at the corresponding symmetric positions. Then,

$$\tilde{y} = [a_1, b_1][a_2, b_2] \dots [a_i, a_j] \dots [b_i, b_j] \dots [a_n, b_n],$$

and the corresponding element y has non-zero entries at the positions $(a_i, a_j), (b_i, b_j)$, as well as at their symmetric positions. This is an *ed*-rise for x.

Therefore, the covering relations of $\tilde{F}_{2n}$ are among the covering relations of F_{2n}, hence the proof is finished.

□

As a corollary of Corollary 49 we see that

Corollary 51. *For $m \in \mathbb{N}$, let $[m]_q$ denote its q-analogue $1 + q + \cdots + q^{m-1}$. Then, the length-generating function $\sum_{x \in F_{2n}} q^{\ell_{F_{2n}}(x)}$ of F_{2n} is equal to*

$$[2n-1]_q!! := [2n-1]_q[2n-3]_q \cdots [3]_q[1]_q.$$

Proof. Follows from Corollary 2.2 of [13]. □

We should mention here that the conclusion of the above corollary is obtained by other combinatorial methods by A. Avni in his M.Sc. thesis at Bar-Ilan University.

It turns out there is another simple characterization of $\ell_{F_{2n}}$, which seems to be known to the experts. Although it is not difficult to prove, since we could not locate it in the literature, we record its proof here for the sake of completeness.

Proposition 52. *Let $x \in S_{2n}$ be a fixed point free involution. Then*

$$\widetilde{inv}(x) = \ell_{F_{2n}}(x) = \ell_{\tilde{F}_{2n}}(x) = \frac{inv(x) - n}{2},$$

where $inv(x) = |\{(i,j) : \ i < j \text{ and } x(i) > x(j)\}|$.

Proof. Let $x \in F_{2n}$. The second equality is shown to be true in Chapter 7. The first equality follows from Proposition 6.2 of [12]. It remains to show the third equality.

In Lemma 34 we show that $\ell_{I_{2n}}(x) - \ell_{F_{2n}}(x) = n$. On one hand, we know that $\ell_{I_{2n}}(x) = \frac{exc(x)+inv(x)}{2}$, where $exc(x) = |\{i : i < x(i)\}|$ (see [6]). On the other hand, if $x \in F_{2n}$, then $exc(x) = n$. Therefore,

$$\ell_{F_{2n}}(x) = \ell_{I_{2n}}(x) - n = \frac{n + inv(x)}{2} - n = \frac{inv(x) - n}{2}.$$

$\square$

Example 53. *Let $x = (1,8)(2,6)(3,5)(4,7) \in S_8$. Counting inversions in the word 18263547 we obtain $\widetilde{inv}(x) = 10$. On the other hand, written in one line notation $x = 86573241$. We see that $inv(x) = 24$. We are in S_8, so $n = 4$. Indeed we have $10 = \frac{24-4}{2}$.*

Chapter 8

The order complexes of F_{2n} and PF_n

8.1 The order complex of F_{2n}

This section is joint work with Can and Cherniavsky [7].

Recall that the Möbius function is equal to the reduced Euler characteristic of the topological realization of the order complex of the poset P, and moreover, this number is found by counting the number of maximal chains in P.

In this section by applying these considerations to $P = F_{2n}$, we prove that the order complex $\Delta(F_{2n})$ is homotopy equivalent to a ball of dimension $n(n-1)-2$.

Theorem 54. *The order complex $\Delta(F_{2n})$ triangulates a ball of dimension $n(n-1)-2$.*

Proof. We know from [23] that if in a pure shellable complex Δ each $\dim \Delta - 1$ dimensional face lies in at most two maximal faces, then Δ triangulates a sphere or a ball.

We also know that the Bruhat-Chevalley poset I_{2n} is Eulerian, hence every interval of length 2 has 4 elements. Since F_{2n} is a subposet of I_{2n}, every interval of length 2 in F_{2n} has at most 4 elements. This implies that in the order complex $\Delta(F_{2n})$, each $\dim \Delta(F_{2n}) - 1$ dimensional face lies in at most two maximal faces, hence it triangulates a sphere or a ball of dimension

$$\dim \Delta(F_{2n}) = \ell(F_{2n}) = 2\binom{n}{2} - 2 = n(n-1) - 2.$$

To see that it is a ball, we show that the reduced Euler characteristic of $\Delta(F_{2n})$ is 0. Thus, by the discussion above, it is enough to show that there is no maximal chain $j_{2n} \leftarrow x_1 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow w_0$ such that $f(j_{2n}, x_1) > f(x_2, x_1) > \cdots > f(x_{m-1}, w_0)$, where $f : C(F_{2n}) \to \mathit{\Gamma}$ is the EL-labeling that is constructed implicitly in the proof of Theorem 36. Indeed, f is obtained from Incitti's EL-labeling $g : C(I_{2n}) \to \mathit{\Gamma}$, by reversing the order of the totally ordered set $\mathit{\Gamma}$ of pairs (i, j), $1 \leq i, j \leq 2n$ ordered lexicographically. Therefore, it is enough to show that there is no maximal chain

$$\mathfrak{c}: \ j_{2n} \leftarrow x_1 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow w_0 \tag{8.1}$$

in F_{2n} such that $g(j_{2n}, x_1) < g(x_2, x_1) < \cdots < g(x_{m-1}, w_0)$. On the other hand, since g is an EL-labeling, we see that $\mathfrak{c}$ is the unique such chain whose sequence of labels is lexicographically smallest among all such chains in the interval $[j_{2n}, w_0]$ in I_{2n}. Thus, the proof is finished once we show that $\mathfrak{c}$ does not lie in F_{2n}.

It is easy to verify our claim directly in the case of $n = 2$. For $n \geq 3$, let $C(j_{2n})$ denote the set

$$C(j_{2n}) = \{g(j_{2n}, z) \in \mathit{\Gamma} : \ z \in I_{2n} \text{ and } j_{2n} = (1,2)(3,4)\cdots(2n-1,2n) \leftarrow z\}.$$

Observe that $\min C(j_{2n}) = g(j_{2n}, z) = (1,3)$ with $z = (1,4)(5,6)(7,8)\cdots(2n-1,2n)$. Therefore, x_1 of $\mathfrak{c}$ has to be equal to z. Since z has a fixed point, $\mathfrak{c}$ does not lie in F_{2n}.

□

8.2 The order complex of PF_n

We would like to thank Yonah Cherniavsky for pointing out the following fact.

Similarly to the F_{2n} case we see that $\Delta(PF_n)$ triangulates a ball or a sphere of dimension

$$\dim \Delta(PF_n) = l(PF_n) = n + (n-1) + \cdots + 1 - n = \binom{n}{2}.$$

Theorem 55. *The order complex $\Delta(PF_n)$ triangulates a ball of dimension $\binom{n}{2}$.*

Proof. To see that it is a ball, we show that the reduced Euler characteristic of $\Delta(PF_n)$ is 0. Recall that the reduced Euler characteristic of $\Delta(PF_n)$ is equal to $\mu([\hat{0},\hat{1}])$ where $\hat{0}$ and $\hat{1}$ are the minimal and maximal elements of PF_n respectively and μ is the Möbius function.

We see that for every n the highest three levels consist of a chain. Recall the $n = 4$ case illustrated in Figure 5.4. It immediately follows from the definition of the Möbius function that it is equivalent to show $\mu([\hat{0},\hat{1}]) = 0$ for the opposite poset.

We will prove by induction that $\mu([0,z]) = 0$ for all z if $l(z) > 1$. Recall that the Möbius function is defined by $\mu([x,x]) = 1\ \forall x$ and $\mu([x,y]) = -\sum_{x\le z<y}\mu([x,z])$. When $l(z) = 2$ then $\mu([\hat{0},z]) = -\mu([\hat{0},b]) - \mu([\hat{0},\hat{0}]) = -(-1) - 1 = 0$ where b is the only element such that $\hat{0} < b < z$. If z is an element with $l(z) = n+1$ then $\mu([\hat{0},z]) = -\sum_{\hat{0}\le c<z}\mu([\hat{0},c]) = -\sum_{b\le c<z} 0 - (-1) - 1 = 0$ by the induction hypothesis. In particular $\mu([\hat{0},\hat{1}]) = 0$ which concludes the proof.

$\square$

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Biography

The author was born on July 29, 1986 in Mainz, Germany. He started his studies in mathematics at Technische Universität Darmstadt in 2006 where he stayed until he was admitted to the Ph.D. program at Tulane in 2008.