Topological Galois theory of Riemann surfaces

An Abstract

Submitted On the fourteenth day of August 2020,

To The Department of Mathematics

In Partial Fulfillment Of The Requirements

Of The School of Science and Engineering

Of Tulane University

For The Degree Of

Master of Science

By

Dejun Zhang

Approved: ______________________

Mahir Bilen Can, Ph.D.
Chairman

Scott Alister McKinley, Ph.D.

Michael Joyce, Ph.D.

Abstract

There is a deep analogy between the theory of covering spaces and the theory of field extensions. Indeed, for many theorems about the Galois groups of field extensions there are analogous statements for the fundamental groups of covering spaces. The purpose of this thesis is to present an expository account of the connections between these two subjects of active research in mathematics.

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Chapter 1

Introduction

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The theory of covering spaces has important applications in many branches of mathematics, such as fundamental groups and Riemann surfaces. Furthermore, the analogy between the theory of covering spaces and the theory of field extensions is mentioned in many textbooks. For example, in [1, Section 1.3], Hatcher describes the basic ideas of Galois theory in the context of covering spaces. In his blog [2], Tao explains the Galois correspondence for manifolds. A synthesis of the theory of Riemann surfaces and covering spaces can be found in [3], where an elegant description of the ramified covers and respective field extensions of meromorphic functions is presented. The reference [4] also deals with these topics in a more technical manner. These developments witness the importance of the main ideas of Galois theory in geometry. The purpose of our thesis is to give a down-to-earth presentation of the circle of ideas and concepts that we mentioned before. Here we will focus on the analogy and we hope to be able to explain the intuitive idea of the profound mechanism behind the aforementioned works. Although we will avoid some important but technical concepts (such as algebraic varieties, schemes, and so on), we will provide some general remarks to explain them.

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Chapter 2

Fundamental groups and covering spaces

2.1 Fundamental groups

In this section, we will introduce the concept of a fundamental group and collect some of its properties.

Definition 2.1.1. A *path* in X is a continuous map $\gamma : I \to X$ for $I = [0,1]$. If the end points $\gamma(1) = \gamma(0)$, then γ is said to be *a loop based at* $\gamma(0)$. The *inverse path* of γ is defined by $\bar{\gamma}(s) := \gamma(1-s)$.

Definition 2.1.2. Let $\gamma_0 : I \to X$ and $\gamma_1 : I \to X$ be two paths such that $\gamma_0(0) = \gamma_1(0) =: x_0$ and $\gamma_0(1) = \gamma_1(1) =: x_1$. We say that γ_0 and γ_1 are *path-homotopic* if there is a continuous map $\Gamma : I \times I \to X$ satisfying the following two conditions:

(1) $\Gamma(s,0) = \gamma_0(s)$, $\Gamma(s,1) = \gamma_1(s)$,

(2) $\Gamma(0,t) = x_0$, $\Gamma(1,t) = x_1$.

In this case, we will write $\gamma_0 \simeq \gamma_1$.

It is readily seen that the path homotopy induces an equivalence relation on the set of paths in X, and we can use $[\gamma]$ to denote the class containing γ.

Definition 2.1.3. Let γ and β be two paths with $\gamma(1) = \beta(0)$. Then the concatenation of paths γ and β is defined by

$$(\gamma * \beta)(s) = \begin{cases} \gamma(2s) & s \in \left[0, \frac{1}{2}\right], \\ \beta(2s-1) & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We want to multiply the equivalence classes of paths.

Lemma 2.1.4. *The concatenation of paths can be defined for suitable equivalence classes of paths as well.*

Proof. Assuming $\gamma_1 * \beta_1$ and $\gamma_2 * \beta_2$ are both defined. Let $\gamma_1 \simeq \gamma_2$ and $\beta_1 \simeq \beta_2$ via Γ and B, respectively. The function

$$H(s,t) = \begin{cases} \Gamma(2s) & s \in \left[0, \frac{1}{2}\right], \\ B(2s-1) & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

is a homotopy between $\gamma_1 * \beta_1$ and $\gamma_2 * \beta_2$. □

Now we set

$$\pi_1(X, x_0) := \{[\gamma] : \ \gamma \text{ is a loop based at } x_0\}.$$

Since concatenation of two loops based at the same point is again a loop, concatenation induces a binary operation on this set. The checking of other group axioms is omitted here and given in [5, Theorem 51.2].

Definition 2.1.5. The set $\pi_1(X, x)$ equipped with the concatenation defined in Lemma 2.1.4 is called the *fundamental group of X based at x*.

Definition 2.1.6. A topological space X is called *path connected* if for every pair of points x_0, x_1, there is a path such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. X is called *locally path connected at* x if for every open neighborhood U of x, there is a neighborhood of x contained in U that is path connected.

Given a path connected space X, every loop based at x_0 can be "transferred" to a loop based at x_1 via a path between x_0 and x_1. By using this transfer of the loops we obtain an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

Remark 2.1.7. In the sequel, we will mainly consider Riemann surfaces. Such spaces are path connected and locally path connected. We make the convention that all spaces considered in this thesis enjoy two conditions.

Example 2.1.8. Let $x \in \mathbb{R}^n$. Then we have $\pi_1(\mathbb{R}^n, x) = \{[c_x]\}$, where c_x is the *constant path* based at x defined by $c(s) \equiv x$. In fact, for any convex subspace $C \subseteq \mathbb{R}^n$ with $x \in C$, we have $\pi_1(C, x) = \{[c_x]\}$. To see this, let $[\gamma_1]$ and $[\gamma_2] \in \pi_1(C, x)$, there is a *straight line homotopy* defined by $\Gamma(s, t) = t\gamma_1(s) + (1 - t)\gamma_2(s)$.

Definition 2.1.9. A topological space X is called *simply connected* if X is path connected and of a trivial fundamental group.

Theorem 2.1.10. *In a simply connected space, any two paths sharing the same initial and end points are homotopic.*

Proof. Let γ_1 and γ_2 be two paths from x_0 to x_1. Then $\gamma_1 * \bar{\gamma}_2$ will be homotopic to the constant path based at x_0. It follows that $\gamma_1 \simeq \gamma_2$. □

Definition 2.1.11. For a continuous map $f : X \to Y$ with $y = f(x)$, the function $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ defined by $f_*([\gamma]) = [f \circ \gamma]$ is well-defined.

Indeed, f_* is a homomorphism and called the induced homomorphism of the continuous map f with marked point x. It can be checked that such homomorphisms enjoy the following properties:

(1) If $h : (X, x) \to (Y, y), k : (Y, y) \to (Z, z)$ are continuous maps with marked points specified, then $(k \circ h)_* = k_* \circ h_*$.

(2) If i is the identity map on X. It induces the identity homomorphism on $\pi_1(X, x) \to \pi_1(X, x)$.

we can now define the category of *marked topological spaces* $\mathbf{Top}_*$. Its objects $\mathbf{Top}_*$ are the marked spaces like (X, x). Its morphisms are marked continuous maps like $f : (X, x) \to (Y, y)$. In this setting, if $x' \neq x$ is another element from $f^{-1}(y)$, $f_{x'} : (X, x') \to (Y, y)$ is different from $f_x : (X, x) \to (Y, y)$.

The category to be associated with $\mathbf{Top}_*$ is the category of groups $\mathbf{Grp}$, with groups as its objects and group homomorphisms as its morphisms. Recall from our earlier discussion of fundamental groups that to each marked topological space we can assign a group via

$$(X, x) \rightsquigarrow \pi_1(X, x).$$

Furthermore, we saw that each marked continuous map $f : (X, x) \to (Y, y)$ gives a group homomorphism between the corresponding fundamental groups. In summary, we have the following theorem:

Theorem 2.1.12. *The fundamental group functor is a covariant functor $\pi_1 : \boldsymbol{Top}_* \to \boldsymbol{Grp}$. As an object functor, it assigns a marked space (X, x) with its fundamental group space $\pi_1(X, x)$. As a morphism functor, it assigns a marked continuous map $f : (X.x) \to (Y, y)$ to its induced homomorphism $\pi_1(f) = f_* : \pi_1(X, x) \to \pi_1(Y, y)$.*

2.2 Covering maps

Definition 2.2.1. Given a continuous map $f : X \to Y$, it becomes a *covering map* if for each $y \in Y$, there is a neighborhood U of y that is *evenly covered* by f, in the sense that $f^{-1}(U)$ can be written as a disjoint union of U_α's, where each U_α is homeomorphic with U via f. In this case, X is said to be a covering space over the base space Y.

Example 2.2.2. Let $\mathbb{S}^1 = \{z \in \mathbb{C} : \ |z| = 1\}$ be the unit circle in complex plane $\mathbb{C}$. The map $f : \mathbb{R} \to \mathbb{S}^1$ given by $\theta \mapsto exp(2\pi i\theta)$ is a covering map. Besides, given any $n \in \mathbb{N}$, the map $f_n : \mathbb{S}^1 \to \mathbb{S}^1$ with $z \mapsto z^n$ is a covering map.

Theorem 2.2.3. *For a covering map $f : X \to Y$, if for some $y \in Y$ the index $|f^{-1}(y)| = d$, then $|f^{-1}(y')| = d$ for all $y' \in Y$. In this case, d is defined to be the degree of the covering map f.*

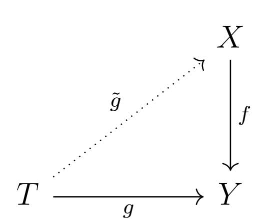
Proof. We assumed Y is connected. Let $S = \{y' \in Y : \ |f^{-1}(y')| = d\}$. S is not empty. Next we prove S is open. Given $y' \in S$, there is a neighborhood U of y' that is evenly covered. It follows $|f^{-1}(y_U)| = d$ for all $y_U \in U$. We can show $Y - S$ is open by a similar argument, thus the closeness of S is justified. □

Remark 2.2.4. In the proof above, a trick called the *continuity method* is applied. That is, if we can prove some nonempty subset in a connected space is both closed and open, then this subset must be the whole space.

It is natural to inspect the corresponding homomorphism induced by a marked covering map. For this purpose, let us introduce the notion of lifting property.

Definition 2.2.5. Given a covering map $f : X \to Y$ and a continuous map $g : T \to Y$, a *lift* of g is a map $\tilde{g} : T \to X$ with $f \circ \tilde{g} = g$.

Diagrammatically, a lift is presented as follows:



Our discussion will be decomposed into 3 stages:

(1) $T = I$.

(2) $T = I \times I$.

(3) T is a space that is path connected and locally path connected.

Theorem 2.2.6. *Given a path $g : I \to Y$ with $g(0) = y$, then for each $x_i \in f^{-1}(y)$ there is a unique lift $\tilde{g} : I \to X$ with $\tilde{g}(0) = x_i$.*

Proof. Given a fixed x_i, let $\tilde{T} \subset [0, 1]$ be the set of t such that the restricted $g_{[0,t]}$ can be lifted a path beginning at x_i. $\tilde{T}$ is not empty. Given $t \in \tilde{T}$, there is a neighborhood V of $g(t)$ that is evenly covered. Let U be the unique component containing $\tilde{g}(t)$. Then by continuity we have $g([t, t + \delta]) \subset V$ for some positive δ and $\tilde{g}$ is then uniquely extended to $[t, t + \delta]$ on U. It follows $\tilde{T}$ is open. For the closeness, consider a sequence $\{t_n\}$ converging to $t_0 \in [0, 1]$. There is a neighborhood V of $g(t_0)$ that is evenly covered. Then for sufficiently large n we have $g([t_n, t_0]) \subset V$. Again by choosing the suitable component, we can extend $\tilde{g}$ to include t_0. This justified the closeness. □

With a similar treatment on $I \times I$, we can prove the second case:

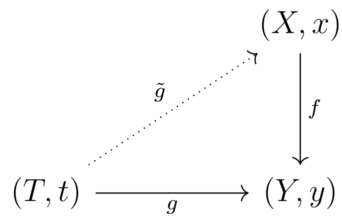
Theorem 2.2.7. *Given a continuous map $g : I \times I \to Y$ with $g(0, 0) = y$, then for each $x_i \in f^{-1}(y)$ there is a unique lift $\tilde{g} : I \times I \to \tilde{X}$ with $\tilde{g}(0, 0) = x_i$.*

The next corollary is of extra significance.

Corollary 2.2.8. *If g is a path homotopy, then $\tilde{g}$ is also a path homotopy.*

Proof. If g is a homotopy, then by definition $\tilde{g}(\{0\} \times I)$ and $\tilde{g}(\{1\} \times I)$ are connected subsets in the discrete set $f^{-1}(y)$. It follows they both contain only one element, implying that $\tilde{g}$ is a path homotopy. □

Theorem 2.2.9. *([5, Lemma 79.1]) Given a marked continuous map $g : (T, t) \to (Y, y)$, then there exists a unique (marked) lift $\tilde{g} : (T, t) \to (X, x)$ if and only if $\pi_1(g)(\pi_1(T, t)) \subset \pi_1(f)(\pi_1(X, x))$.*



Theorem 2.2.10. *(The lifting correspondence) For a marked covering map $f_i : (X, x_i) \to (Y, y)$ and $[\gamma] \in \pi_1(Y, y)$, if $\tilde{\gamma}$ is the lift of γ beginning at x_i, then there is a well-defined function $\phi_{f_i} : \pi_1(Y, y) \to f^{-1}(y)$ defined by $[\gamma] \mapsto \tilde{\gamma}(1)$.*

Proof. To show this function is well-defined, we can choose another $\beta \in [\gamma]$, then the respective lifts $\tilde{\beta}$ and $\tilde{\gamma}$ starting at x_i are path homotopic. It follows $\tilde{\beta}(1) = \tilde{\gamma}(1)$. □

Definition 2.2.11. The function ϕ_{f_i} is called the lifting correspondence derived from the marked covering map $f_i : (X, x_i) \to (Y, y)$.

Now let us collect some properties of the lifting correspondence.

Theorem 2.2.12. *Let $f_i : (X, x_i) \to (Y, y)$ be a marked covering map.*

(1) $\phi_{f_i} : \pi_1(X, x_i) \to f^{-1}(y)$ is surjective. Moreover, if X is simply connected, it is bijective.

(2) The induced homomorphism $\pi_1(f_i) : \pi_1(X, x_i) \to \pi_1(Y, y)$ is injective.

(3) Let H_i be the image $\pi_1(f_i)(\pi_1(X, x_i))$. ϕ_{f_i} induces a bijective map Φ_{f_i} from the collection of right cosets $\pi_1(Y, y)/H_i$ to $f^{-1}(y)$.

(4) The image H_i can also be given by $H_i = \{[\gamma] : \ \gamma$ lifts to a loop $\tilde{\gamma}$ based at $x_i\}$.

Proof. It is assumed that all spaces to be considered here are path connected and locally path connected.

(1) For $x_j \in f^{-1}(y)$, there is a path $\tilde{\beta}$ from x_i to x_j. Then $\beta = f \circ \tilde{\beta}$ is a loop based at (Y, y) with $\phi_{f_i}([\beta]) = x_j$. Next we assume X is simply connected. It suffices to show the injectivity. Let $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ be lifts of γ_1 and γ_2, respectively. If $\tilde{\gamma_1}(1) = \tilde{\gamma_2}(1) \in f_i^{-1}(y)$, then $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ are path homotopic by the simply connected condition. Composing this homotopy with f we have a homotopy between γ_1 and γ_2.

(2) If $\pi_1(f_i)([\tilde{\gamma}])$ is equal to the identity, then by definition there is a path homotopy between $[f_i \circ \tilde{\gamma}]$ and the constant path at y. By lifting both of them, $\tilde{\gamma}$ is path homotopic to the constant path based at x_i. Thus the homomorphism is of a trivial kernel.

(3) Define the function $\Phi_{f_i}(H_i[\gamma]) := \phi_{f_i}([\gamma]) = \tilde{\gamma}(1)$. Next we show Φ_{f_i} is well-defined. Given another $[\beta] = [h \circ \gamma] \in H_i * [\gamma]$, then by definition $h = f_i \circ \tilde{h}$ with $\tilde{h}$ a loop based at x_i. It follows the lifts $\tilde{\beta}$ and $\tilde{h} * \tilde{\gamma}$ are path homotopic and $\tilde{\gamma}(1) = \tilde{\beta(1)}$. The surjectivity follows from the surjectivity of ϕ_{f_i}. It remains to check the injectivity. Suppose $\Phi_{f_i}[H_i * \gamma] = \Phi_{f_i}[H_i * \beta]$, then by definition $\tilde{\beta}(1) = \tilde{\gamma}(1)$. It follows tha $[\tilde{\beta}] = [\tilde{h} * \tilde{\gamma}]$. Composing both sides with f_i we have a path homotopy between $[\beta]$ and $[h * \gamma]$. Thus $[\beta] \in H_i * [\gamma]$.

(4) If $[\gamma] \in H_i$, then γ is path homotopic to $f_i \circ \eta$ with a loop η based at x_i, thus the lift $\tilde{\gamma}$ will be path homotopic to η. It follows $\tilde{\gamma}(1) = x_i$ and $\tilde{\gamma}$ is also a loop based at x_i. For the converse, let $\tilde{\gamma}$ be the lift of γ with $\tilde{\gamma}$ a loop based at x_i. It follows $\gamma = f_i \circ \tilde{\gamma}$ and $[\gamma] \in H_i$.

□

Remark 2.2.13. Let us pause for a while and pay attention to claim (2). It shows a marked covering map $f_i : (X, x_i) \to (Y, y)$ is associated with a monomorphism $\pi_1(f_i)$, from which $\pi_1(X, x_i)$ can be identified as a subgroup of of $\pi_1(Y, y)$.

Given $x_i \neq x_j$ from $f^{-1}(y)$, the question naturally arises how ϕ_{x_i} and ϕ_{x_j} are related.

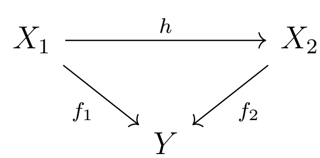
Theorem 2.2.14. *([5, Lemma 79.3]) Given $f_i : (X, x_i) \to (Y, y)$ and $f_j : (X, x_j) \to (Y, y)$ that originate from the (unmarked) covering map $f : X \to Y$, let $H_i := \pi_1(f_i)(\pi_1(X, x_i))$ and $H_j := \pi_1(f_j)(\pi_1(X, x_j))$. Then we have:*

(1) H_i and H_j are conjugate.

(2) If H is conjugate to H_i, there exists some $x_j \in f^{-1}(y)$ such that $H_j = H_i$ for $H_j = \pi_1(f_j)(\pi_1(X, x_j))$.

Next we consider the equivalence of covering spaces.

Definition 2.2.15. Two covering maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are called equivalent if there is a homeomorphism $h : X_1 \to X_2$ such that $f_1 = f_2 \circ h$.



Now we introduce a criterion for checking the equivalence for two marked covers. Roughly speaking, since X_1 and X_2 are homeomorphic via h, then there is an isomorphism $\pi_1(h) : \pi_1(X_1, x_1) \to \pi_1(X_2, x_2)$. So we can transfer every subgroup of $\pi_1(X_1, x_1)$ into that of $\pi_1(X_2, x_2)$ and vice versa. In this manner, we have reduced the case such that Theorem 2.2.14 can be applied. Thus we have the following criterion:

Theorem 2.2.16. *([5, Theorem 79.4]) Two marked covers $f_1 : (X_1, x_1) \to (Y, y)$ and $f_2 : (X_2, x_2) \to (Y, y)$ are equivalent if and only if $\pi_1(f_1)(\pi_1(X_1, x_1))$ and $\pi_1(f_2)(\pi_1(X_2, x_2))$ are conjugate.*

Example 2.2.17. The fundamental group of $\mathbb{S}^1$ is isomorphic to $\mathbb{Z}$ and there are countably many covering spaces of $\mathbb{S}^1$, up to equivalence.

Proof. Given $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ with base point $1 \in \mathbb{C}$. For the loops defined by $\gamma_n(\theta) = exp(2\pi in\theta)$, with $\theta \in [0,1]$ and $n \in \mathbb{Z}$, we have the contour integral $\oint_{\gamma_n} \frac{1}{z} = 2n\pi i$. Thus by Cauchy's integral theorem, γ_n is not homotopic to γ_m if $m \neq n$. The set map $f : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ given by $f([\gamma_n]) := n$ is then bijective. It is an isomorphism since $[\gamma_{(n+m)}] = [\gamma_n * \gamma_m]$. For the classification of covers for $\mathbb{S}^1$ up to equivalence, note that for a positive integer n, the map $f_n : \mathbb{S}^1 \to \mathbb{S}^1$ with $z \mapsto z^n$ is a covering map. Besides, there is a trivial case $f_0 : \mathbb{R} \to \mathbb{S}^1$ given by $\theta \to exp(2\pi i\theta)$. These exhaust all covering spaces of $\mathbb{S}^1$, up to equivalence. □

In the above example, $f_0 : \mathbb{R} \to \mathbb{S}^1$ is quite special since this covering map corresponds to the trivial subgroup.

Definition 2.2.18. A covering space that is simply connected is called a universal covering space.

Chapter 3

Topological Galois theory

In this part we will present the topological Galois theory. To begin with, let us review some basics of classical Galois theory.

3.1 Notes from field theory

Definition 3.1.1. Given a field extension X/Y, the set $Aut(X/Y)$ of automorphisms of X fixing Y is a group under function composition. The degree of extension X/Y is denoted by $[X : Y]$. If X/Y is a finite extension, then X is a Galois extension of Y if $|Aut(X/Y)| = [X : Y]$. In this case, $Aut(X/Y)$ is called the Galois group of X/Y and denoted by $Gal(X/Y)$.

Theorem 3.1.2. *([6, Theorem 14.14] The main theorem of Galois theory) Let X/Y be a Galois extension and $G = Gal(X/Y)$, then there is a one to one correspondence between the set of subfields Z of X that contain Y and the set of subgroups H of G. The subfield Z is sent to the set H_Z, which is the set of all elements of G fixing Z. The subgroup H is sent to the subfield X^H of X, which is the fixed field by H. Moreover, under this correspondence we have:*

(1) If Z_1, Z_2 corrspond to H_1, H_2, respectively, then $Z_1 \subset Z_2$ if and only if $H_2 \subset H_1$.

(2) The degree of field extension $[X : Z] = |H|$ and $[Z : Y] = |G : H|$, which is the index of H in G.

(3) X/Z is always Galois, with Galois group $Gal(X/Z) = H$.

(4) Z is Galois over Y if and only if H is normal in G. In this case, $Gal(Z/Y)$ is isomorphic to the quotient group G/H.

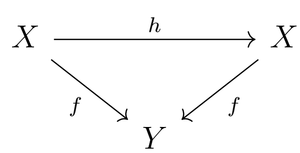
Example 3.1.3. Here is an example to illustrate this theorem. Let $Y = \mathbb{C}(z)$ be the quotient field of variable z with coefficients from $\mathbb{C}$. If $w = z^{\frac{1}{n}}$, then the simple extension $X = Y(w)$ is a Galois extension with $Gal(X/Y) \cong \mathbb{Z}/n\mathbb{Z}$ and $[X : Y] = n$. All intermediate fields are given by $Z = \mathbb{C}(w^{\frac{n}{m}})$ with m dividing n.

3.2 Covering transformations

The next notion reminds us of the group of automomorphism $Aut(X/Y)$ in field theory.

Definition 3.2.1. An equivalence of covering map $f : X \to Y$ with itself is called a *covering transformation*. Given a covering map $f : X \to Y$, the set $Aut(X/Y)$ of all its covering transformations is a group under the function composition.

Diagrammatically we have:



Diagrams like this appear frequently in field theory, except that f is now a covering map and h is a homeomorphism instead of a field extension and an automorphism, respectively.

Theorem 3.2.2. *Given a covering map $f : X \to Y$ and $h \in Aut(X/Y)$, if $h(x) = x$ for some $x \in X$, then h is the identity map.*

Proof. Let $X' = \{x \in X : \ h(x) = x\}$. The remaining is done by the continuity method and I omit it here. □

It follows that given $f_i : (X, x_i) \to (Y, y)$, the function $\Psi : Aut(X/Y) \to f^{-1}(y)$ given by $h \mapsto \Psi(h) = h(x_i)$ is injective. Recall that by claim (3) of Theorem 2.2.12, there is a bijection from the collection of right cosets of H_i to $f^{-1}(y)$. These observations motivates our correlation of $\pi_1(Y, y)/H_i$ and $Aut(X/Y)$.

Theorem 3.2.3. *([5, Lemma 81.1].) The image of the function Ψ equals the image $\Phi(N(H_i)/H_i)$, where $N(H_i)$ is the normalizer of H_i in $\pi_1(Y, y)$.*

Thus we have a bijection $\Phi_{f_i}^{-1} \circ \Psi$. Moreover, we can prove it is an isomorphism by checking the homomorphism property. It should be noted such an isomorphism depends on the marking covering map $f_i : (X, x_i) \to (Y, y)$.

Theorem 3.2.4. *([5, Theorem 81.2]) $\iota_i := \Phi_{f_i}^{-1} \circ \Psi : Aut(X, Y) \to N(H_i)/H_i$ is an isomorphism.*

Two cases deserve extra attention:

(1) If the normalizer $N(H_i)$ is all of $\pi_1(Y, y)$, then the isomorphism is defined on the whole $\pi_1(Y, y)/H_i$.

(2) If H_i is trivial, then there is an isomorphism between $Aut(X/Y)$ with $\pi_1(Y, y)$.

For these two cases, we immediately have the next two corollaries.

Corollary 3.2.5. *H_i is normal in $\pi_1(Y, y)$ if and only if for x_i, $x_j \in f^{-1}(y)$, there exists $h \in Aut(X/Y)$ such that $h(x_i) = x_j$. In this case, $f : X \to Y$ is called a Galois covering map and $Aut(X/Y)$ is usually written as $Gal(X/Y)$.*

Corollary 3.2.6. *If U_0 is a universal cover, then $Aut(U_0/Y)$ is isomorphic to $\pi_1(Y, y)$ via ι_i.*

3.3 Quotient spaces

Definition 3.3.1. Given a space X with G and a group of homeomorphisms of X with itself, the quotient map $p : X \to X/G$ is given by the $x \mapsto \{g(x) : \ g \in G\}$. If for any $x \in X$, there is a neighborhood U of x such that $g(U) \cap U = \emptyset$ whenever g is not the identity element, then the action of G on X is called *properly discontinuous.*

Theorem 3.3.2. *([5, Theorem 81.5]) The quotient map $p : X \to Z = X/G$ is a covering map if and only if the action of G is properly discontinuous. In this case, the covering map p is Galois with $Gal(X/Z) = G$.*

Theorem 3.3.3. *([5, Theorem 81.6]) Given $f : X \to Y$ a Galois covering map with $G = Gal(X/Y)$, then there is a homeomorphism $k : X/G \to Y$ such that $f = k \circ p$, where p is given by the quotient map $q : X \to X/G$.*

$$\begin{array}{ccc} X & \longrightarrow & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ X/G & \xrightarrow{k} & Y \end{array}$$

Theorem 3.3.4. *([5, Theorem 22.2]) Let $p : X \to Z$ be a quotient space. Let Y be a space and $f : X \to Y$ be a map that is constant on each $p^{-1}(z)$, for $z \in Z$. Then f induces a map g such that $f = g \circ p$. In this case, we have:*

(1) g is continuous if and only if f is continuous.

(2) g is a quotient map if and only if f is a quotient map.

$$\begin{array}{ccc} X & \xrightarrow{p} & Z \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ Y & & \end{array}$$

3.4 The main theorem of topological Galois theory

Definition 3.4.1. Let $f : X \to Y$ be a Galois covering map. Given two covering maps $p : X \to Z$ and $g : Z \to Y$, if $f = g \circ p$, then Z is an intermediate covering space lying between $f : X \to Y$.

The diagram in Theorem 3.3.4 can serve as an illustrate here. Roughly speaking, an intermediate is a covering space that is itself a base space. We can compare it with an intermediate field. Recall that an intermediate field Z is a subfield of X and a field extension of Y.

Lemma 3.4.2. *([5, Lemma 80.2]) Let f be a covering map and g, p be continuous maps such that $f = g \circ p$, then g is a covering map if and only if p is a covering map.*

Now we have got all preparation done. It is time to present the main theorem of topological Galois theory. Again, the diagram in Theorem 3.3.4 will be of great help for our discussion in the following.

Theorem 3.4.3. *Given $f : X \to Y$ and $G = Gal(X/Y)$, if $H \subset G$, then the quotient space $Z = X/H$ defined by $p : X \to Z = X/H$ is an intermediate covering space. That is, we have two covering maps $p : X \to Z$ and $g : Z \to Y$ satisfying $f = g \circ p$. Moreover, $p : X \to Z = X/H$ is always a Galois covering map with $Gal(X/Z) = H$.*

Proof. As a subgroup of G, the action of H on X is also properly discontinuous, then by Theorem 3.3.2, the quotient map $p : X \to Z = X/H$ is a Galois covering map with $H = Gal(X/Z)$. By Theorem 3.3.4, f induces a quotient map $g : Z = X/H \to Y$ such that $f = g \circ p$. Now it suffices to show g is also a covering map. However, this follows immediately from Lemma 3.4.2 as the quotient map g is surely continuous. □

Theorem 3.4.4. *For Z an intermediate covering spaces lying between $f : X \to Y$, the covering map $g : Z \to Y$ is Galois if and only if $Gal(X/Z)$ is normal in $Gal(X/Y)$. In this case, $Gal(Z/Y) \cong Gal(X/Y)/Gal(X/Z)$.*

Proof. One direction is relatively easy. Given $H = Gal(X/Z)$ normal in G, then by 3.3.3 we have homeomorphisms $Z \cong X/H$ and $Y \cong X/G$. It follows G/H is a group of homeomorphisms on Z and we can define a quotient space of Z by action of G/H. More precisely, we can check that the original quotient map $X \to X/G$ can be factored into two quotient maps $X \to Z(\cong X/H) \to Y(\cong Z/(G/H))$. Note that the second quotient map $Z \to Y = Z/(G/H)$ is identical with the covering map $Z \to Y$, thus the quotient map $Z \to Y(\cong Z/(G/H))$ is a covering map. Then by 3.3.2 the covering map is Galois. For the other side, if $g : Z \to Y$ is Galois, then for $h \in Gal(X/Y)$, we can define a unique $h' \in Gal(Z/Y)$ corresponding to h. More precisely, fix $x \in X$, let h' be the unique homeomorphism sending $p(x)$ to $p(h(x))$. Thus we have a set map $Gal(X/Y) \to Gal(Z/Y)$ with $h \mapsto h'$ defined as before. Moreover, this set map is a homomorphism, with Gal(X/Z) as its kernel, thus we have $Gal(Z/Y) \cong Gal(X/Y)/Gal(X/Z)$ by isomorphism theorem of group theory. □

In light of the main theorem of Galois theory, we can now present the following:

Theorem 3.4.5. *(The main theorem of topological Galois theory) Let $X \to Y$ be a Galois covering map and $G = Gal(X/Y)$, then there is a one to one correspondence between intermediate covering spaces Z lying between $X \to Y$ and subgroups H of G. Under this correspondence, an intermediate covering space Z is sent to the set $H_Z = Aut(X/Z)$ and a subgroup H is sent to an intermediate covering space X/H lying between $X \to Y$. Moreover, under this correspondence,*

(1) If Z_1, Z_2 corrspond to H_1, H_2, respectively, then $Z_1 \to Z_2$ if and only if $H_1 \subset H_2$.

(2) (Assuming the degree of $X \to Y$ is finite) The degree of covering map $X \to Z$ is $|H|$ and that of $Z \to Y$ is $|G : H|$.

(3) $X \to Z$ is always Galois, with Galois group $Gal(X/Z) = H_Z$.

(4) Z is Galois over Y if and only if H is a normal subgroup of G. In this case, the Galois group $Gal(Z/Y)$ is isomorphic to the quotient group G/H.

Proof. Claims (3) and (4) are already proved before. It remains to check cliam (1) and (2).

(1) Given $H_1 \subset H_2$ and $Z_1 = X/H_1$ and $Z_2 = X/H_2$, there is a continuous map $g : Z_1 \to Z_2$. This continuous map is a covering map by Lemma 3.4.2.

(2) Let us consider the marked covering map $p : (X, x) \to (Z, z)$. By 3.2.5, there is an isomophism between $H = Gal(X/Z)$ and $\pi_1(Z, z)/\pi_1(p)(\pi_1(X, x))$. Note that the index of $\pi_1(p)(\pi_1(X, x))$ in $\pi_1(Z, z)$ is the same as the degree d_1 of the covering map $X \to Z$. By the same token, for $f : (X, x) \to (Y, y)$, there is an isomorphism between $G = Gal(X/Y)$ and $\pi_1(Y, y)/\pi_1(p)(\pi_1(X, x))$, where the index of $\pi_1(p)(\pi_1(X, x))$ in $\pi_1(Y, y)$ is the same as the degree d of the covering mao $X \to Z$. Let d, d_1, d_2 be the degrees of $X \to Y$ and $X \to Z$ and $Z \to Y$, respectively. Then $|G| = d = d_1 d_2$ and $d_1 = |H|$. It follows $d_2 = |G|/|H| = |G : H|$.

□

Example 3.4.6. Given the unit circle $\mathbb{S}^1$, the covering map $\mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^n$ is a Galois covering map with $Gal(\mathbb{S}^1/\mathbb{S}^1) \cong \mathbb{Z}/n\mathbb{Z}$. All intermediate covering spaces arise from maps $z \mapsto z^m$ with m dividing n.

Chapter 4

Classification of covering spaces

In this part, our former results will be used to make a classification of covering spaces for a fixed base space. To make our argument simpler, we will assume extra conditions on the base space.

Definition 4.0.1. A space Y is *semilocally simply connected* if for any $y \in Y$, there exist an open U containing Y such that the inclusion map $i : (U, b) \to (Y, y)$ induces a trivial homomorphism.

Note that a Riemann surface is semilocally simply connected since it is locally identical with an open set in $\mathbb{C}$.

Theorem 4.0.2. *A space Y admits a universal covering map if and only if Y is path connected, locally path connected, and semilocally simply connected.*

Theorem 4.0.3. *(The classification of covering maps) Let (Y, y) be fixed as a base space, then there is a one to one correspondence between subgroups of $\pi_1(Y, y)$ and the intermediate covering spaces Z lying between $U_0 \to Y$. Moreover, a normal subgroup is sent to a Galois covering space Z over Y.*

Proof. Combining Corollary 3.2.6 and main theorem of topological Galois theory, there is an isomophism $\iota_0 : Gal(U_0/Y) \to \pi_1(Y, y)$ and an bijection between subgroups H of $Gal(U_0/Y)$ and imtermediate covering spaces U_0/H. Given $\iota_0(H)$ a subgroup of $\pi_1(Y, y)$, it will be sent to an intermediate covering space U_0/H. Conversely, given any intermediate covering Z, it will be sent to $\iota_0(Gal(U_0/Z))$, which is a subgroup of $\pi_1(Y, y)$. Moreover, the normality of subgroup is equivalent to the Galois condition of the respective covering map. □

Definition 4.0.4. An n-dimensional topological torus, which is abbreviated to *n-torus*, and denoted by $\mathbb{T}^n$, is a product manifold of the form $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n copies). An n-dimensional (topological) solid torus, abbreviated to *solid n-torus*, is a product manifold of the form $\mathbb{D} \times \cdots \times \mathbb{D}$ (n copies), where $\mathbb{D}$ is the two dimensional unit disk in $\mathbb{R}^2$.

Remark 4.0.5. Note that $\mathbb{S}^1$ is a deformation retract of the multiplicative group $\mathbb{C}^*$. Therefore, an n-torus is homologous to the n-dimensional complex algebraic group $(\mathbb{C}^*)^n$. In other words, the homology groups of $(\mathbb{C}^*)^n$ and $\mathbb{T}^n$ are identical. Let us also mention that the 1-torus, that is, the unit circle $\mathbb{S}^1$ is isomorphic, as a topological group, to the quotient of the real numbes by the intergers. In other words, we have $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$. Therefore, for every positive integer n, the n-torus $\mathbf{T}^n$ can be identified with the product group $\mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}$ (n copies).

Example 4.0.6. There are 3 covering spaces with degree 2 of $\mathbb{T}^2$, up to equivalence. As mentioned before, $\mathbb{R}^2$ is a universal covering space for the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and $\pi_1(\mathbb{T}^2, t) \cong \mathbb{Z} \times \mathbb{Z}$. The classification of covering maps amounts to the classification of all subgroups of $\mathbb{Z} \times \mathbb{Z}$ since $\mathbb{Z} \times \mathbb{Z}$ is abelian. Now we exhibit the 3 cases. The first is $p_1 : \mathbb{T}^2 \to \mathbb{T}^2 : (z, w) \mapsto (z^2, w)$, where z and w are in complex notation, with the correspondent subgroup of $2\mathbb{Z} \times \mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}$. The second is p_2 given by $(z, w) \mapsto (z, w^2)$ with subgroup $\mathbb{Z} \times 2\mathbb{Z}$. The third is given by subgroup $\{(m, n) :\ m + n$ is an even integer$\}$, which is the kernel of the homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ where the two generators (1,0) and (0,1) sent to the nontrivial element $1+2\mathbb{Z}$. These 3 cases exhaust all possible subgroups of index 2 of $\mathbb{Z} \times \mathbb{Z}$.

Next we define the monodromy representation of a covering map $f : X \to Y$ of finite degree d. Given $y \in Y$, let $f^{-1}(y) = \{x_1, x_2, \dots, x_d\}$. A path γ from $[\gamma] \in \pi_1(Y, y)$ can be lifted to $\tilde{\gamma_1}, \tilde{\gamma_2}, \dots \tilde{\gamma_d}$ beginning at $x_1, x_2, \dots, x_d$ respectively, and ending at $\tilde{\gamma_1}(1), \tilde{\gamma_2}(1), \dots, \tilde{\gamma_d}(1)$, respectively. In this manner, γ induces a set function between $f^{-1}(y)$ to itself. Actually this induced set function is a permutation on $f^{-1}(y)$ since we can reverse the process by taking the inverse of $[\gamma]$. Note that the function depends only on the class $[\gamma]$. Besides, we can label the elements $\{x_1, x_2, \dots, x_d\}$ by $\{1, 2, \dots, d\}$. Thus we have associated each $[\gamma]$ with a permutation in the symmetric group. In summary, we have a well-defined function:

$$\rho_f : \pi_1(Y, y) \to S_d. \tag{4.1}$$

Moreover, this is a group homomorphism by noting that $\rho([\gamma_1] * [\gamma_2]) = \rho([\gamma_1]) \circ \rho([\gamma_2])$. Let us formulate our discussion as follows:

Definition 4.0.7. If $f : X \to Y$ is a covering map of degree d, then the map $\rho_f : \pi_1(Y, y) \to S_d$ as defined in (4.1) is a group homomorphism, which is called the *monodromy representation* of f.

Recall that a subgroup H of S_d is called a *transitive subgroup* if for every number $i, j \in \{1, \dots, d\}$, there is a permutation $\tau \in H$ such that $\tau(i) = j$.

Theorem 4.0.8. *If $f : X \to Y$ is a covering map of degree d, then the image of the monodromy representation $\rho_f : \pi_1(Y, y) \to S_d$ is a transitive subgroup.*

Proof. Since X is path connected, then for any two x_i and x_j, there exists a path β from x_i and x_j. Then the composite $f \circ \beta$ will be the path based at y that sends x_i to x_j. It yields that $\rho_f([f \circ \beta])$ sends i to j. □

The next theorem justifies the converse:

Theorem 4.0.9. *Given a space Y and a group homomorphism $\rho : \pi_1(Y, y) \to S_d$ with a transitive image, then there is a covering map $f_\rho : X \to Y$, of which the monodromy representation is exactly the homomorphism ρ.*

Proof. For a fixed $i \in \{1, 2...d\}$, $H_i = \{[\gamma] \in \pi_1(Y, y) : \ \rho([\gamma])(i) = i\}$ is a subgroup of $\pi_1(Y, y)$. By Theorem 4.0.3, H induces a covering map $f_\rho : Z \to Y$ such that the fundamental group of Z can be identified as the subgroup $H_i \subset \pi_1(Y, y)$. It follows the monodromy representation of f_ρ is exactly ρ. □

Example 4.0.10. Now let us make a classification of the covering maps with degree 2 of M_g, where M_g is the orientable surface of genus g. The construction of M_g is given in [1, Page 5]. By definition $\pi_1(M_g, m)$ is a free group generated by $2g$ generators $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ subjected to the single relation $a_1 b_1 a_1^{-1} b_1^{-1}, \ldots, a_g b_g a_g^{-1} b_g^{-1} = e$, where e is the identity element. To generate a transitive subgroup, it is required that at least one permutation is mapped to a non-identity element. It follows there are $2^{2g} - 1$ covering maps with degree 2, up to equivalence. If $g = 1$, the case degenerates to the case of $\mathbb{T}^2$ and there are 3 non-equivalent covering spaces, which is already known from Example 4.0.6.

Chapter 5

Riemann surfaces

Roughly speaking, a Riemann surface is a topological space that is locally isomorphic to an open set in $\mathbb{C}$ with respect to its analytic structure. To make sense of this, we introduce the definition of a complex atlas.

Definition 5.0.1. A *complex atlas* defined on a space X is a collection of homeomorphisms $\phi_i : U_i \to \phi_i(U_i)$, where $\phi_i(U_i)$ is an open subset in $\mathbb{C}$, satisfying:

(1) The collection $\mathcal{U} = \{U_i : \ i \in I\}$ is an open cover of X.

(2) For any $i_1, i_2 \in I$, the *transition map* $\phi_{i_2} \circ \phi_{i_1} : \phi_{i_1}^{-1}(U_{i_1} \cap U_{i_2}) \to \phi_{i_2}(U_{i_1} \cap U_{i_2})$ is holomorphic.

In this case, each hemeomorphism $\phi_i \in \mathcal{U}$ is called a *chart*. A chart ϕ is called *centered at x* if $\phi(x) = 0$.

Definition 5.0.2. Two complex atlases $\{\phi_i\}_{i\in I}$ $\{\psi_j\}_{j\in J}$ are *compatible* if for any $i \in I, j \in J$, the transition map $\psi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ is holomorphic.

Given a space X, compatibility defines an equivalence relation on the collection of complex atlases on it. A complex structure is then defined to be an equivalence class containing some given atlas.

Definition 5.0.3. A *Riemann surface* is a second countable, connected, and Hausdorff topological space, equipped with a complex structure.

Example 5.0.4. Many familiar notions we met in complex analysis are Riemann surfaces. The complex plane $\mathbb{C}$, the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$, and the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \ Im(z) > 0\}$ are Riemann surfaces. Actually any connected open subset U in $\mathbb{C}$ is a Riemann surface with the atlas consisting of a single chart $id : U \to U$ given by an identity map.

Many interesting manifolds in geometry can be obtained from Riemann surfaces.

A *(complex) abelian variety* is a complete (complex) algebraic variety together with a compatible group structure. A one dimensional abelian variety is called an *elliptic curve*. An excellent introduction to this topic can be found in [7, Chapter IV, §4]. Every complex elliptic curve can be realized as a quotient of the abelian group $(\mathbb{C}, +)$ by a discrete subgroup $L \leqslant \mathbb{C}$. More precisely, L is a free abelian subgroup of rank 2. (Free abelian groups are called *lattices*, but this name should not be confused with the notion of a lattice that is commonly studied in algebraic combinatorics.) An elliptic curve acquires the structure of a quotient manifold. Furthermore, the cosets $z + L$, where z is in the 'fundamental mesh of L', are in bijection with the elements of a suitably parametrized two dimensional torus.

Example 5.0.5. Let L denote the following discrete subgroup of $\mathbb{C}$:

$$L = \{2\pi(a + ib) : \ a, b \in \mathbb{Z}\}.$$

Then the fundamental mesh of L in $\mathbb{C}$ is the rectangle $Mesh(L) := \{(t, s) \in \mathbb{R}^2 : 0 \leq t, s < 2\pi\}$. Every coset $z + L$ in $\mathbb{C}/L$ is uniquely represented by a point (t, s) in

$Mesh(L)$. Let φ denote the following smooth function:

$$\varphi : Mesh(L) \longrightarrow \mathbb{R}^3$$
$$(t, s) \longmapsto (2 + \cos(t)) \cos(s + \pi/2), (2 + \cos(t)) \sin(s + \pi/2), \sin(t))$$

The image of φ is a two dimensional (embedded) torus in $\mathbb{R}^3$ as shown in Figure 5.1, which we borrowed from the website: `http://pgfplots.net/tikz/examples/torus/`. We thank *cmhughes* for making this figure publicly available.

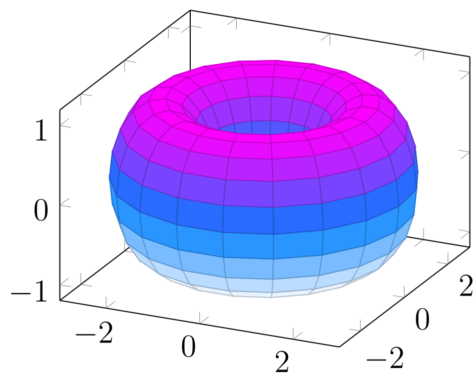


Figure 5.1: A 2-torus in $\mathbb{R}^3$.

As we hinted in the previous paragraph, from the abstract group theory point of view, every elliptic curve is a 2-torus. In other words, if X is an elliptic curve, then as an abstract group X is isomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. A proof of this fact can be found in [7, Theorem 4.16]. The 2-torus has a distinguished place in low-dimensional topology also. For example, the following manifold is diffeomorphic to $\mathbb{T}^2$,

$$Z := \{(x, y, z, w) \in \mathbb{S}^3 : \ x^2 + y^2 = \frac{1}{2} \ \text{ and } \ z^2 + w^2 = \frac{1}{2}\}.$$

It can be shown without much difficuiculty that Z separates the 3-sphere into two solid 2-tori.

Theorem 5.0.6. *Every 2-torus is a Riemann surface.*

Proof. Given the quotient map $p : \mathbb{C} \to \mathbb{C}/L$ with $z \mapsto [z] = z + L$, for each fixed z there is a small disc D_z centered at z such that no two points of D_z can differ by an element in L. It readily follows the rescrition of quotient map on to D_z is open, continuous and bijective, thus a homeomorphism. Now for each $[z] \in \mathbb{C}/L$, let $\phi_{[z]} : p(D_z) \to D_z$ be the inverse of p on D_z. It is readily checked $\phi_{[z]}$ is a chart. For $\phi_{[z_1]}$ and ϕ_{z_2}, their transition map $T := \phi_{[z_2]} \circ \phi_{[z_1]}^{-1}$ is a translation. Thus the transition map is holomorphic. □

Recall that in complex analysis the notation ∞ is used to denote 'infinity', which is an element out of the complex plane $\mathbb{C}$. By one-point compactification, the set $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ can be visualized as a sphere given by $\mathbb{S}^2 \subset \mathbb{R}^3$.

Remark 5.0.7. The space $\mathbb{P}$ is an example of a projective space. Let k be a field. The n-dimensional projective space over k, denoted $\mathbb{P}^n(k)$, is the algebraic scheme whose points are given by the one dimensional vector subspaces in k^{n+1}. In particular, $\mathbb{P}$ is equal to $\mathbb{P}^1(\mathbb{C})$.

The following observation justifies why $\mathbb{P}$ is called a Riemann sphere.

Example 5.0.8. A Riemann sphere $\mathbb{P}$ is a Riemann surface. The charts are defined on $U_1 = \mathbb{C}$ given by $\phi_1(z) = z$ and $U_2 = \mathbb{P} - \{0\}$ given by $\phi_2(z) = \frac{1}{z}$, respectively. The transition map $\phi_2 \circ \phi_1^{-1} : \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ with $z \mapsto \frac{1}{z}$ is holomorphic.

Before we introduce the notion of a holomorphic map between Riemann surfaces, let us recall firstly inverse function theorem in complex analysis.

Theorem 5.0.9. *(inverse function theorem) Let $f : U \to \mathbb{C}$ be a holomorphic map. For $z_0 \in U$, if $f'(z_0) \neq 0$, then f is locally invertible and its inverse is holomorphic.*

Proof. We assume that the reader is familiar with the inverse function theorem of real variables. Since it is a holomorphic map, $f(x,y) = u(x,y) + iv(x,y)$ can be viewed as a smooth function from $\mathbb{R}^2$ to $\mathbb{R}^2$. The jacobian matrix of f is given by $J_f(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. By Cauchy-Riemann equations, the determinant $det(J_f) = (u_x)^2 + (u_y)^2 = f'(z_0)^2 \neq 0$ at z_0. So $f : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible in real sense . For the inverse $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$, its Jacobian matrix $J_{f^{-1}}$ at $f(z_0)$ is of the form of $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then by Cauchy-Riemann equations, f^{-1} is holomorphic. □

Now we show that in complex analysis, any non-constant holomorphic map can be modified locally into a power map. This observation will help us understand nonconstant holomorphic maps between general Riemann surfaces.

Theorem 5.0.10. *If $w = f(z)$ is a non-constant holomorphic map at z_0, with k the smallest integer such that the k-th derivative $f^{(k)}(z_0)$ is nonzero, then there exist new variable $\tilde{z}$ and $\tilde{w}$, in terms of z and w respectively, such that the power function $\tilde{w} = \tilde{z}^k$ holds.*

Proof. f can be expanded near z_0 into Taylor series $f(z) = f(z_0) + \sum_k^\infty a_n(z-z_0)^n$. Let $h(z) = \sum_k^\infty a_n(z-z_0)^{(n-k)}$, then $h(z)$ is holomorphic at z_0 with $h(z_0) \neq 0$. From complex analysis, we know $h(z)$ induces a branch of k-th root $\sqrt[k]{h(z)}$ locally at z_0. Now for $\tilde{z} = (z-z_0)\sqrt[k]{h(z)}$, $\tilde{z}'(z_0)$ is nonzero. Thus this change of variable is locally invertible at z_0 by inverse function theorem. Another change of variable is given by a translation $\tilde{w} = w - f(z_0)$. By computation we have $\tilde{w} = \tilde{z}^k$. □

Now we generalize the notion of holomorphic maps into Riemann surfaces.

Definition 5.0.11. A map $f : X \to Y$ between Riemann surfaces is *holomorphic at* x if there exist charts ϕ defined on $U \subset X$ containing x and ψ defined on $V \subset Y$

containing $f(x)$ with the composition $\psi \circ f \circ \phi^{-1}$ is holomorphic in the sense of complex analysis.

Definition 5.0.12. A holomorphic map that is bijective is called an *isomorphism.* An isomorphism from a Riemann surface to itself is called an *automorphism.* The set of automorphisms of a given Riemann surface X forms a group under function composition. The automorphism group of X will be denoted by $Aut(X)$.

We will describe the automorphism group of a simply connected open subset of $\mathbb{C}$. To this end, let us mention a result that was first discovered by Riemann.

Theorem 5.0.13. *([8, Section 6.1])(Riemann mapping theorem) If U is a non-empty open proper subset of $\mathbb{C}$ that is simply connected, then there is an isomorphism $f : U \to \mathbb{D}$.*

Example 5.0.14. Recall that $\mathbb{H}$ denotes the upper half plane in $\mathbb{C}$. We can verify that $\mathbb{H}$ and $\mathbb{D}$ are isomorphic via the Möbius transformatios $z \mapsto \frac{z-i}{z+i}$.

Example 5.0.15. Below are examples of groups of automorphisms.

(1) $Aut(\mathbb{C}) = \{f(z) = az + b : \ a, b \in \mathbb{C}\}$

(2) $Aut(\mathbb{P}) = \{f(z) = \frac{az+b}{cz+d} : \ a, b, c, d \in \mathbb{C}, ad - bc \neq 0\}$

(3) $Aut(\mathbb{H}) = \{f(z) = \frac{az+b}{cz+d} : \ a, b, c, d \in \mathbb{R}, ad - bc \ = 1\}$

These groups play a key role in many branches of mathematics. For example, $Aut(\mathbb{P})$ is the familiar group of Möbius transformatios. Besides, $Aut(\mathbb{H})$ appears frequently in any text of modular forms. From [9, Chapter 2] and [9, Chapter 5], we can find many details for $Aut(\mathbb{P})$ and $Aut(\mathbb{H})$, respectively.

A holomorphic map between Riemann surfaces is locally a holomorphic function in the sense of complex analysis. Then by applying Theorem 5.0.10 to a suitable transition map, we immediately have the following generalization:

Theorem 5.0.16. *(Local normal form) For $f : X \to Y$ a non-constant holomorphic maps between Riemann surfaces X and Y. Let $a \in X$ and $b = f(a) \in Y$. Then there is a nonzero $k \in \mathbb{N}$ and charts $\phi_1 : U_1 \to V_1$ centered at p and $\phi_2 : U_2 \to V_2$ centered at $f(a)$ such that the composition $\phi_2 \circ f \circ \phi_1^{-1}(\tilde{z}) = \tilde{z}^k$.*

Remark 5.0.17. Our discussion about the local normal form reveal the essence of a non-constant holomorphic map between Riemann surface.can be viewed locally as a power map $\tilde{f} : \mathbb{D} \to \mathbb{D}, \tilde{z} \mapsto \tilde{w} = \tilde{z}^k$. For $\tilde{w} = 0$, since $\tilde{f}'(0) = 0$, the power map is not invertible. For each $\tilde{w} \neq 0$ with $\tilde{f}^{-1}(\tilde{w}) = \{\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_k\}$, the power map is locally invertible at $\tilde{w}$ and each $\tilde{z}_i$ induces one branch of the inverse of the power map. This observation also justifies the uniqueness of the integer k for a fixed $a \in X$.

Definition 5.0.18. Given a non-constant holomorphic map $f : X \to Y$, the *multiplicity of f at x*, denoted by $mult_x(f)$, is the unique positive integer k_x given by the local normal form at x. x is said to be a ramification point if $mult_x(f) \geq 2$ and in this case $f(x)$ is a branch point. Otherwise f is said to be it unramified at x. Ramification locus R_f is the set of ramification points and branch locus B_f is the set of branch points.

Theorem 5.0.19. *If $f : X \to Y$ is a non-constant holomorphic map, then its ramification and branch loci have discrete topologies.*

Proof. It suffices to show that the ramification locus has discrete topology. For $r \in R$, we know that, by the local normal form, there exists a neighborhood U of r, such that $mult_x(f) = 1$ for $x \in U - \{r\}$. □

The ramification locus R is a discrete closed subset. Moreover, if X is compact, then R is also compact, thus R is a finite set since any open covering of R admits a finite subcovering. It follows for a holomorphic maps between compact Riemann surfaces, its ramification and branch loci are both finite set. In this case, the following theorem of counting justifies the concept of the *degree* of a holomorphic map.

Theorem 5.0.20. *Let $f : X \to Y$ be holomorphic maps between compact Riemann surfaces. The number*

$$\textstyle\sum_{x \in f^{-1}(y)} mult_x(f)$$

is independent of y and is defined to the degree of the holomorphic map f.

Proof. For a fixed $y' \in Y - B$ with $f^{-1}(y')$, let $Y' = \{y \in Y - B : \ |f^{-1}(y)| = d\}$. Then Y' is nonempty. It remains to show Y' is closed and open. For any $y \in Y'$ with $f^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$. For each x_i there exists a chart defined on U_i containing x_i, such that the local norm form is an identity map. These U_i's can be small enough such that they are disjoint. Then $V = f(\cap U_i)$ will be an open set containing y, thus Y' is open. To prove Y' is closed, we can apply the same argument and show that the complement $Y - Y'$ is open. Next, if $b \in B$ with $f^{-1}(b) = \{x_1, x_2, \ldots, x_n\}$, there exists an open set V containing b homeomorphic to the open punctured disk $\mathbb{D}^\circ = \mathbb{D} - \{0\}$ such that $f^{-1}(V)$ is a finite disjoint union $V_1, \ldots, V_n$, where $n < d$. Each component V_i contains a unique x_i. At each x_i, f is locally like a power map $\mathbb{D} \to \mathbb{D}$ with $z \mapsto z^{mult_{x_i}(f)}$. By summing up, $\sum mult_{x_i}(f) = |f^{-1}(y)|$, for $y \in Y - B$ close to b. $\square$

Chapter 6

An explanation of the analogy

By analogy, we have presented what we may call *main theorem of topological Galois theory of Riemann surfaces.* Roughly speaking, a covering map is like a field extension, while the group of covering transformations is like a group of automorphism. However, a subtle explanation of the story is always more satisfactory rather than a facile analogy. To achieve this, at least we should look for a field extension to begin with.

Definition 6.0.1. Given a compact Riemann surface X and a finite set $S \subset X$, f is *meromorphic* on X if $f : X - S \to \mathbb{C}$ is a holomorphic map, and moreover for all charts $\phi : U \to \mathbb{C}$ the composite $f \circ \phi^{-1}$ is meromorphic in the sense of complex analysis.

There is another equivalent way to define a meromorphic map: a meromorphic map on a compact Riemann surface X is a holomorphic map from X to $\mathbb{P}$.

Theorem 6.0.2. *Given X a compact Riemann surface, the set $\mathcal{M}(X)$ of all meromorphic maps defined on X is a field.*

Proof. It is readily to see that $\mathcal{M}(X)$ is a ring under function addition and multiplication. Let us show that for any nonzero f, there exist an inverse $\frac{1}{f}$. It suffices to show that there are only finite zeros of f. Assuming a sequence $\{x_i\}$ converging to x

such that $f(x_i) = 0$, then by identity principle of complex analysis, there is a open set containing x such that $f \equiv 0$ on it, contradicting our assumption. □

Example 6.0.3. We are already familiar with meromorphic maps on Riemann sphere $\mathbb{P}$. Given $p(z)$ and $q(z)$ two polynomials with complex coefficients that are co-prime, let $f(z) = \frac{p(z)}{q(z)}$. If $z \in \mathbb{C}$ and $q(z) \neq 0$, $f(z)$ will be a complex number. If otherwise $z = \infty$ or $q(z) = 0$, $f(z)$ is defined to be the limit of $f(z')$ as z' approaches z. In this setting, the quotient $f(z)$ is a meromorphic map.

A meromorphic function in the sense of complex analysis is equivalent to a rational function in the sense of algebraic geometry.

Theorem 6.0.4. *([9, Theorem 1.4]) A map $f : \mathbb{P} \to \mathbb{P}$ is meromorphic if and only if it is rational.*

Theorem 6.0.5. *([4, Theorem 3.3.3])(Riemann's Existence Theorem) Let X be a compact Riemann surface, and let $x_1, x_2, \ldots, x_n$ be points from X. Let $a_1, a_2, \ldots, a_n \in \mathbb{C}$. Then there exists a meromorphic function f on X such that $f(x_i) = a_i$ $(1 \leq i \leq n)$ and f is holomorphic in a neighborhood around x_i.*

We will not present the proof of this theorem here. Nevertheless, let us point out that for the Riemann sphere $X = \mathbb{P}$, f is easily constructed by using Lagrange interpolation formula.

By Theorem 6.0.2, we know that every compact Riemann surface comes with a field that consists of meromorphic functions on the surface. We will explain the connection between these fields of meromorphic functions and the covering maps of Riemann surfaces.

Definition 6.0.6. A continuous map $f : X \to Y$ between two compact topological surfaces is called a *ramified covering* if there is $B = \{b_1, b_2, \ldots, b_n\} \subset Y$ such that the restriction $f^\circ : X - f^{-1}(B) \to Y - B$ is a covering map with finite degree.

Theorem 6.0.7. *If $f : X \to Y$ is a holomorphic map between compact Riemann surfaces, then f is a ramified covering.*

Proof. We already proved this result in the proof of Theorem 5.0.20. □

Theorem 6.0.8. *Let Y be a compact Riemann surface and X° be a topological space. Let B be a finite subset $B = \{b_1, b_2, \ldots, b_n\} \subset Y$. If $f^\circ : X^\circ \to Y - B$ is a covering map of finite degree d, then there exist a compact Riemann surface X containing X° such that f° extends to a holomorphic map $f : X \to Y$.*

Proof. We will prove our assertion in two steps.

(1) Given $x \in X^\circ$, then for $y = f^\circ(x) \in Y - B$ there is a chart $\phi_y : U_y \to \mathbb{D}$ centered at y. We can assume U_y is evenly covered by f°. Now $(f^\circ)^{-1}(U_y) = \cup_{i \in I} U_i$ for $1 \leq i \leq d$. Note that x is contained in a unique U_i and the composite $\phi_y \circ f^\circ$ will be a chart defined on U_i. In this manner, we can define a chart for each $x \in X^\circ$. The transition map between two charts is a composite of holomorphic maps, which is also holomorphic (actually the transition map is locally like the identity map on $\mathbb{D}$). If there is another complex structure on X satisfying our assumption, then the local normal form at any $x \in X$ should also be given by a map locally like the identity map, since otherwise ramification points will be introduced into X°, contradicting the assumption that f° is topological a covering map. This observation justifies the uniqueness of complex structure of X°.

(2) Now we have a holomorphic map $f^\circ : X^\circ \to Y - B$, which is topological a covering map. It remains to extend it to a holomorphic map $f : X \to Y$. Given a fixed $b \in B$, there is a chart $\phi_b : U_b \to \mathbb{D}$ centered at b. We can further assume that U_b contains no other $b' \in B$. Let U° be the punctured open set $U - \{b\}$. Then by definition the restricted $f^\circ : (f^\circ)^{-1}(U^\circ) \to U^\circ$ is a covering

map of degree d. Let $U_1^\circ, \ldots, U_n^\circ$ be the connected component of $(f^\circ)^{-1}(U^\circ)$, then $n < d$ and each component U_i° is itself a covering space of U°. Thus each U_i° is homeomorphic to the punctured open disc $\mathbb{D}^\circ$. It can be proved that $\mathbb{D}^\circ$ and the unit circle $\mathbb{S}^1$ are of the same homotopy type, thus its fundamental group is also isomorphic to $\mathbb{Z}$. Moreover, the classification of covering spaces of $\mathbb{D}^\circ$ is the same as that of $\mathbb{S}^1$, which is already discussed in Example 2.2.17. More precisely, $f_m : \mathbb{D}^\circ \to \mathbb{D}^\circ$ given by $z \mapsto z^m$ for each $m \in \mathbb{N}$ exhaust all its covering spaces up to equivalence. It follows that the covering map $U_i^\circ \to \mathbb{D}^\circ$ is equivalent to $\psi^\circ : \mathbb{D}^\circ \to \mathbb{D}^\circ$ given by $z \mapsto z^{k_i}$, where k_i is given by the local geometry on each U_i°. For each U_i°, we add an abstract point x_i to X° to fill in the hole of $\mathbb{D}^\circ$. In this manner, we have defined the chart near x_i on $U_i^\circ \cup \{x_i\}$ and it can be shown that the local normal from is given by $z \mapsto z^{k_i}$. The resulting space X is c0mpact since every open covering of it admits a finite subcollection covering X.

□

Thanks to Theorem 6.0.8, the notions of 'ramified covering' and 'holomorphic map' on compact Riemann surfaces can be used interchangeably. This correspondence is one of the main motivations for our thesis. To properly explain the algebraic nature of this topological/complex analytic correspondence, we introduce an automorphism group:

Definition 6.0.9. Let $f : X \to Y$ be a non-constant holomorphic map. We define the set $Aut(X/Y)$ of automorphism of $h \in Aut(X)$ such that $f \circ h = f$.

It is readily to see $Aut(X/Y)$ is a group under function composition. Also associated with $f : X \to Y$, there is a homomorphism $\tilde{f} : \mathcal{M}(Y) \to \mathcal{M}(X)$ defined by $m \mapsto m \circ f$.

Lemma 6.0.10. *Let $f : X \to Y$ be a non-constant holomorphic map of degree d between compact Riemann surfaces. Every meromorphic function $m \in \mathcal{M}(X)$ satisfies a polynomial of degree d over $\mathcal{M}(Y)$.*

Proof. Given B the branch locus, if $y \notin B$, then by definition of a covering map there is a neighborhood of y such that $f^{-1}(U_y)$ is a disjoint union of V_i's for $1 \leq i \leq d$. Let s_i be the section of f mapping U homeomorphic to V_i. Thus the composite $m_i := m \circ s_i$ is a meromorphic map on U_y. Now we set

$$M = \prod (t - m_i) = t^d + a_{n-1}t^{d-1} + \cdots + a_0.$$

Note that each a_i is meromorphic on U_y, thus M is also meromorphic on U_y. Now assume there is another $y' \notin B$ with corresponding M' meromorphic on U' arising in such a manner. If $U'_{y'} \cap U_y \neq \emptyset$, then M' and M have to be identical on $U' \cap U$. With this observation we can extend the meromorphic map M to all of $Y - B$. Next let us deal with the branch locus. If $y \in B$, there is a chart $m_y : U_y \to \mathbb{C}$ centered at y. The composite $m_y \circ f$ is holomorphic at $x \in f^{-1}(y)$ with $m_y \circ f(x) = 0$. By definition m is meromorphic on X, its poles are of finite orders and we can find sufficiently large k such that $(m_y \circ f)^k m$ is holomorphic at $x \in f^{-1}(y)$ and bounded on a punctured neighborhood of each x. By computation $((m_y \circ f)^k m) \circ s_i = m_y^k m_i$, which is bounded on the punctured neighborhood $U_y - \{y\}$. It follows for each a_i, the function $m_y^{kd} a_i$ is also bounded on the punctured neighborhood $U_y - \{y\}$. Thus y is a removable singularity and we can extend $m_y^{kd} a_i$ to the whole U_y. It follows $a_i \in \mathcal{M}(Y)$ and $M \in \mathcal{M}(Y)[t]$. It remains to show that m satisfies the polynomial M. Given y and U_y not containing branch points, then for the polynomial $\tilde{f}(M) = t^d + (a_{d-1} \circ f)t^{d-1} + \cdots + (a_0 \circ f)$, we have $\tilde{f}(M)(m) \circ s_i = (\tilde{f}(M) \circ s_i)(m \circ s_i) = 0$ on U_y, thus $\tilde{f}(M)(m) = 0$ on V_i. This implies for every $y \notin B$, $\tilde{f}(M)(m) = 0$. We can further extend this claim to include branch points. Thus m satisfies the polynomial

M on all of Y. □

Theorem 6.0.11. *In the above notation, the field extension* $[\mathcal{M}(X) : \tilde{f}(\mathcal{M}(Y))]$ *is of degree* d*.*

Proof. Given $y \notin B$ and $f^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$, by Riemann's Existence Theorem 6.0.5, there is a $m \in \mathcal{M}(X)$ which is holomorphic at each x_i with respective $m(x_i)$ attained distinctly. By Lemma 6.0.10, m satisfies an irreducible polynomial $m^n + \tilde{f}(a_{n-1})m^{n-1} + \cdots + \tilde{f}(a_0)$ over $\mathcal{M}(Y)$ with $n \leq d$. If each a_i is holomorphic at y, then $t^n + \cdots + a_0(y)$ will have $m(x_i)$'s as roots. Thus it has d distinct roots and it follows $n = d$. If some a_i has a pole at y, we can still make adjustment and pick another y' sufficiently close to y such that $y' \notin B$. In this setting, no $x' \in f^{-1}(y')$ will induce a pole and we can apply the previous argument. By the primitive element theorem in field, every finite extension is simple in characteristic 0. Also note that the field of meromorphic functions is an extension of the complex number field $\mathbb{C}$. Thus another $g \in \mathcal{M}(X)$, we have $\mathcal{M}(Y)(m, g) = \mathcal{M}(Y)(h)$ for some $h \in \mathcal{M}(X)$. Since h is also at most of degree d over $\mathcal{M}(Y)$, we must have the equality $\mathcal{M}(Y)(h) = \mathcal{M}(Y)(m)$. It follows $g \in \mathcal{M}(Y)(m)$. That is to say m already generates $\mathcal{M}(X)$ and $\mathcal{M}(X) \cong \mathcal{M}(Y)(m)$. □

Theorem 6.0.11 shows for a fixed compact Riemann surface Y together with its field of meromorphic functions $\mathcal{M}(Y)$, we can consider a compact Riemann surfaces X together with a holomorphic map $f : X \to Y$ and the respective field $\mathcal{M}(X)$ which is a field extension of $\mathcal{M}(Y)$. We want to point out that such a process is reversible.

Theorem 6.0.12. *([4, Proposition 3.3.8]) Given a compact Riemann surface* Y *and a finite field extension* L *of* $\mathcal{M}(Y)$*, there is a compact Riemann surface* X_L *associated with a holomorphic map onto* Y*, such that* $\mathcal{M}(X_L) \cong L$ *as a finite extension over* $\mathcal{M}(Y)$*.*

$$\begin{array}{ccc} X_L & & L \cong M(X_L) \\ \Big\downarrow f & & \Big\downarrow M(f) \\ Y & & M(Y) \end{array}$$

In the the above result, if d denotes the degree of the covering map $f : X_L \to Y$, then d is also the degree of the corresponding field extension $\mathcal{M}(X)/\mathcal{M}(Y)$. On one hand, we have the fact that $X \to Y$ is Galois if and only if $Aut(X/Y)$ contains d elements (given by Theorem 6.0.7 and Theorem 6.0.8 in this thesis). On the other hand, $\mathcal{M}(X)/\mathcal{M}(Y)$ is Galois if and only if the degree of the extension is the same as the index of $Aut(\mathcal{M}(X)/\mathcal{M}(Y))$ which is isomorphic with $Aut(X/Y)$ (given by the definition of a Galois extension in algebra). In conclusion, if f is a ramified Galois covering map, then the corresponding field extension is a Galois field extension. The proofs of these assertions can be found in [4, Section 3.3].

Nearing the end of this chapter, we will point out another important fact: The *Riemann Existence Theorem* says that a compact Riemann surface admits a non-constant meromorphic function. It follows from this theorem that if Y is a compact Riemann surface, then the field of meromorphic functions $\mathcal{M}(Y)$ is a finite field extension of $\mathbb{C}(t)$. More conceptually stated, there is an anti-equivalence between the category of connected Riemann surfaces with non-constant holomorphic maps as morphisms and the category of finite field extensions of $\mathbb{C}(t)$ with morphisms the $\mathbb{C}$-algebra homomorphisms.

Chapter 7

Further remarks

There are several main directions of active research that is directly related to the topics of this thesis. We will mention two of these directions. Throught this chapter, we are not going to present every notions in a rigorous manner since many of them are quite subtle and can not be explained throughly in a single chapter.

The classical results that we reviewed just now show that every finite group G can be realized as the fundamental group of a Riemann surface. Indeed, let $S := \{x_1, \ldots, x_n\}$ be n distinct points from $\mathbb{P}^1(\mathbb{C})$. Then the fundamental group of the complement of S in $\mathbb{P}^1(\mathbb{C})$ based at a point $x_0 \in \mathbb{P}^1(\mathbb{C}) - S$ has the following presentation:

$$\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x_0) = \langle \gamma_1, \ldots, \gamma_n | \gamma_1 \cdots \gamma_n = 1 \rangle, \tag{7.1}$$

where each generator γ_i can be represented by a closed path around the point x_i passing through x_0. It is not difficult to show that the right hand side of (7.1) is the free group F_{n-1} on $n-1$ generators. It follows that every finite group arises as a finite quotient of the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) - S, x_0)$. In particular, every finite group occurs as the finite Galois group of some finite Galois extension $L/\mathbb{C}(t)$. Equivalently, every finite group occurs as the automorphism group $Aut(Y/\mathbb{P}^1(\mathbb{C}))$ for some finite branched covering $Y \to \mathbb{P}^1(\mathbb{C})$. This result is a special case of a much more general statement, which is known as the *Abhyankar's conjecture for affine curves.* This conjecture is in fact a theorem. Its proof is a culmination of the works of Serre, Harbater, and Raynaud.

Theorem 7.0.1. *Let k be an algebraically closed field of characteristic $p > 0$, and let X be a smooth projective curve of genus g defined over k. Let B be a set of points of X such that $|B| = r$, and let U denote the complement of B in X. Then a finite group G is the Galois group of an unramified cover of U if and only if $G/p(G)$ has a generating set of size at most $2g + r - 1$, where $p(G)$ is the (normal) subgroup of G that is generated by its p-subgroups.*

For a detailed explanation of the proof, see [10, Section 3]. Also in this article, the following open problem is discussed:

Conjecture 7.0.2. *(Abhyankar's affine arithmetical conjecture). A finite group G occurs as the Galois group of an unramified cover of the affine line over $\mathbb{F}_q$ if and only if $G/p(G)$ is cyclic.*

The connection between algebraic curves and Riemann surfaces has a far reaching generalization, which we will briefly explain. Roughly speaking, the aforementioned existence theorem of Riemann says that compact Riemann surfaces are algebraic. More precisely, every compact Riemann surface can be recognized as a smooth projective algebraic curve defined over $\mathbb{C}$, and vice versa. In higher dimensions such a perfect correspondence does not hold. Nevertheless there are still strong organic ties between (complex) algebraic varieties and complex manifolds. Indeed, for every scheme X of finite type over $\mathbb{C}$, there is an *associated complex analytic space* X_h. (For the definition of an associated analytic space as well as for the unexplained algebraic geometry terminology, we refer the reader to [7, Appendix B].) In fact, the association, $X \mapsto X_h$ is functorial. In dimension one, $X \mapsto X_h$ is an isomorphism of categories between the smooth projective curves over $\mathbb{C}$ and the category of compact Riemann surfaces. Note that every Riemann surface is a complex manifold of dimension one. A generalization of the Riemann's existence theorem in higher dimensions is obtained by Chow [11]: If X' is a compact analytic subspace of the complex manifold $\mathbb{P}^n(\mathbb{C})$, then there is a projective subscheme $X \subseteq \mathbb{P}^n(\mathbb{C})$ with $X_h = X'$. Another result, which is due to Chow and Kodaira [12] states that a 2-dimensional compact complex manifold with two algebraically independent meromorphic functions comes from a projective algebraic surface. Note that, in general, a compact complex manifold need not have a non-constant meromorphic function.

Let us mention another generalization of the Riemann's existence theorem which is first obtained by Grauert and Remmert (and further generalized by Grothendieck): Let X be a normal scheme of finite type over $\mathbb{C}$, and let $\mathfrak{X}'$ be a normal complex analytic space together with a finite morphism $\mathfrak{f} : \mathfrak{X}' \to X_h$. Then there is a unique normal scheme X' and a finite morphism $g : X' \to X$ such that $X'_h \cong \mathfrak{X}'$ and $g_h = \mathfrak{f}$. A corollary of this theorem will help us to finish our presentation by tying up the loose ends: For an scheme X defined over $\mathbb{C}$, let $\pi_1^{\text{alg}}(X)$ denote the *algebraic fundamental group* defined as the inverse limit of the Galois groups of finite étale covers of X. Then, it is shown by Grothendieck that, for a connected algebraic scheme X of finite type over $\mathbb{C}$, $\pi_1^{\text{alg}}(X)$ is isomorphic to the completion $\widehat{\pi_1^{\text{top}}(X_h)}$ of the usual fundamental group of X_h with respect to subgroups of finite index, [13, Exposé XII, Corollaire 5.2]. The following open question is asked by Serre (see [4, Remark 5.7.5]):

Question 7.0.3. *(Serre). Is there a (smooth) projective variety X defined over $\mathbb{C}$ such that $\pi_1^{alg}(X) = \{1\}$ but $\pi_1^{top}(X_h) \neq 1$?*

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Biography

Dejun Zhang is a master student majoring in mathematics at Tulane university. His research interest mainly focuses on covering theory and Riemann surfaces.