Diagonal Orbits in Double Flag Varieties

AN ABSTRACT
SUBMITTED ON THE FORTH DAY OF MAY, 2020
TO THE DEPARTMENT OF MATHEMATICS
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TULANE UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
BY

DocuSigned by:
Tien Minh Le
97AD2D4B9EAE4D7...
Tiến Minh Lê

APPROVED:

DocuSigned by:
Mahir Bilen Can
BB2C84D915194D8...
Mahir Bilen Can, Ph.D.
Director

DocuSigned by:
Tai Huy Ha
C1341E6963A2498...
Tài Huy Hà, Ph.D.

DocuSigned by:
Michael O. Joyce
522411935120499...
Michael O. Joyce, Ph.D.

DocuSigned by:
Rafal Komendarczyk
81F4DAFDA994481...
Rafal Komendarczyk, Ph.D.

DocuSigned by:
Robert L. Herbert
3895B78ECB56419...
Robert L. Herbert, Ph.D.

Abstract

Let G be a connected reductive complex algebraic group. We study the inclusion posets of diagonal G-orbit closures in a product of two partial flag varieties. In this dissertation, we show some results for $G = SL_n$ and $G = SO_{2n}$. If the diagonal action is of complexity zero, then the poset is a graded lattice. If the diagonal action is of complexity one, then the poset is isomorphic to one of the finite number of posets that we determine explicitly.

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I hereby declare that the materials and results in this dissertation, unless accompanied by specific references, are original and have not been published elsewhere.

New Orleans, May 4th, 2020

Tiến Minh Lê

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Finally, I dedicate this dissertation to my wife, my son, my daughter, and my parents, for their endless love and support.

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Chapter 1

Introduction

Let W be a Coxeter group which is generated by a set S. Let I and J be subsets of S. Let W_I and W_J be subgroups generated by I and J; then, W_I and W_J are parabolic subgroups of W.

Given any $w \in W$, we define a double coset as:

$$W_I w W_J := \{w_i w w_j : \ w_i \in W_I, w_j \in W_J\}.$$

The original problem that we have been working on is finding the conditions of parabolic subgroups so that the double flag variety will carry a lattice structure under Bruhat order. In order to achieve it, we have tried many different methods, including, but not limited to, studying the W-orbits of the double flag varieties.

Let G be a connected reductive complex algebraic group, and let B be a Borel subgroup in G. Let X be an irreducible complex algebraic G-variety. The complexity of the action $G : X$, denoted by $c_G(X)$, is defined as the codimension of a general B-orbit in X. This notion plays an important role in the study of the embedding of homogeneous spaces. Among all homogeneous spaces of G, the ones with complexity of at most one form the most remarkable subclass, see the seminal article of Panyushev [**?**], as well as the book of Timashev [**?**], dedicated to the study of such embeddings.

An enduring problem in representation theory is how to decompose the tensor products of irreducible representations of G. Let λ_I and λ_J be two dominant weights corresponding to the irreducible representations V_I, V_J of G, and let P_I, P_J denote the corresponding parabolic subgroups that arise as the stabilizer subgroups of highest weight vectors $v_I \in V_I$, $v_J \in V_J$. There is a close relationship between the decomposition of $V_I \otimes V_J$ as a G-module and the polynomial invariants of the diagonal action of G on the double flag variety $X := G/P_I \times G/P_J$. By using the coordinate ring of the affine cone over the double flag varieties, in [?], Littelmann obtained a precise description of the decomposition of the tensor products of two fundamental representations of a simple group. This progress motivated the works [?], [?], and [?]. In [?], Stembridge classified all multiplicity-free tensor products of irreducible representations of semisimple complex Lie groups, hence, he classified the parabolic subgroups P_I, P_J such that $c_G(G/P_I \times G/P_J) = 0$. Finally, in [?], Ponomareva classified all double flag varieties of complexity one. In the same paper, Ponomareva showed (by examples) how one could use the results of Brion [?] and Timashev [?] for decomposing the spaces of global sections of line bundles on the double flag varieties of complexity ≤ 1.

In [?], for $G = SL_n$ (Type A_n), Can shows that if $c_G(X) = 0$, then the inclusion poset of G-orbit closures in X is a particular kind of lattice. In general, such a poset (inclusion order on the closures of diagonal G-orbits in X) can be very complicated. It may not even be a graded poset. However, it always has a unique minimal element and a unique maximal element. In [?], we show that if $c_G(X) = 1$, then the inclusion poset of G-orbit closures in X is a finite poset. We also classify them to 28 different non-isomorphic posets.

In this dissertation, we analyze the case of $G = SO_{2n}$ (Type D_n). It turns out that if $c_G(X) = 0$, then the inclusion poset of G-orbit closures in X is a lattice. If

$c_G(X) = 1$, then the inclusion poset of G-orbit closures in X is a lattice and is a finite poset.

Chapter 2

Type A_n

2.1 Preliminaries

Let G be complex a semisimple algebraic group, let B be a Borel subgroup in G, and let T be a maximal torus of G that is contained in B. We denote by Φ the root system corresponding to the pair (G, T), and we denote by Δ the set of simple roots determined by B. A parabolic subgroup P of G is said to be *standard with respect to* B if $B \subseteq P$. In this case, P is uniquely determined by a subset $I \subseteq \Delta$.

Let $N_G(T)$ denote the normalizer subgroup of T in G. The *Weyl group* $W := N_G(T)/T$ of G is a Coxeter group, and we denote its Coxeter generating system corresponding to Δ by

$$R(\Delta) := \{s_\alpha \in W : \ \alpha \in \Delta\}.$$

The elements of $R(\Delta)$ are called the *simple reflections* relative to B. If the set of simple roots we are using is fixed, then we will denote $R(\Delta)$ by R to ease our notation.

We will interchangeably use the letters I and J to denote subsets of Δ and the corresponding subsets of simple reflections in $R(\Delta)$. The *length* of an element $w \in W$, denoted by $\ell(w)$, is the minimal number of simple reflections $s_{\alpha_i} \in R(\Delta)$ that is needed for the equality $w = s_{\alpha_1} \cdots s_{\alpha_k}$ hold true. In this case, the product $s_{\alpha_1} \cdots s_{\alpha_k}$ is called a *reduced expression* for w.

The *Bruhat-Chevalley order* on W is be defined by declaring $v \leq w$ $(w, v \in W)$ if a reduced expression of v is obtained from a reduced expression $s_{\alpha_1} \cdots s_{\alpha_k} = w$ by deleting some of the simple reflections s_{α_i} in w. More geometrically, the Bruhat-Chevalley order is given by $v \leq w \iff B\dot{v}B/B \subseteq \overline{B\dot{w}B/B}$. Here, $\dot{v}$ and $\dot{w}$ are any representatives of v and w in $N_G(T)$, respectively. The sets $B\dot{v}B/B, B\dot{w}B/B$ denote the B-orbits of $\dot{v}, \dot{w}$ in G/B, and the bar on $B\dot{w}B/B$ indicates the Zariski closure. In this notation, $\ell(w)$ is equal to the dimension of the orbit $B\dot{w}B/B$.

Let G be a classical matrix group with entries in $\mathbb{C}$, and let B denote its Borel subgroup consisting of upper triangular matrices. The parabolic subgroups of G containing B have block-triangular structure, and they are determined by the sizes of the diagonal blocks. Following Ponomareva's notation from [**?**], if P is a parabolic subgroup containing B, then we will denote by $Bl(P)$ the sequence $(p_1, \dots, p_r)$, where p_i denotes the size of the i-th block in P_I. For example, if P is the Borel subgroup of upper triangular matrices in SL_n, then each diagonal block of P is a 1×1 matrix, therefore, $Bl(P_I)$ is the sequence $(1, 1, \dots, 1)$ with n entries.

Let consider the cases where $G = SL_n$. We take B as the Borel subgroup of upper triangular matrices, and we take T as the maximal torus of diagonal matrices in B. The Weyl group W of SL_n is denoted by S_n, which is isomorphic to the symmetric group of permutations of $\{1, \dots, n\}$. The elements of the set of simple roots relative to B, that is $\Delta_{n-1} := \{\alpha_1, \dots, \alpha_{n-1}\}$, is ordered so that the i-th simple reflection s_{α_i} $(1 \leq i \leq n-1)$ is the simple transposition $s_i \in S_n$ that interchanges i and $i+1$. Thus we set

$$R_{n-1} := R(\Delta_{n-1}) = \{s_1, \dots, s_{n-1}\}.$$

If a permutation w in S_n is given in one-line notation $w = w_1 \dots w_n$, then its length is equal to the cardinality of the following set: $\{1 \leq i < j \leq n : \; w_i > w_j\}$.

An important fact that we repeatedly use in our paper is that SL_n is the stabilizer subgroup in SL_{n+1} of the standard basis vector e_{n+1} of $\mathbb{C}^{n+1}$, where SL_{n+1} acts by its defining representation. In particular, by using this identification of SL_n as a subgroup of SL_{n+1}, we will use the following containments without further mentioning in the sequel:

$$\Delta_{n-1} \hookrightarrow \Delta_n, \; R_{n-1} \hookrightarrow R_n, \quad \text{and } S_n \hookrightarrow S_{n+1} \text{ (as a subgroup).}$$

Let X_1 and X_2 be two G-varieties. Let $x_i \in X_i$ $(1 \leq i \leq 2)$ be two points in general positions. If $G_i \subset G$ denotes the stabilizer subgroup of x_i in G, then $\text{Stab}_G(x_1 \times x_2)$ coincides with the stabilizer in G_1 of a point in general position from G/G_2 (or, equivalently, with the stabilizer in G_2 of a point in general position from G/G_1), see [**?**]. As a special case, we consider the G-variety $X := G/P_1 \times G/P_2$. The proof of the following lemma is from [**?**, Lemma 2.1].

Lemma 2.1.1. *The poset of G-orbit closures in X is isomorphic to the poset of P_2-orbit closures in G/P_1.*

From now on we assume that P_1 and P_2 are standard parabolic subgroups with respect to B. If I and J are the subsets of $R := R(\Delta)$ (or, of Δ) that determine P_1 and P_2, respectively, then we will write P_I (resp. P_J) in place of P_1 (resp. P_2). The Weyl groups of P_I and P_J are denoted by W_I and W_J, respectively. The set of (W_I, W_J)-double cosets in W is denoted by $W_I\backslash W/W_J$.

It follows from Bruhat-Chevalley decomposition that the set of B-orbits in G/P_J are in a bijection with the set of minimal length left coset representatives for W/W_J, which we denote by W^J. The set of minimal length right coset representatives for $W_I\backslash W$ is denoted by ${}^IW$. In a similar way, $W_I\backslash W/W_J$ is in a bijection with the set of P_I-orbits in G/P_J, see [**?**, Section 21.16]. Let w be an element from W, and let $[w]$

denote the double coset $W_I w W_J$. Let

$$\pi : W \to W_I \backslash W / W_J$$

denote the canonical projection onto the set of (W_I, W_J)-double cosets. Then the preimage in W of every double coset in $W_I \backslash W / W_J$ is an interval with respect to Bruhat-Chevalley order, therefore, there is a unique maximal and a unique minimal element, see [?]. Moreover, if $[w]$ and $[w']$ are two elements from $W_I \backslash W / W_J$, w_1 and w_2 are their maximal length elements, respectively, then $[w] \leq [w']$ if and only if $w_1 \leq w_2$. (This can be seen directly by a geometric argument, but see [?] also.) The set of (W_I, W_J)-cosets a natural combinatorial partial ordering defined by

$$[w] \leq [w'] \iff w \leq w' \iff w_1 \leq w_2$$

where $[w], [w'] \in W_I \backslash W / W_J$ and w_1 and w_2 are the maximal elements, $w_1 \in [w]$ and $w_2 \in [w']$. There is a geometric interpretation of this partial: If O_1 and O_2 are two P_I-orbits in G/P_J with the corresponding double cosets $[w_1]$ and $[w_2]$, respectively, then $O_1 \subseteq \overline{O_2}$ if and only if $w_1 \leq w_2$. The bar on O_2 stands for the Zariski closure in G/P_J.

Let $[w]$ $(w \in W)$ be an element from $W_I \backslash W / W_J$ such that $\ell(w) \leq \ell(v)$ for all $v \in [w]$. Such minimal length double coset representatives are parametrized by the set ${}^I W \cap W^J$. From now on, we denote ${}^I W \cap W^J$ by $W^-_{I,J}$. Set $H = I \cap wJw^{-1}$. Then $uw \in W^J$ for $u \in W_I$ if and only if u is a minimal length coset representative for W_I/W_H. In particular, every element of $W_I w W_J$ has a unique expression of the form uwv with $u \in W_I$ is a minimal length coset representative of W_I/W_H, $v \in W_J$ and

$$\ell(uwv) = \ell(u) + \ell(w) + \ell(v). \tag{2.1}$$

Let s_i denote the i-th simple reflection, and let w be permutation in S_n. Let $w = w_1 \ldots w_n$ be the one-line notation for w. We call the number i a *right descent* for w if $w_i > w_{i+1}$. Equivalently, i is a right descent if $\ell(ws_i) < \ell(w)$. In a similar way, the integer i is said to be a *right ascent* if $w_i < w_{i+1}$, equivalently, $\ell(ws_i) > w$.

The following characterization of $W^-_{I,J}$ is useful for our purposes: For $w \in W$, the *right ascent set* is defined as

$$\mathrm{Asc}_R(w) = \{s \in R : \ \ell(ws) > \ell(w)\}.$$

The *right descent set*, $\mathrm{Des}_R(w)$ is the complement $R - \mathrm{Asc}_R(w)$. Similarly, the *left ascent set* of w is

$$\mathrm{Asc}_L(w) = \{s \in R : \ \ell(sw) > \ell(w)\} \ \ (= \mathrm{Asc}_R(w^{-1})).$$

Then

$$W^-_{I,J} = \{w \in W : \ I \subseteq \mathrm{Asc}_L(w) \ \text{ and } J \subseteq \mathrm{Asc}_R(w)\} \tag{2.2}$$

$$= \{w \in W : \ I^c \supseteq \mathrm{Des}_R(w^{-1}) \ \text{ and } J^c \supseteq \mathrm{Des}_R(w)\}. \tag{2.3}$$

Also, we will need the distinguished set of maximal length representatives for each double coset

$$W^+_{I,J} = \{w \in W : \ I \subseteq \mathrm{Des}_R(w^{-1}) \ \text{ and } J \subseteq \mathrm{Des}_R(w)\} \tag{2.4}$$

$$= \{w \in W : \ I^c \supseteq \mathrm{Asc}_R(w^{-1}) \ \text{ and } J^c \supseteq \mathrm{Asc}_R(w)\}. \tag{2.5}$$

For a proof of this characterization of $W^+_{I,J}$, see [**?**, Theorem 1.2(i)].

Remark 2.1.2. *The restriction of the Bruhat-Chevalley order to the sets* $W^-_{I,J}$ *and* $W^+_{I,J}$ *give isomorphic posets.*

Remark 2.1.3. *Let θ denote the involution of the set R_{n-1} that is defined by $s_i \mapsto s_{n-i}$ for $i \in \{1, \ldots, n-1\}$. Then $W^-_{I,J}$ (respectively $W^+_{I,J}$) and $W^-_{\theta(I),\theta(J)}$ (respectively $W^+_{\theta(I),\theta(J)}$) are isomorphic as posets.*

2.2 Some results for Type A

Let n and m be two positive integers such that $n < m$. We define W and V by

$$W := S_n = \langle s_1, \ldots, s_{n-1} \rangle \text{ and } V := S_m = \langle s_1, \ldots, s_{m-1} \rangle.$$

Clearly, W is a parabolic subgroup of V. Let I and J be two subsets from R_{n-1}, and let W_I and W_J denote the corresponding parabolic subgroups of W. In a similar way we denote by K and L two subsets from R_{m-1}, and we let V_K and V_L denote the corresponding parabolic subgroups in V.

Finally, let us denote by Γ the Dynkin diagram corresponding to R_m. Let D be a subset of R_m, and let $s_i \in D$ be a simple reflection. We will denote by Γ_D the subdiagram that is determined by the subset D, and we will denote by $\Gamma_D(s_i)$ the connected component containing s_i of Γ_D.

The following theorems are what Can and I have found during the process of making the paper [?]. They are not related to the main theorem **??**, but they may be useful for future analysis of these posets.

Theorem 2.2.1. *If $|I^c| > 2$ and $|J^c| > 3$, then $W^-_{I,J}$ is not a graded poset.*

Theorem 2.2.2. *If $|I^c| \leq 2$ and $|J^c| \leq 3$, then $W^-_{I,J}$ is a graded poset.*

Theorem 2.2.3. *If $I^c = J^c = \{s_1, s_n\}$, then $W^-_{I,J}$ is a double diamond shape.*

Theorem 2.2.4. *If $I^c = K^c$ and $J^c = L^c$, then we have $W^-_{I,J} \subset V^-_{K,L}$.*

Let t denote the number of nodes in $\Gamma_{I \cap J}(s_{n-1})$. If $t > \lfloor m/2 \rfloor$, then $W^-_{I,J} = V^-_{K,L}$. Finally, if $s_{n-1} \in I^c \cap J^c$, then $W^-_{I,J} \subsetneq V^-_{K,L}$.

Theorem 2.2.5. *(This is a better version of Theorem **??**) If $I^c = K^c$ and $J^c = L^c$, then we have $W^-_{I,J} \subset V^-_{K,L}$.*

Let t denote the number of nodes in $\Gamma_{I \cap J}(s_{n-1})$. Let $h = |I^c \cup J^c|$.

If $2t > \lceil m - 2 - \frac{h}{2} \rceil$, then $W^-_{I,J} = V^-_{K,L}$.

Finally, if $s_{n-1} \in I^c \cap J^c$, then $W^-_{I,J} \subsetneq V^-_{K,L}$.

Proof. To show $W^-_{I,J} \subset V^-_{K,L}$ we assume that there exists an element $w \in W^-_{I,J} \setminus V^-_{K,L}$. By our hypothesis $W_I \subseteq V_K$ and $W_J \subseteq V_L$. Since $w \notin V^-_{K,L}$, we see that w is not of minimal length in the double coset $V_K w V_L$; let $v \in V_K w V_L$ be the element with the minimal length, hence $v \in V^-_{K,L}$. Since $V_K w V_L$ is an interval in V and since $w \in V_K w V_L$, we see that $v \leq w$, hence $v \in W$. By (**??**) we know that there exists unique elements $a \in V_K$ and $b \in V_L$ such that $w = avb$ and

$$\ell(w) = \ell(a) + \ell(v) + \ell(b). \tag{2.6}$$

If the reduced expressions of a has a simple reflection $s_i \in K$ with $i > n-1$, then the reduced expression of b must also have the same simple reflection to cancel the one in a; otherwise, we would have $avb \notin W$. Let a' be the element of V_K that is obtained by deleting s_i from a, and let b' be the element of V_L that is obtained from b by deleting s_i. Then we obtain $w = a'vb'$. But this contradicts with (**??**). It follows that both of the permutations a and b are from W, that is, s_i with $i > n-1$ does not appear in their reduced expressions. Now, since $I^c = K^c$, we see that $V_K \cap W = W_I$, and similarly, $V_L \cap W = W_J$. Therefore, a, hence a^{-1}, are from W_I, and b, hence b^{-1} are from W_J. In particular, we see that $v = a^{-1} w b^{-1} \in W_I w W_J$. This contradicts with the fact that w is the unique minimal length representative of the interval $W_I w W_J$. This finishes the proof of the fact that $W^-_{I,J} \subset V^-_{K,L}$.

Next, we will prove the second claim. Since $I \subset K$ and $J \subset L$, we have $I \cap J \subseteq K \cap L$.

Let $v = v_1v_2\ldots v_m$ be an element from $V^-_{K,L} = \{v \in V : K \subseteq \mathrm{Asc}_L(v)$ and $L \subseteq \mathrm{Asc}_R(v)\}$. Let t denote the cardinality of $\Gamma_{K\cap L}(s_{m-1})$. Observe that

$$\Gamma_{K\cap L}(s_{m-1}) = \Gamma_{K\cap L}(s_{n-1}).$$

The condition $L \subseteq \mathrm{Asc}_R(v)$ implies that $v_i < v_{i+1}$ for $s_i \in L$, and the condition $K \subseteq \mathrm{Asc}_L(v)$ implies that i comes before $i+1$ in v for $s_i \in K$. Consequently, we have that $v_{m-t} < v_{m-t+1} < \cdots < v_m$, and that for $j = 0, \ldots, t-1$ the number $m-t+j$ appears before $m-t+j+1$ in $v = v_1 \ldots v_n$.

Now, consider $2t > \lceil m - 2 - \frac{h}{2} \rceil$, it is equivalent to $t+1 > \lceil m - (t+1) - \frac{h}{2} \rceil$. By contradiction, suppose that m does not appears at the m-th position. Then, none of the numbers $m-t+j$ for $j = 0, \ldots, t$ (the biggest $t+1$ numbers) appears in any of the last $(t+1)$ positions. Also, if $s_j \in (K^c \cup L^c)$, then either $v_j > v_{j+1}$ or j comes after $j+1$. If $v_j > v_{j+1}$, then v_{j+1} is not equal to any of those biggest $t+1$ numbers. If j comes after $j+1$, then $j+1$ cannot appear in the last $t+1$ positions. In the most extreme case (in the sense of satisfying the most numbers of elements in $K^c \cup L^c$), $j+1$ can appear in v_{k+1}, where $s_k \in (K^c \cup L^c)$ and $k \neq j$. So, every two elements in $K^c \cup L^c$ can make at least another 1 position not possible to be one of the biggest $t+1$ numbers (besides the last $t+1$ positions). Thus, there are at least $(t+1) + \lfloor \frac{h}{2} \rfloor$ positions that are not possible for the biggest $t+1$ numbers. We have to use $\lfloor \frac{h}{2} \rfloor$ because if h is an odd number, the element $s_j \in (K^c \cup L^c)$, for j is the maximum, may not contribute any new position not possible for those biggest $t+1$ numbers. It brings a contradiction to $t+1 > \lceil m - (t+1) - \frac{h}{2} \rceil$.

Now we have that m appears at the m-th position, that is, $v_m = m$. Since all elements of $V^-_{K,L}$ has this property, removing m from the one-line notation of v ($v \in V^-_{K,L}$) does not effect the Bruhat-Chevalley order on $V^-_{K,L}$. In other words, if $K' = K \setminus \{s_{m-1}\}$, $L' = L \setminus \{s_{m-1}\}$, then $V^-_{K,L} = V'^-_{K',L'}$, where V' is the parabolic subgroup generated by the simple reflections from R_{m-2}. Now we repeat this process for $V'^{-}_{K',L'}$ and remove $m-1$ from the one-line notation of its elements. Clearly, we can repeat this as long as $t' > s'$, where s' is the cardinality of $\Gamma_{K'\cap L'}(s_{m-2})$. This finishes the proof of our second claim.

To prove our last claim, it is enough to find a double coset $V_K v V_L$ whose minimal length representative is not contained in $W^{-}_{I,J}$.

Let $w \in W$ denote the unique maximal element of the poset $W^{-}_{I,J}$. Since s_{n-1} is in $I^c \cap J^c$, it is either a left descend or it is a right descend for w. Let us first assume that $w_{n-1} > w_n$ where $w = w_1 \dots w_n \dots w_m$ is the one-line notation for w; this means that s_{n-1} is a right descend. In this case, let v denote $ws_{n-1}s_ns_{n-1}$, which is equal to $ws_ns_{n-1}s_n$. If $v = v_1 \dots v_n \dots v_m$ is the one-line notation for v, then $w_{n+i} = n+i$ for $i = 1, \dots, m-n$, and v is obtained from w by interchanging w_{n-1} with $n+1$. In other words,

$$v_j = \begin{cases} w_j & \text{if } j < n-1 \\ n+1 & \text{if } j = n-1 \\ w_n & \text{if } j = n \\ w_{n-1} & \text{if } j = n+1 \\ j & \text{if } j > n+1. \end{cases}$$

This means that $W_I v W_J \cap W = \emptyset$, and furthermore, $\ell(v) = \ell(w) + 1$. Therefore, we see that v is the minimal length representative of the coset $V_K v V_L$, hence our proof is complete in this case.

Next, we assume that $w_{n-1} < w_n$. This means that s_{n-1} is a left descend. In this case, we let v denote $s_{n-1}s_ns_{n-1}w$, which is equal to $s_ns_{n-1}s_nw$. If $v = v_1 \dots v_n \dots v_m$ is the one-line notation for v, then v is obtained from w by interchanging $n-1$ with

$n+1$. In other words,

$$v_j = \begin{cases} w_j & \text{if } j \notin \{n-1, n+1\} \\ n+1 & \text{if } w_j = n-1 \\ n-1 & \text{if } w_j = n+1. \end{cases}$$

This means that $W_I v W_J \cap W = \emptyset$, and furthermore, $\ell(v) = \ell(w) + 1$. Therefore, we see that v is the minimal length representative of the coset $V_K v V_L$, hence our proof is complete in this case also.

□

Corollary 2.2.6. *We maintain the notation and the hypothesis of Theorem* **??**. *Then* $W^-_{I,J}$ *is a lower interval in* $V^-_{K,L}$.

Proof. We already know that $W^-_{I,J} \subseteq V^-_{K,L}$. To prove our claim, it suffices to show that $v \not\leq w$ if $w \in W^-_{I,J}$ and $v \in V^-_{K,L} \setminus W^-_{I,J}$. Let us assume the contrary statement that there exist $v \in V^-_{K,L} \setminus W^-_{I,J}$ and $w \in W^-_{I,J}$ such that $v \leq w$. But this means that the reduced expression of v is a subexpression of the reduced expression of w. In particular, $v \in W$. The argument in the first paragraph of the proof of Theorem **??** shows that $v \in W^-_{I,J}$, which is a contradiction. This finishes the proof of the corollary. □

For easing our notation, whenever it is clear from the context, we will denote the simple transposition s_i by its index i.

The following theorem is the main result from [**?**].

Theorem 2.2.7. *Let* G *denote* SL_{n+1} *and let* P_I *and* P_J *be two standard parabolic subgroups of* G. *If* $G/P_I \times G/P_J$ *is a spherical double flag variety, then the inclusion poset* $(Z, \subseteq)$ *of* G*-orbit closures is either a chain or one of the "ladder lattices" as depicted in Figure* **??**. *More precisely, we have*

1. *if* $|I^c| = |J^c| = 1$*, then* Z *is isomorphic to a chain;*

2. *if* $|I^c| = 1$ *and* $J^c = \{s_j, s_{j+1}\}$ $(1 \leq j \leq n-1)$*, then* Z *is isomorphic to a chain;*

3. *if* $I^c \in \{\{s_2\}, \{s_{n-1}\}\}$ *and* $J^c = \{s_p, s_q\}$ $(1 < p < p+1 < q < n)$*, then the Hasse diagram of* Z *is as in (E) in Figure* **??***;*

4. *if* $I^c \in \{\{s_1\}, \{s_n\}\}$ *and* $|J^c| \geq 2$ *(but* $J^c \neq \{s_j, s_{j+1}\}$ $(1 \leq j \leq n-1)$*), then* Z *is isomorphic to a chain;*

5. *if* $I^c \in \{\{s_2\}, \ldots, \{s_{n-1}\}\}$*, and* $J^c = \{s_1, s_j\}$ *or* $J^c = \{s_j, s_n\}$ *with* $2 < j < n-1$*, then*

   (a) *the Hasse diagram of* Z *is as in (A) in Figure* **??** *for* $2 < j \leq i$ *and* $i + j - 2 < n$*;*

   (b) *the Hasse diagram of* Z *is as in (B) in Figure* **??** *for* $2 < j \leq i$ *and* $i + j - 2 \geq n$*;*

   (c) *the Hasse diagram of* Z *is as in (C) in Figure* **??** *for* $j > i \geq 2$ *and* $i + j - 2 < n$*;*

   (d) *the Hasse diagram of* Z *is as in (D) in Figure* **??** *for* $j > i \geq 2$ *and* $i + j - 2 \geq n$*.*

2.3 Type A, Complexity 0

The following theorem **??** is the corollary of the main result of Can's paper [**?**].

Theorem 2.3.1. *Let* G *denote* SL_n *and let* X *be a double flag variety* $G/P_I \times G/P_J$*. If* $c_G(X) = 0$*, then the inclusion poset of* G*-orbit closures in* X *is a lattice.*

In this section, we will prove the theorem by using the method that we use on the paper [**?**]. It is a different method from what Can uses in his original paper [**?**].

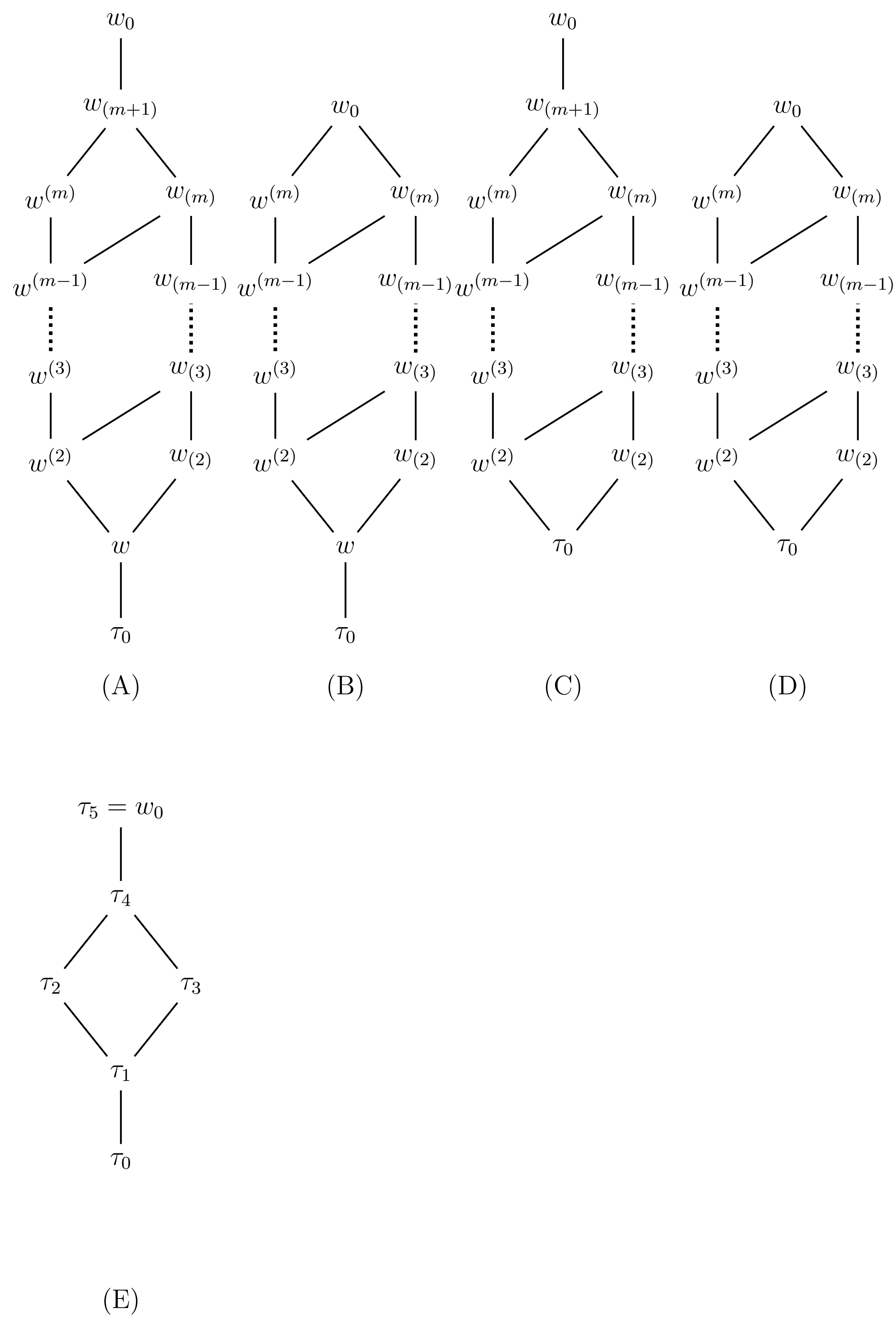


Figure 2.1: The ladder posets.

Table **??** is from Ponomavera's paper [**?**]. It summazies all the cases of complexity 0 for $G = SL_n$.

Number of blocks	$Bl(W_I)$	$Bl(W_J)$
$2,2$	(p_1, p_2)	(q_1, q_2)
$2,3$	(p_1, p_2)	$(1, q_2, q_3)$
	(p_1, p_2)	$(q_1, 1, q_3)$
	$(2, p_2)$	$(q_1, q_2, q_3),\ q_1, q_2 \geq 2$
$2, s$	$(1, p_2)$	$(q_1, q_2, ..., q_s)$

Table 2.1: The list of all complexity 0 double flag varieties for SL_n.

2.3.1 $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (q_1, q_2)$.

Let $w = w_1 \dots w_n \in W^-_{I,J}$. Since $I^c = \{s_{p_1}\}$, and $J^c = \{s_{q_1}\}$, we have:

(i) for $i \in \{1, \dots, p_1 - 1, p_1 + 1, \dots, n-1\}$, i comes before $i+1$ in w;

(ii) $w_1 < \cdots < w_{q_1}$, $w_{q_1+1} < \cdots < w_n$.

In condition (i), let's introduce the notation: $1 \to 2 \to 3$. It means 1 appears before 2, 2 appears before 3. Then, the condition (i) is equivalent to:

$$1 \to \cdots \to p_1, p_1 + 1 \to \cdots \to n.$$

Suppose that $p_1 \geq q_2 + 1$. Consider the position of number 1 in w. 1 cannot be in the segment $w_{q_1+1} \dots w_n$, because there are at least $p_1 - 1 \geq q_2$ numbers appearing after 1. Thus, 1 is in the segment $w_1 \dots w_{q_1}$, and $w_1 = 1$. By removing 1 from $w \in W^-_{I,J}$ and reducing all the remaining entries by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1 - 1, p_2)$ and $Bl(W_{J'}) = (q_1 - 1, q_2)$. With the condition $p_1 \geq q_2 + 1$, we know that $q_1 \geq 2$, which makes $q_1 - 1 \geq 1$. Therefore, we can assume that $p_1 \leq q_2$ (1).

Suppose that $p_1 \geq q_1 + 1$. Consider the position of number p_1 in w. p_1 cannot be in the segment $w_1 \dots w_{q_1}$, because there are at least $p_1 - 1 \geq q_1$ numbers appearing before p_1. Thus, p_1 is in the segment $w_{q_1+1} \dots w_n$. Now, consider the position of number $q_1 - 1$ in w. If $p_1 - 1$ is in the segment $w_{q_1+1} \dots w_n$ with p_1, then $p_1 - 1$ must appear in front of p_1. If $p_1 - 1$ is in the segment $w_1 \dots w_{q_1}$, since there are $p_1 - 1$ numbers in this segment with q_1 positions, we have $q_1 = p_1 - 1$ and $w = id$. In both cases, $p_1 - 1$ and p_1 always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing p_1 from $w \in W^-_{I,J}$ and reducing all entries $w_j > p_1$ by 1, we obtain a poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1 - 1, p_2)$ and $Bl(W_{J'}) = (q_1, q_2 - 1)$, Therefore, we can assume that $p_1 \leq q_1$ (2).

Suppose that $p_2 \geq q_1 + 1$. Consider the position of number n in w. n cannot be in the segment $w_1 \dots w_{q_1}$, because there are at least $p_2 - 1 \geq q_1$ numbers appearing before n. Thus, n is in the segment $w_{q_1+1} \dots w_n$, and $w_n = n$. By removing n from $w \in W^-_{I,J}$, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1, p_2 - 1)$ and $Bl(W_{J'}) = (q_1, q_2 - 1)$. With the condition $p_2 \geq q_1 + 1$, we know that $q_2 \geq 2$, which makes $q_2 - 1 \geq 1$. Therefore, we can assume that $p_2 \leq q_1$ (3).

Suppose that $p_2 \geq q_2 + 1$. Consider the position of number $p_1 + 1$ in w. $p_1 + 1$ must be in the segment $w_1 \dots w_{q_1}$ because there are at least $p_2 - 1 \geq q_2$ numbers appearing after $p_1 + 1$. Now, consider the position of number $p_1 + 2$ in w. If $p_1 + 2$ is in the first segment, then $p_1 + 2$ must appear right after $p_1 + 1$. If $p_1 + 2$ is in the last segment, then $w = id$ because there are at least $p_2 - 2 \geq q_2 - 1$ numbers appearing after $p_1 + 2$. In both cases, $p_1 + 1$ and $p_1 + 2$ always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing $p_1 + 1$ from $w \in W^-_{I,J}$ and reducing all entries $w_j > p_1 + 1$

by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1, p_2 - 1)$ and $Bl(W_{J'}) = (q_1 - 1, q_2)$, Therefore, we can assume that $p_2 \le q_2$ (4).

Since $n = q_1 + q_2 = p_1 + p_2$, (2) implies that $q_2 \le p_2$. Combine it with (4), we have $p_2 = q_2$. Similarly, using (1), (2), (3), and (4), we have $p_1 = p_2 = q_1 = q_2 =: a$.

Now, our case becomes $Bl(W_I) = (a, a)$ and $Bl(W_J) = (a, a)$. Our conditions (i) and (ii) become:

(i) $1 \to \cdots \to a, a+1 \to \cdots \to 2a$;

(ii) $w_1 < \cdots < w_a$, $w_{a+1} < \cdots < w_{2a}$.

Suppose that $a \ge 2$. Consider the positions of numbers 1 and n in w. From conditions (i) and (ii), we have $1 \in \{w_1, w_{a+1}\}$ and $n \in \{w_a, w_{2a}\}$. If $1 = w_1$, then $n = w_n$. If $1 = w_{a+1}$, then $n = w_a$. By removing n from $w \in W^-_{I,J}$, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (a, a-1)$ and $Bl(W_{J'}) = (a, a-1)$. After that, since we have $a > a - 1$, we can again use one of the above arguments to reduce to $Bl(W_{I'}) = (a-1, a-1)$ and $Bl(W_{J'}) = (a-1, a-1)$.

The reduction arguments show that all of the cases are isomorphic to only one case:

(A) $Bl(W_I) = (1, 1)$, $Bl(W_J) = (1, 1)$.

2.3.2 $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (1, q_2, q_3)$.

Let $w = w_1 \dots w_n \in W^-_{I,J}$. Since $I^c = \{s_{p_1}\}$, and $J^c = \{s_1, s_{q_2+1}\}$, we have:

(i) for $i \in \{1, \dots, p_1 - 1, p_1, \dots, n-1\}$, i comes before $i+1$ in w

$$\iff 1 \to \cdots \to p_1, p_1 + 1 \to \cdots \to n;$$

(ii) $w_2 < \cdots < w_{q_2+1}$, $w_{q_2+2} < \cdots < w_n$.

Suppose that $p_1 \geq q_2 + 2$. Consider the position of p_1 in w. p_1 must be in the last segment because there are at least $p_1 - 1 \geq q_2 + 1$ numbers appearing before p_1. Now, consider the position of $p_1 - 1$ in w. If $p_1 - 1$ is in the last segment, then it must appears right in front of p_1. If $p_1 - 1$ is not in the last segment, then $w = id$ because there are at least $p_1 - 2 \geq q_2$ numbers appearing before $p_1 - 1$ and in this case $p_1 - 1$ must be equal to w_{q_2+1}. Thus, $p_1 - 1$ and p_1 always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing p_1 from $w \in W^-_{I,J}$ and reducing all entries $w_j > p_1$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1 - 1, p_2)$ and $Bl(W_{J'}) = (1, q_2, q_3 - 1)$. Therefore, we can assume that $p_1 \leq q_2 + 1$ (1).

Suppose that $p_1 \geq q_3 + 2$. Consider the position of 2 in w. 2 cannot be in the last segment because there are at least $p_1 - 2 \geq q_3$ numbers appearing after 2. Also, 2 cannot be w_1 because there is number 1 appearing before 2. So, 2 is in the segment $w_2 \dots w_{q_2+1}$. Now, consider the position of number 1 in w. 1 cannot be in the last segment because there are at least $p_1 - 1 \geq q_3 + 1$ numbers appearing after 1. If 1 is in the same segment $w_2 \dots w_{q_2+1}$ with 2, then $1 = w_2$ and $2 = w_3$. If 1 is not in the segment $w_2 \dots w_{q_2+1}$, then $1 = w_1$ and $2 = w_2$. Thus, 1 and 2 always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing 2 from $w \in W^-_{I,J}$ and reducing all entries $w_j > 2$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1 - 1, p_2)$ and $Bl(W_{J'}) = (1, q_2 - 1, q_3)$. Therefore, we can assume that $p_1 \leq q_3 + 1$ (2).

Suppose that $p_2 \geq q_3 + 2$. Consider the position of $p_1 + 2$ in w. $p_1 + 2$ must be in the segment $w_2 \dots w_{q_2+1}$ because there are at least $p_2 - 2 \geq q_3$ numbers appearing after $p_1 + 2$, and number $p_1 + 1$ always appears before $p_1 + 2$. Now, consider the position of $p_1 + 3$ in w. If $p_1 + 3$ is in the same segment with $p_1 + 2$, then it must appears right after $p_1 + 2$. If $p_1 + 3$ is in the last segment, then $w = id$ because there are at least $p_2 - 3 \geq q_3 - 1$ numbers appearing after $p_1 + 3$ and in this case $p_1 + 3 = w_{q_2+2}$. Thus, $p_1 + 2$ and $p_1 + 3$ always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing $p_1 + 2$ from $w \in W^-_{I,J}$ and reducing all entries $w_j > p_1 + 1$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1, p_2 - 1)$ and $Bl(W_{J'}) = (1, q_2 - 1, q_3)$. Therefore, we can assume that $p_2 \leq q_3 + 1$ (3).

Suppose that $p_2 \geq q_2 + 2$. Consider the position of number n in w. n must be in the last segment because there are at least $p_2 - 1 \geq q_2 + 1$ numbers appearing before n. It implies $n = w_n$. By removing n from $w \in W^-_{I,J}$, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1, p_2 - 1)$ and $Bl(W_{J'}) = (1, q_2, q_3 - 1)$. Therefore, we can assume that $p_2 \leq q_2 + 1$ (4).

Since $n = p_1 + p_2 = q_2 + q_3 + 1$, (3) implies that $q_2 \leq p_1$. Combine it with (1), we have either $p_1 = q_2$ or $p_1 = q_2 + 1$. Let $a := q_2$ and $b := q_3$. We have two cases:

(Case 1) $Bl(W_I) = (a, b + 1)$, $Bl(W_J) = (1, a, b)$;

(Case 2) $Bl(W_I) = (a + 1, b)$, $Bl(W_J) = (1, a, b)$.

Case 1: $Bl(W_I) = (a, b + 1)$, $Bl(W_J) = (a, 1, b)$.

(2) and (4) imply $a \leq b + 1$ and $b + 1 \leq a + 1$. They imply that either $b = a$ or $b = a - 1$.

Our case 1 now becomes:

(1A) $Bl(W_I) = (a, a + 1)$, $Bl(W_J) = (1, a, a)$;

(1B) $Bl(W_I) = (a, a)$, $Bl(W_J) = (1, a, a - 1)$.

Case 2: $Bl(W_I) = (a + 1, b)$, $Bl(W_J) = (a, 1, b)$.

(2) and (4) imply $a + 1 \leq b + 1$ and $b \leq a + 1$. They imply that either $b = a$ or $b = a + 1$.

Our case 2 now becomes:

(2A) $Bl(W_I) = (a+1, a)$, $Bl(W_J) = (1, a, a)$;

(2B) $Bl(W_I) = (a+1, a+1)$, $Bl(W_J) = (1, a, a+1)$

$\iff Bl(W_I) = (a, a)$, $Bl(W_J) = (1, a-1, a)$.

Summary of all cases we have:

(1A) $Bl(W_I) = (a, a+1)$, $Bl(W_J) = (1, a, a)$;

(1B) $Bl(W_I) = (a, a)$, $Bl(W_J) = (1, a, a-1)$;

(2A) $Bl(W_I) = (a+1, a)$, $Bl(W_J) = (1, a, a)$;

(2B) $Bl(W_I) = (a, a)$, $Bl(W_J) = (1, a-1, a)$.

2.3.3 $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (q_1, 1, q_3)$.

Let $w = w_1 \dots w_n \in W^-_{I,J}$. Since $I^c = \{s_{p_1}\}$, and $J^c = \{s_{q_1}, s_{q_1+1}\}$, we have:

(i) for $i \in \{1, \dots, p_1 - 1, p_1, \dots, n-1\}$, i comes before $i+1$ in w

$\iff 1 \to \cdots \to p_1, p_1 + 1 \to \cdots \to n$;

(ii) $w_1 < \cdots < w_{q_1}$, $w_{q_1+2} < \cdots < w_n$.

Suppose that $p_1 \geq q_1 + 2$. Consider the position of p_1 in w. p_1 must be in the last segment because there are at least $p_1 - 1 \geq q_1 + 1$ numbers appearing before p_1. Now, consider the position of $p_1 - 1$ in w. If $p_1 - 1$ is in the last segment, then it must appears right in front of p_1. If $p_1 - 1$ is not in the last segment, then $w = id$ because there are at least $p_1 - 2 \geq q_1$ numbers appearing before $p_1 - 1$ and in this case $p_1 - 1$ must be equal to w_{q_1+1}. Thus, $p_1 - 1$ and p_1 always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing p_1 from $w \in W^-_{I,J}$ and reducing all entries $w_j > p_1$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1 - 1, p_2)$ and

$Bl(W_{J'}) = (q_1, 1, q_3 - 1)$. Therefore, we can assume that $p_1 \leq q_1 + 1$ (1).

Suppose that $p_1 \geq q_3 + 2$. Consider the position of number 1 in w. 1 must be in the first segment because there are at least $p_1 - 1 \geq q_3 + 1$ numbers appearing after 1. It implies $w_1 = 1$. By removing 1 from $w \in W^-_{I,J}$ and reducing all remaining entries by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1 - 1, p_2)$ and $Bl(W_{J'}) = (q_1 - 1, 1, q_3)$. Therefore, we can assume that $p_1 \leq q_3 + 1$ (2).

Suppose that $p_2 \geq q_3 + 2$. Consider the position of $p_1 + 1$ in w. $p_1 + 1$ must be in the first segment because there are at least $p_2 - 1 \geq q_3 + 1$ numbers appearing after $p_1 + 1$. Now, consider the position of $p_1 + 2$ in w. If $p_1 + 2$ is in the first segment, then it must appears right after $p_1 + 1$. If $p_1 + 2$ is not in the first segment, then $w = id$ because there are at least $p_2 - 2 \geq q_3$ numbers appearing after $p_1 + 2$ and in this case $p_1 + 2$ must be equal to w_{q_1+1}. Thus, $p_1 + 1$ and $p_1 + 2$ always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing $p_1 + 1$ from $w \in W^-_{I,J}$ and reducing all entries $w_j > p_1 + 1$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1, p_2 - 1)$ and $Bl(W_{J'}) = (q_1 - 1, 1, q_3)$. Therefore, we can assume that $p_2 \leq q_3 + 1$ (3).

Suppose that $p_2 \geq q_1 + 2$. Consider the position of number n in w. n must be in the last segment because there are at least $p_2 - 1 \geq q_1 + 1$ numbers appearing before n. It implies $w_n = n$. By removing n from $w \in W^-_{I,J}$, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1, p_2 - 1)$ and $Bl(W_{J'}) = (q_1, 1, q_3 - 1)$. Therefore, we can assume that $p_2 \leq q_1 + 1$ (4).

Since $n = p_1 + p_2 = q_1 + q_3 + 1$, (3) implies that $q_1 \leq p_1$. Combine it with (1), we have either $p_1 = q_1$ or $p_1 = q_1 + 1$. Let $a := q_1$ and $b := q_3$. We have two cases:

(Case 1) $Bl(W_I) = (a, b+1)$, $Bl(W_J) = (a, 1, b)$;

(Case 2) $Bl(W_I) = (a+1, b)$, $Bl(W_J) = (a, 1, b)$.

Case 1: $Bl(W_I) = (a, b+1)$, $Bl(W_J) = (a, 1, b)$.

(2) and (4) imply $a \le b+1$ and $b+1 \le a+1$. They imply that either $b = a$ or $b = a-1$.

Our case 1 now becomes:

(1A) $Bl(W_I) = (a, a+1)$, $Bl(W_J) = (a, 1, a)$;

(1B) $Bl(W_I) = (a, a)$, $Bl(W_J) = (a, 1, a-1)$.

Case 2: $Bl(W_I) = (a+1, b)$, $Bl(W_J) = (a, 1, b)$.

(2) and (4) imply $a+1 \le b+1$ and $b \le a+1$. They imply that either $b = a$ or $b = a+1$.

Our case 2 now becomes:

(2A) $Bl(W_I) = (a+1, a)$, $Bl(W_J) = (a, 1, a)$;

(2B) $Bl(W_I) = (a+1, a+1)$, $Bl(W_J) = (a, 1, a+1)$

$$\iff Bl(W_I) = (a, a),\ Bl(W_J) = (a-1, 1, a).$$

Summary of all cases we have:

(1A) $Bl(W_I) = (a, a+1)$, $Bl(W_J) = (a, 1, a)$;

(1B) $Bl(W_I) = (a, a)$, $Bl(W_J) = (a, 1, a-1)$;

(2A) $Bl(W_I) = (a+1, a)$, $Bl(W_J) = (a, 1, a)$;

(2B) $Bl(W_I) = (a, a)$, $Bl(W_J) = (a-1, 1, a)$.

2.3.4 $Bl(W_I) = (2, p_2)$ and $Bl(W_J) = (q_1, q_2, q_3)$, $q_1, q_2 \geq 2$.

Let $w = w_1 \dots w_n \in W^-_{I,J}$. Since $I^c = \{s_2\}$, and $J^c = \{s_{q_1}, s_{q_1+q_2}\}$, we have:

(i) for $i \in \{1, 3, 4, \dots, n-1\}$, i comes before $i+1$ in w

$\iff 1 \to 2, 3 \to 4 \to \cdots \to n$;

(ii) $w_1 < \cdots < w_{q_1}$, $w_{q_1+1} < \cdots < w_{q_1+q_2}$, $w_{q_1+q_2+1} < \cdots < w_n$.

Suppose that $q_1 \geq 3$. Consider the position of number 4 in w. 4 cannot be in the segment $w_{q_1+q_2+1} \dots w_n$ because there are at least $n-4 = q_1+q_2+q_3-4 \geq q_2+q_3-1$ numbers bigger than 4 and appearing after 4. Thus, 4 must be either in $w_1 \dots w_{q_1}$ or in $w_{q_1+1} \dots w_{q_1+q_2}$. If 4 is in the segment $w_1 \dots w_{q_1}$, then 3 must appear in front of 4. If 4 is in the segment $w_{q_1+1} \dots w_{q_1+q_2}$, then $w_{q_1+1} = 4$ because only 1, 2, and 3 can appear before 4 and $q_1 \geq 3$. In this case, 3 also appears in front of 4. Thus, 3 and 4 always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing 3 from $w \in W^-_{I,J}$ and reducing all entries $w_j > 3$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (2, p_2 - 1)$ and $Bl(W_{J'}) = (q_1 - 1, q_2, q_3)$, $q_1 - 1, q_2 \geq 2$. Therefore, we can assume that $q_1 = 2$.

Suppose that $q_2 \geq 3$. Consider the position of number $m := q_1 + q_2$ in w. m cannot appear in the last segment because there are q_3 numbers bigger than m and appearing after m. m cannot appear in the first segment because there are at least $q_1 + q_2 - 3 \geq q_1$ numbers appearing before m. Thus, m appears in the segment $w_{q_1+1} \dots w_{q_1+q_2}$. Now, consider the position of $m+1$ in w. If $m+1$ appears in the segment $w_{q_1+1} \dots w_{q_1+q_2}$, then $m+1$ must appear right after m. If $m+1$ appears in the segment $w_{q_1+q_2+1} \dots w_n$, then $w_k = k$ for all $k \geq m$. In both cases, m and $m+1$ always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing m from $w \in W^-_{I,J}$ and reducing all entries $w_j > m$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (2, p_2 - 1)$ and $Bl(W_{J'}) = (q_1, q_2 - 1, q_3)$, $q_1, q_2 - 1 \geq 2$.

Therefore, we can assume that $q_2 = 2$.

Suppose that $q_3 \geq 3$. By conditions (i) and (ii), we know that $n \in \{w_{q_1}, w_{q_1+q_2}, w_n\}$. However, n cannot be w_{q_1} or $w_{q_1+q_2}$, because 1 and 2 won't fill up the last q_3 positions in w. Thus, $n = w_n$. By removing n from $w \in W^-_{I,J}$, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (2, p_2 - 1)$ and $Bl(W_{J'}) = (q_1, q_2, q_3 - 1)$, $q_1, q_2 \geq 2$. Therefore, we can assume that $q_3 \leq 2$.

The reduction arguments show that it suffices to consider the following two cases:

(A) $Bl(W_I) = (2, 3)$, $Bl(W_J) = (2, 2, 1)$;

(B) $Bl(W_I) = (2, 4)$, $Bl(W_J) = (2, 2, 2)$;

2.4 Type A, Complexity 1

Theorem **??** is the main result in our paper [**?**] in 2018. This section is the proof of the theorem **??**.

Theorem 2.4.1. *Let G denote SL_n and let X be a double flag variety $G/P_I \times G/P_J$. If $c_G(X) = 1$, then the inclusion poset of G-orbit closures in X is a finite poset.*

Table **??** is from Ponomavera's paper [**?**].

2.4.1 $Bl(W_I) = (3, p_2)$, $p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3)$, $q_1, q_2, q_3 \geq 2$.

Since $I^c = \{s_3\}$, and $J^c = \{s_{q_1}, s_{q_1+q_2}\}$, we see that if $w = w_1 \ldots w_n \in W^-_{I,J}$, then

(i) for $i \in \{1, 2, 4, 5 \ldots, n-1\}$, i comes before $i + 1$ in w

$\iff 1 \to 2 \to 3, 4 \to 5 \to \ldots \to n$;

Number of blocks	$Bl(W_I)$	$Bl(W_J)$
$2,3$	$(3,p_2),\ p_2 \geq 3$ $(p_1,p_2),\ p_1,p_2 \geq 3$ $(p_1,p_2), p_1,p_2 \geq 3$	$(q_1,q_2,q_3),\ q_1,q_2,q_3 \geq 2$ $(2,2,q_3),\ q_3 \geq 2$ $(2,q_2,2),\ q_2 \geq 2$
$2,4$	$(2,p_2),\ p_2 \geq 3$ $(p_1,p_2),\ p_1,p_2 \geq 2$ $(p_1,p_2),\ p_1,p_2 \geq 2$	(q_1,q_2,q_3,q_4) $(1,1,1,q_4)$ $(1,1,q_3,1),\ q_3 \geq 2$
$3,3$	$(1,1,p_3)$ $(1,p_2,1)$	(q_1,q_2,q_3) (q_1,q_2,q_3)

Table 2.2: The list of all complexity 1 double flag varieties for SL_n.

(ii) $w_1 < \cdots < w_{q_1},\ w_{q_1+1} < \cdots < w_{q_1+q_2},\ w_{q_1+q_2+1} < \cdots < w_n$.

Suppose that $q_1 \geq 4$. Consider the position of number 5 in w. 5 cannot be in the segment $w_{q_1+q_2+1} \ldots w_n$ because there are at least $n-5 = q_1+q_2+q_3-5 \geq q_2+q_3-1$ numbers bigger than 5 and appearing after 5. Thus, 5 must be either in $w_1 \ldots w_{q_1}$ or in $w_{q_1+1} \ldots w_{q_1+q_2}$. If 5 is in the segment $w_1 \ldots w_{q_1}$, then 4 must appear in front of 5. If 5 is in the segment $w_{q_1+1} \ldots w_{q_1+q_2}$, then $w_{q_1+1} = 5$ because only 1, 2, 3, and 4 can appear before 5 and $q_1 \geq 4$. In this case, 4 also appears in front of 5. Thus, 4 and 5 always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing 4 from $w \in W^-_{I,J}$ and reducing all entries $w_j > 4$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (3, p_2-1)$ and $Bl(W_{J'}) = (q_1-1, q_2, q_3)$, $q_1-1, q_2, q_3 \geq 2$. Therefore, we can assume that $q_1 \leq 3$.

Suppose that $q_2 \geq 4$. Consider the position of number $m = q_1 + q_2$. m cannot appear in the last segment because there are q_3 numbers bigger than m and appearing after m. m cannot appear in the first segment because there are at least $q_1+q_2-4 \geq q_1$ numbers appearing before m. Thus, m appears in the segment $w_{q_1+1} \ldots w_{q_1+q_2}$. Now, consider the position of $m+1$. If $m+1$ appears in the segment $w_{q_1+1} \ldots w_{q_1+q_2}$, then $m+1$ must appear right after m. If $m+1$ appears in the segment $w_{q_1+q_2+1} \ldots w_n$, then $w_k = k$ for all $k \geq m$. In both bases, m and $m+1$ always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing b from $w \in W^-_{I,J}$ and reducing all entries $w_j > m$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (3, p_2 - 1)$ and $Bl(W_{J'}) = (q_1, q_2 - 1, q_3)$, $q_1, q_2 - 1, q_3 \geq 2$. Therefore, we can assume that $q_2 \leq 3$.

Suppose that $q_3 \geq 4$. By condition (i) and (ii), we know that $n \in \{w_{q_1}, w_{q_1+q_2}, w_n\}$. However, n cannot be w_{q_1} or $w_{q_1+q_2}$, because 1, 2, and 3 won't fill up the last q_3 positions in w. Thus, $n = w_n$. By removing n from $w \in W^-_{I,J}$, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (3, p_2 - 1)$, $p_2 - 1 \geq 3$ and $Bl(W_{J'}) = (q_1, q_2, q_3 - 1)$, $q_1, q_2, q_3 - 1 \geq 2$. Therefore, we can assume that $q_3 \leq 3$.

These reduction arguments show that it suffices to consider the following eight cases only:

(A) $Bl(W_I) = (3, 3)$, $Bl(W_J) = (2, 2, 2)$;

(B) $Bl(W_I) = (3, 4)$, $Bl(W_J) = (2, 2, 3)$;

(C) $Bl(W_I) = (3, 4)$, $Bl(W_J) = (3, 2, 2)$;

(D) $Bl(W_I) = (3, 4)$, $Bl(W_J) = (2, 3, 2)$;

(E) $Bl(W_I) = (3, 5)$, $Bl(W_J) = (2, 3, 3)$;

(F) $Bl(W_I) = (3, 5)$, $Bl(W_J) = (3, 2, 3)$;

(G) $Bl(W_I) = (3, 5)$, $Bl(W_J) = (3, 3, 2)$;

(H) $Bl(W_I) = (3, 6)$, $Bl(W_J) = (3, 3, 3)$.

The Hasse diagrams of these posets are as in Figure **??**.

2.4.2 $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 3$ and $Bl(W_J) = (2, 2, q_3)$, $q_3 \geq 2$.

Let $w = w_1 \ldots w_n \in W^-_{I,J}$. Since $I^c = \{s_{p_1}\}$ and $J^c = \{s_2, s_4\}$, we have:

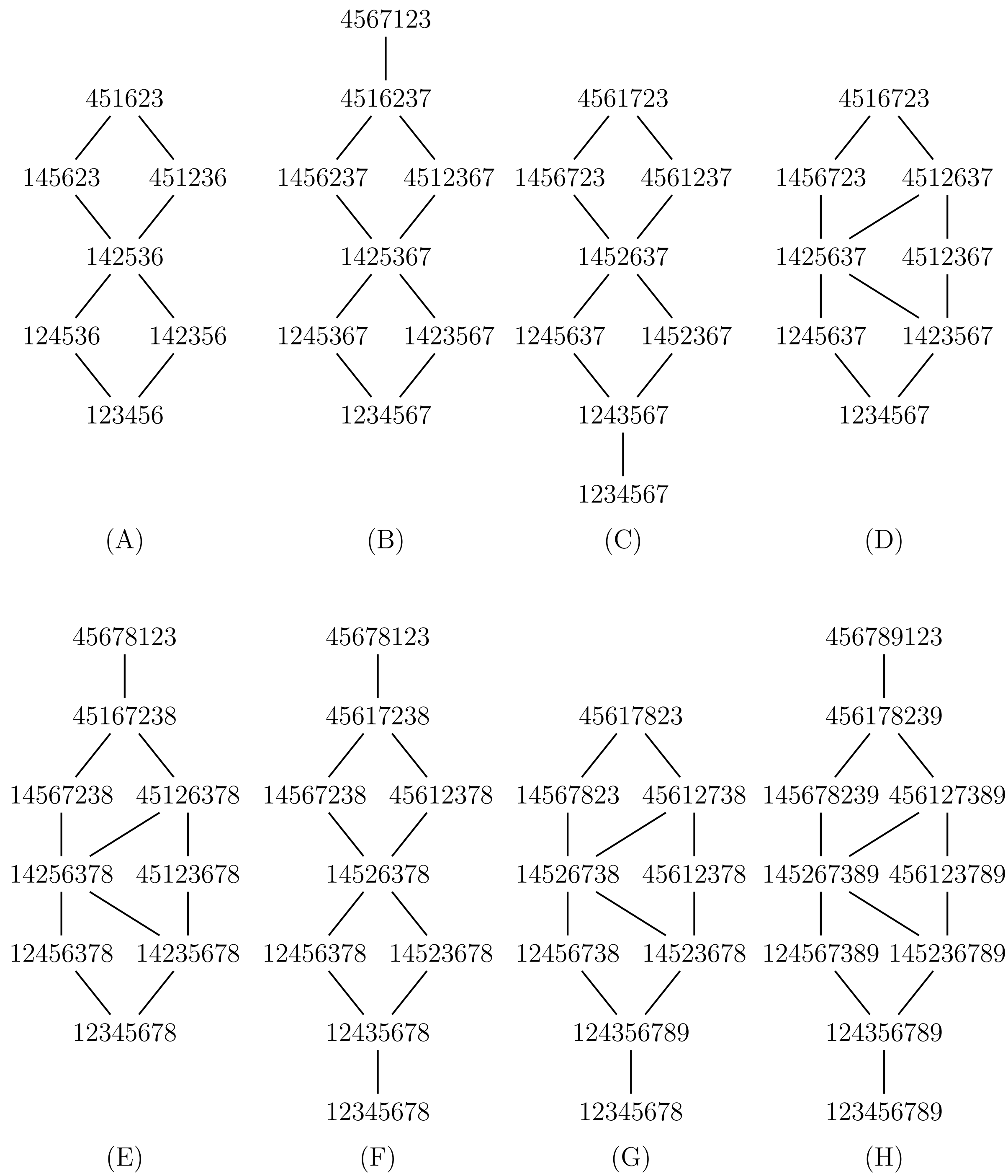


Figure 2.2: $Bl(W_I) = (3, p_2)$, $p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3)$, $q_1, q_2, q_3 \geq 2$.

1. for $i \in \{1, 2, \ldots, p_1 - 1, p_1 + 1, \ldots, n - 1\}$, i comes before $i + 1$ in w

   $\iff 1 \to \cdots \to p_1, p_1 + 1 \to \cdots \to n$;

2. $w_1 < w_2,\ w_3 < w_4,\ w_5 < \cdots < w_n$.

Suppose that $p_1 \geq 5$. Consider the position of number p_1 in w. By condition (i), there are at least $p_1 - 1 \geq 4$ numbers appearing before p_1. It means p_1 must appear in the segment $w_5 \ldots w_n$. Now, consider the position of number $p_1 - 1$. If $p_1 - 1$ is in the segment $w_5 \ldots w_n$, then $p_1 - 1$ must appear right in front of p_1. If $p_1 - 1$ is not in the segment $w_5 \ldots w_n$, then $p_1 = 5$, $p_1 - 1 = 4$, and $w = id$. Thus, $p_1 - 1$ and p_1 always appear as consecutive numbers in $w \in W^-_{I,J}$. By removing p_1 from $w \in W^-_{I,J}$ and reducing all entries $w_j > p_1$ by 1, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1 - 1, p_2)$, $p_1 - 1, p_2 \geq 3$ and $Bl(W_{J'}) = (2, 2, q_3 - 1)$, $q_3 \geq 2$. Therefore, we can assume that $p_1 \leq 4$.

Suppose that $p_2 \geq 5$. Consider the position of number n in w. By condition (i), there are at least $p_2 - 1 \geq 4$ numbers appearing before n. It means n must appear in the segment $w_5 \ldots w_n$. Thus, $n = w_n$. By removing n from $w \in W^-_{I,J}$, we obtain a new poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1, p_2 - 1)$, $p_1, p_2 - 1 \geq 3$ and $Bl(W_{J'}) = (2, 2, q_3 - 1)$, $q_3 - 1 \geq 2$. Therefore, we can assume that $p_2 \leq 4$.

These two reduction arguments show that it suffices to consider the following four cases only:

(A) $Bl(W_I) = (3, 3)$, $Bl(W_J) = (2, 2, 2)$;

(B) $Bl(W_I) = (3, 4)$, $Bl(W_J) = (2, 2, 3)$;

(C) $Bl(W_I) = (4, 3)$, $Bl(W_J) = (2, 2, 3)$;

(D) $Bl(W_I) = (4, 4)$, $Bl(W_J) = (2, 2, 4)$.

The Hasse diagrams of these posets are as in Figure **??**.

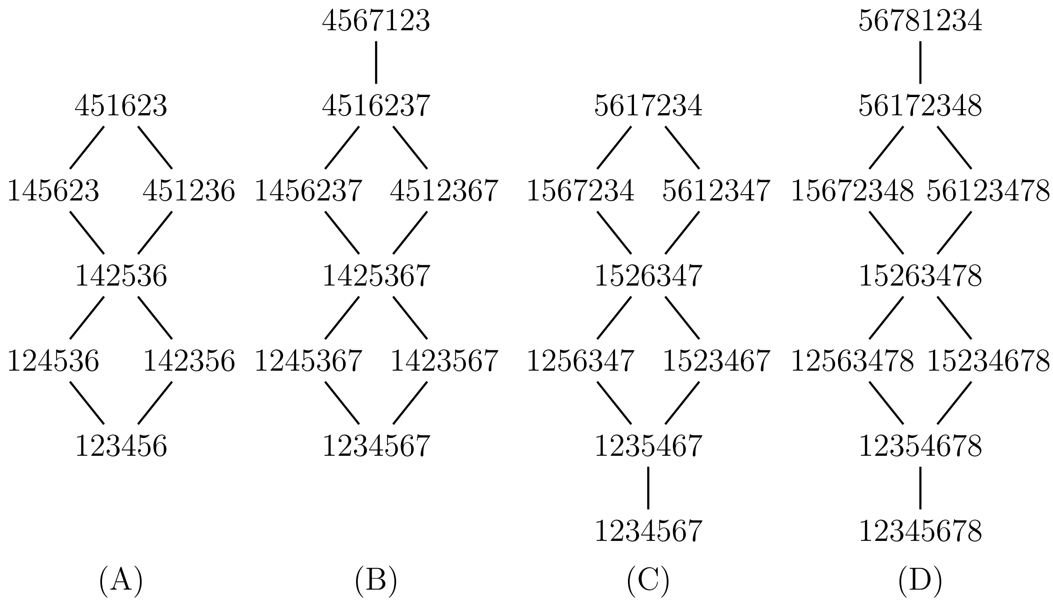


Figure 2.3: $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 3$ and $Bl(W_J) = (2, 2, q_3)$, $q_3 \geq 2$.

2.4.3 $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 3$ and $Bl(W_J) = (2, q_2, 2)$, $q_2 \geq 2$.

First, we assume that $p_1 \geq 5$. Since $I^c = \{s_{p_1}\}$, $J^c = \{s_2, s_{n-2}\}$ in R_{n-1}, we see that if $w = w_1 \ldots w_n \in W^-_{I,J}$, then

(i) for $i \in \{1, 2, \ldots, p_1 - 1, p_1 + 1, \ldots, n-1\}$, i comes before $i+1$ in w

$\iff 1 \to ... \to p_1, p_1 + 1 \to ... \to n;$

(ii) $w_1 < w_2,\ w_3 < \cdots < w_{n-2},\ w_{n-1} < w_n$.

We look for the positions of $p_1 - 3$ and $p_1 - 2$. Since $p_1 \geq 5$, we see from condition (i) that $p_1 - 2$ appears in the segment $w_3 w_4 \ldots w_{n-2}$. If $w_k = p_1 - 2$ for some $k > 3$, then we see that $p_1 - 3$ must also be in the same segment, hence, we must have that $w_{k-1} = p_1 - 3$. If $w_3 = p_1 - 2$, then, by conditions (i) and (ii), we have only one choice that $p_1 = 5$, and $p_1 - 3 = 2 = w_2$. In both of these two cases we see that $p_1 - 3$ must come immediately before $p_1 - 2$ in every $w \in W^-_{I,J}$. Therefore, by removing

$p_1 - 2$ from w and reducing every entry which is greater than $p_1 - 2$ by 1, we do not change the structure of the underlying poset; we obtain a poset $W^-_{I',J'}$ in S_{n-1} such that $Bl(W_{I'}) = (p_1 - 1, p_2)$, $p_1 - 1, p_2 \geq 3$ and $Bl(W_{J'}) = (2, q_2 - 1, 2)$, $q_2 - 1 \geq 2$. In other words, we can assume that $p_1 \leq 4$.

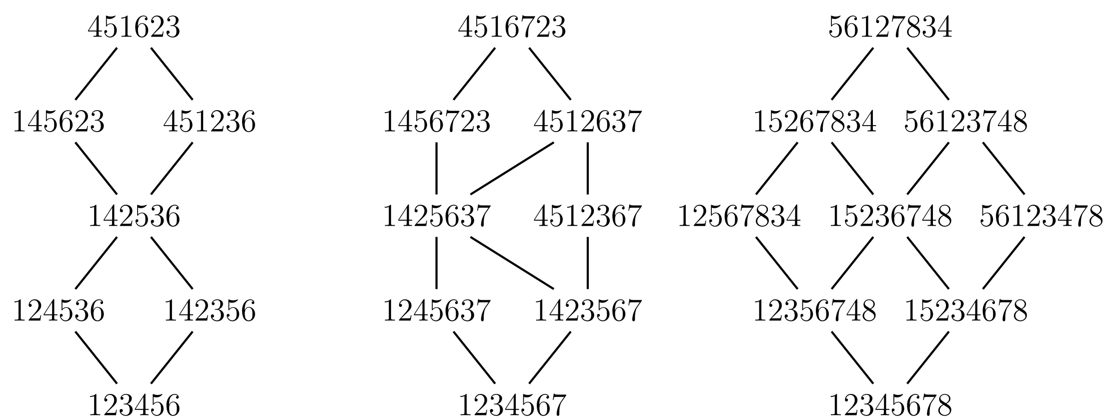
For $p_2 \geq 5$, we repeat the same arguments after applying θ to I and J. Therefore, without loss of generality we can assume that $3 \leq p_1, p_2 \leq 4$. This reduction argument shows that our poset is isomorphic to one of the following three cases:

(A) $Bl(W_I) = (3, 3)$, $Bl(W_J) = (2, 2, 2)$;

(B) $Bl(W_I) = (3, 4)$, $Bl(W_J) = (2, 3, 2)$;

(C) $Bl(W_I) = (4, 4)$, $Bl(W_J) = (2, 4, 2)$.

The Hasse diagrams of these posets are as in Figure **??**.



(A) (B) (C)

Figure 2.4: $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 3$ and $Bl(W_J) = (2, q_2, 2)$, $q_2 \geq 2$.

2.4.4 $Bl(W_I) = (2, p_2)$, $p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3, q_4)$.

Let us first assume that $q_4 \geq 3$. Since $I^c = \{s_2\}$, $J^c = \{s_{q_1}, s_{q_1+q_2}, s_{q_1+q_2+q_3}\}$ in R_{n-1}, we see that if $w = w_1 \dots w_n \in W^-_{I,J}$, then

(i) for $i \in \{1, 3, 4, \ldots, n-1\}$, i comes before $i+1$ in w

$\iff 1 \to 2, 3 \to \ldots \to n$;

(ii) $w_1 < \cdots < w_{q_1}$, $w_{q_1+1} < \cdots < w_{q_1+q_2}$, $w_{q_1+q_2+1} < \cdots < w_{q_1+q_2+q_3}$, and $w_{q_1+q_2+q_3+1} < \cdots < w_n$.

This implies that $n \in \{w_{q_1}, w_{q_1+q_2}, w_{q_1+q_2+q_3}, w_n\}$. By (i) we know that n is preceded by $3, \ldots, n-1$, which prevents the possibilities $n \in \{w_{q_1}, w_{q_1+q_2}, w_{q_1+q_2+q_3}\}$. Therefore, $w_n = n$. Thus, by removing n from $w \in W^-_{I,J}$, we do not change the structure of the underlying poset; we obtain a poset $W^-_{I',J'}$ in S_{n-1}, which is isomorphic to $W^-_{I,J}$, such that $Bl(W_{I'}) = (2, p_2 - 1)$, $p_2 - 1 \geq 3$ and $Bl(W_{J'}) = (q_1, q_2, q_3, q_4 - 1)$. In other words, we can assume without loss of generality that $1 \leq q_4 \leq 2$.

We proceed with the assumption that $q_3 \geq 3$. Then we look at the relative positions of the numbers $m := q_1 + q_2 + q_3$ and $m + 1$ in w. Since we assumed that $1 \leq q_4 \leq 2$, we have $n \in \{w_{m+1}, w_n\}$. If $n = w_{m+1}$, then the following implication is obvious:

$$w_k = m \implies w_{k+1} = m + 1.$$

On the other hand, if $n = w_n$, then since $q_3 \geq 3$, we know that $m + 1$ has to appear in the following segment of w: $w_{q_1+q_2+1} \ldots w_{q_1+q_2+q_3}$. In particular, we have one of the following cases:

$$w_{q_1+q_2+q_3-i} = m \quad \text{and} \quad w_{q_1+q_2+q_3-i+1} = m + 1$$

for $i = 0, 1$. Therefore, m and $m+1$ appear as consecutive terms in w, furthermore, m appears in $w_{q_1+q_2+1} \ldots w_{q_1+q_2+q_3}$. In this case, by removing m from w and reducing every number greater than m by 1, we obtain a poset $W^-_{I',J'}$ in S_{n-1}, which is isomorphic to $W^-_{I,J}$, such that $Bl(W_{I'}) = (2, p_2 - 1)$, $p_2 - 1 \geq 3$ and $Bl(W_{J'}) = (q_1, q_2, q_3 - 1, q_4)$, $q_3 - 1 \geq 2$. In other words, we can assume without loss of generality that $1 \leq q_3 \leq 2$

as well.

Next, we proceed with the assumptions that $q_2 \geq 3$ and $1 \leq q_3, q_4 \leq 2$. In this case, after applying the involution θ to I and J, we assume that $Bl(W_I) = (p_2, 2)$, $p_2 \geq 2$ and $Bl(W_J) = (q_4, q_3, q_2, q_1)$, where $q_2 \geq 3$ and $1 \leq q_3, q_4 \leq 2$. In other words, we have one of the following four possibilities for the first few terms of J:

1. $s_1, s_3, s_5, s_6 \in J$ and $s_2, s_4 \notin J$, or
2. $s_1, s_4, s_5 \in J$ and $s_2, s_3 \notin J$, or
3. $s_2, s_4, s_5 \in J$ and $s_1, s_3 \notin J$, or
4. $s_3, s_4 \in J$ and $s_1, s_2 \notin J$.

In the first case, we have that

$$w_k = 4 \implies w_{k+1} = 5$$

for some $k \geq 1$. In the second case, we have

$$w_k = 3 \implies w_{k+1} = 4$$

for some $k \geq 1$. In the third case, we have

$$w_k = 3 \implies w_{k+1} = 4$$

for some $k \geq 1$. Finally, in the fourth case, we have

$$w_k = 2 \implies w_{k+1} = 3$$

for some $k \geq 1$. In all of these cases, removing w_{k+1} from w and reducing every number that is greater than w_{k+1} by 1 give a poset $W^-_{I',J'}$ in S_{n-1}, which is isomorphic to $W^-_{I,J}$, such that $Bl(W_{I'}) = (p_2 - 1, 2)$ and $Bl(W_{J'}) = (q_4, q_3, q_2 - 1, q_1)$. In other words, we can assume without loss of generality that $1 \leq q_2 \leq 2$.

Finally, we assume that $q_1 \geq 3$ and $1 \leq q_2, q_3, q_4 \leq 2$. The proof of this case develops similar to the previous case; we apply θ to I and J; we assume that $Bl(W_I) = (p_2, 2)$, $p_2 \geq 2$ and $Bl(W_J) = (q_4, q_3, q_2, q_1)$, where $q_1 \geq 3$ and $1 \leq q_2, q_3, q_4 \leq 2$. This time we have 8 possibilities, instead of 4 as in the previous case. In each of these eight cases, we consider the simple reflection s_j with smallest index j among the elements of J associated to its block of size q_1. Then, as in the previous case,

$$w_k = j - 1 \implies w_{k+1} = j$$

for some $k \geq 1$. Therefore, removing j from w and reducing every number that is greater than j by 1 give a poset $W^-_{I',J'}$ in S_{n-1}, isomorphic to $W^-_{I,J}$, such that $Bl(W_{I'}) = (p_2 - 1, 2)$, $p_2 - 1 \geq 3$ and $Bl(W_{J'}) = (q_4, q_3, q_2, q_1 - 1)$. In other words, we can assume without loss of generality that $1 \leq q_1 \leq 2$.

We know now that $W^-_{I,J}$, where $Bl(W_I) = (2, p_2)$, $p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3, q_4)$, is isomorphic to one of the following cases:

(A1) $Bl(W_I) = (2, 2)$ and $Bl(W_J) = (1, 1, 1, 1)$;

(A2) $Bl(W_I) = (2, 3)$ and $Bl(W_J) = (1, 1, 1, 2)$;

(A3) $Bl(W_I) = (2, 3)$ and $Bl(W_J) = (1, 1, 2, 1)$;

(A4) $Bl(W_I) = (2, 3)$ and $Bl(W_J) = (1, 2, 1, 1)$;

(A5) $Bl(W_I) = (2, 3)$ and $Bl(W_J) = (2, 1, 1, 1)$;

(A6) $Bl(W_I) = (2, 4)$ and $Bl(W_J) = (1, 1, 2, 2)$;

(A7) $Bl(W_I) = (2, 4)$ and $Bl(W_J) = (1, 2, 1, 2)$;

(A8) $Bl(W_I) = (2, 4)$ and $Bl(W_J) = (2, 1, 1, 2)$;

(A9) $Bl(W_I) = (2,4)$ and $Bl(W_J) = (1,2,2,1)$;

(A10) $Bl(W_I) = (2,4)$ and $Bl(W_J) = (2,1,2,1)$;

(A11) $Bl(W_I) = (2,4)$ and $Bl(W_J) = (2,2,1,1)$;

(A12) $Bl(W_I) = (2,5)$ and $Bl(W_J) = (1,2,2,2)$;

(A13) $Bl(W_I) = (2,5)$ and $Bl(W_J) = (2,1,2,2)$;

(A14) $Bl(W_I) = (2,5)$ and $Bl(W_J) = (2,2,1,2)$;

(A15) $Bl(W_I) = (2,5)$ and $Bl(W_J) = (2,2,2,1)$;

(A16) $Bl(W_I) = (2,6)$ and $Bl(W_J) = (2,2,2,2)$.

The Hasse diagrams of these posets are as in Figure **??**.

2.4.5 $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1,1,1,q_4)$.

We consider this situation in two different cases:

(a) $Bl(W_I) = (2,2)$ and $Bl(W_J) = (1,1,1,1)$;

(b) $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1,1,1,q_4)$, $q_4 \geq 2$.

We explain the reduction argument for (b); we claim that we can assume $2 \leq p_1, p_2 \leq 3$.

First, we assume that $p_2 \geq 4$. Since $I^c = \{s_{p_1}\}$, $J^c = \{s_1, s_2, s_3\}$ in R_{n-1}, we see that if $w = w_1 \dots w_n \in W^-_{I,J}$, then

(i) for $i \in \{1, 2, \dots, p_1 - 1, p_1 + 1, \dots, n-1\}$, i comes before $i+1$ in w

$$\iff 1 \to \dots \to p_1, p_1 + 1 \to \dots \to n;$$

(ii) $w_4 < \dots < w_n$.

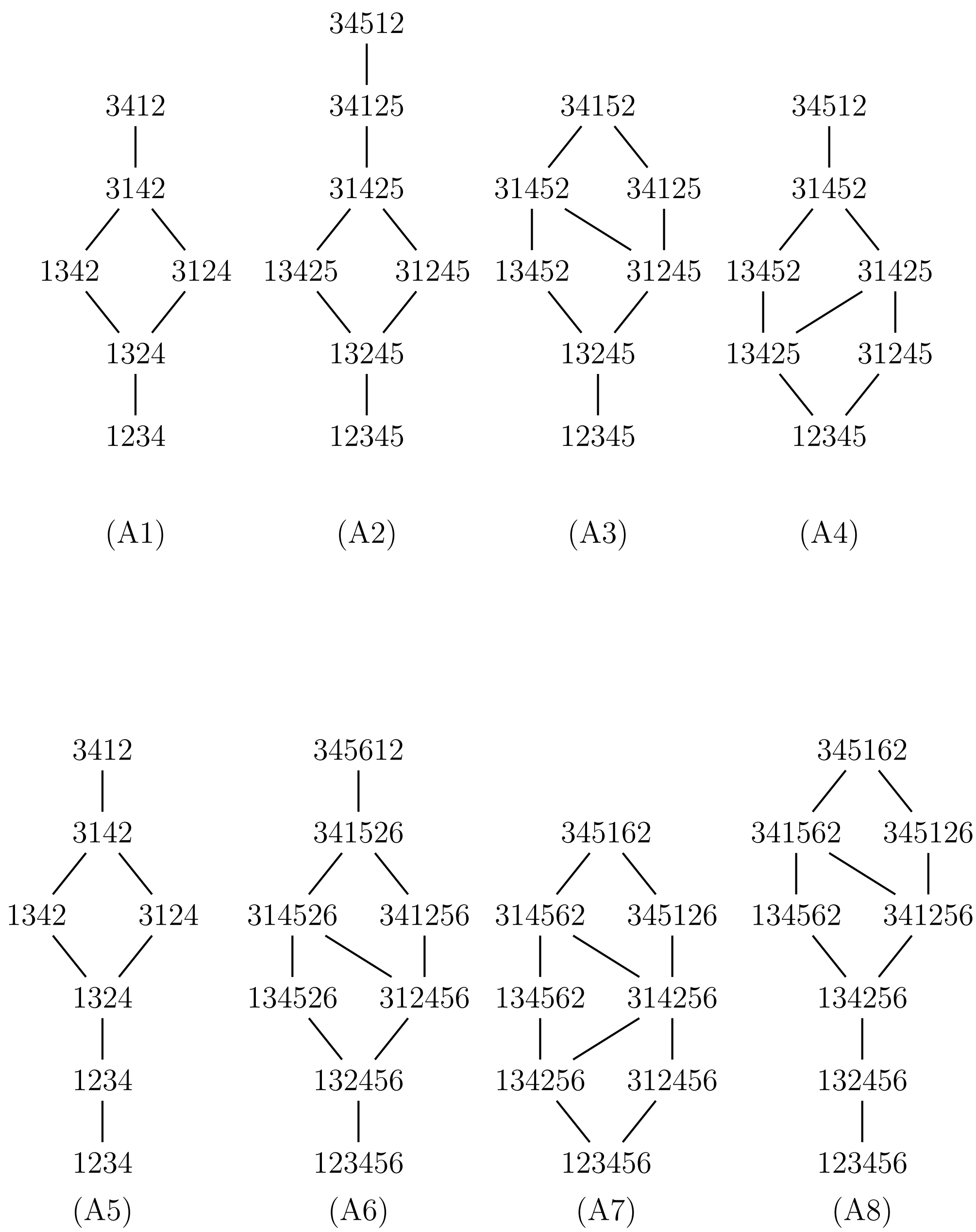


Figure 2.5: First eight cases of $Bl(W_I) = (2, p_2)$, $p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3, q_4)$

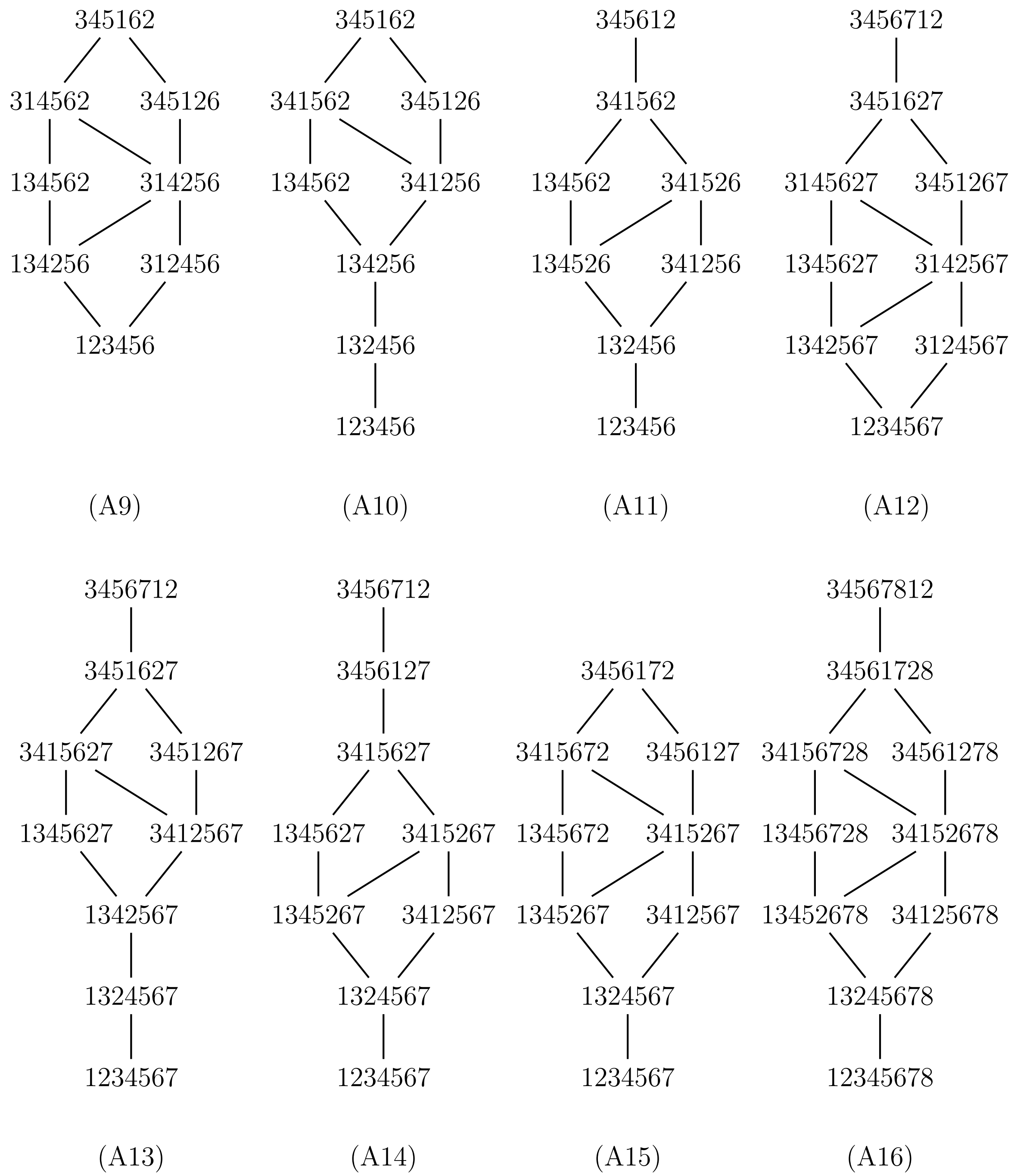


Figure 2.6: The second eight cases of $Bl(W_I) = (2, p_2)$, $p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3, q_4)$

Therefore, $n \in \{w_1, w_2, w_3, w_n\}$. But there are at least $p_2 - 1 \geq 3$ numbers before n in w, therefore, n cannot appear in $\{w_1, w_2, w_3\}$. This means that n is equal to w_n. Now we see that removing n from w, for all $w \in W^-_{I,J}$ gives us an isomorphic poset $W^-_{I',J'}$, where $Bl(W_{I'}) = (p_1, p_2 - 1)$, $p_2 - 1, p_1 \geq 2$ and $Bl(W_{J'}) = (1, 1, 1, q_4 - 1)$, $q_4 - 1 \geq 2$.

We now proceed with the assumption that $p_1 \geq 4$. In this case, we look at the relative positions of numbers 3 and 4. If 3 appears in the segment $w_4 w_5 \ldots w_n$, then 3 is immediately followed by 4 since there are no descents in this portion of w. On the other hand, if 3 does not appear in the segment $w_4 w_5 \ldots w_n$, then it can only appear at w_3 since in this case it has to be preceded by 1 and 2 by condition (i). But then, 4 has to appear as w_4, otherwise, there would be a descent in $w_4 w_5 \ldots w_n$. This argument shows that the numbers 3 and 4 appear in w consecutively. Hence, if we remove 4 from w, and reduce every number greater than 4 by 1, then we do not change the Bruhat-Chevalley order. In other words, we obtain a poset $W^-_{I',J'}$, isomorphic to $W^-_{I,J}$, where $Bl(W_{I'}) = (p_1 - 1, p_2)$, $p_1 - 1, p_2 \geq 2$ and $Bl(W_{J'}) = (1, 1, 1, q_4 - 1)$, $q_4 - 1 \geq 2$. Therefore, we can assume that $p_1 \leq 3$.

As a consequence we conclude that in this case we have the following possibilities:

(A) $Bl(W_I) = (2, 2)$ and $Bl(W_J) = (1, 1, 1, 1)$;

(B) $Bl(W_I) = (2, 3)$ and $Bl(W_J) = (1, 1, 1, 2)$;

(C) $Bl(W_I) = (3, 2)$ and $Bl(W_J) = (1, 1, 1, 2)$;

(D) $Bl(W_I) = (3, 3)$ and $Bl(W_J) = (1, 1, 1, 3)$.

The Hasse diagrams of the resulting posets are depicted in Figure **??**.

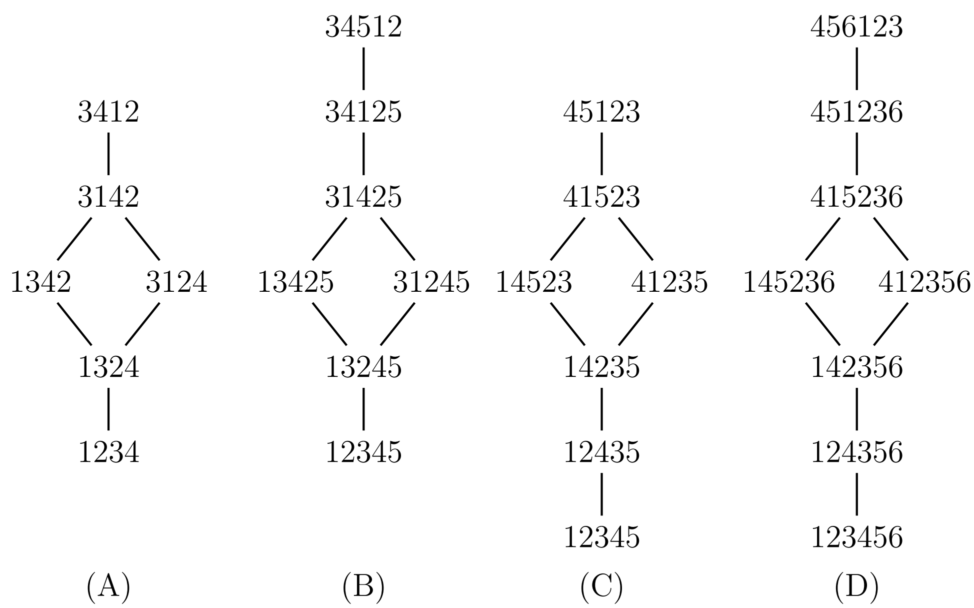


Figure 2.7: $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1, 1, 1, q_4)$.

2.4.6 $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1, 1, q_3, 1)$, $q_3 \geq 2$.

By arguing as in the previous cases, we see that all subcases reduces to one of the following three subcases:

(A) $Bl(W_I) = (2, 3)$ and $Bl(W_J) = (1, 1, 2, 1)$;

(B) $Bl(W_I) = (3, 2)$ and $Bl(W_J) = (1, 1, 2, 1)$;

(C) $Bl(W_I) = (3, 3)$ and $Bl(W_J) = (1, 1, 3, 1)$.

The Hasse diagrams of the possible posets, that are denoted by (A), (B), and (C) are depicted in Figure **??**.

2.4.7 $Bl(W_I) = (1, 1, p_3)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

We will assume that $p_2 \geq 2$. Once again by the appropriate reduction arguments as in the previous cases we see that we can assume $q_1, q_2, q_3 \leq 2$. Hence, we have the following possibilities:

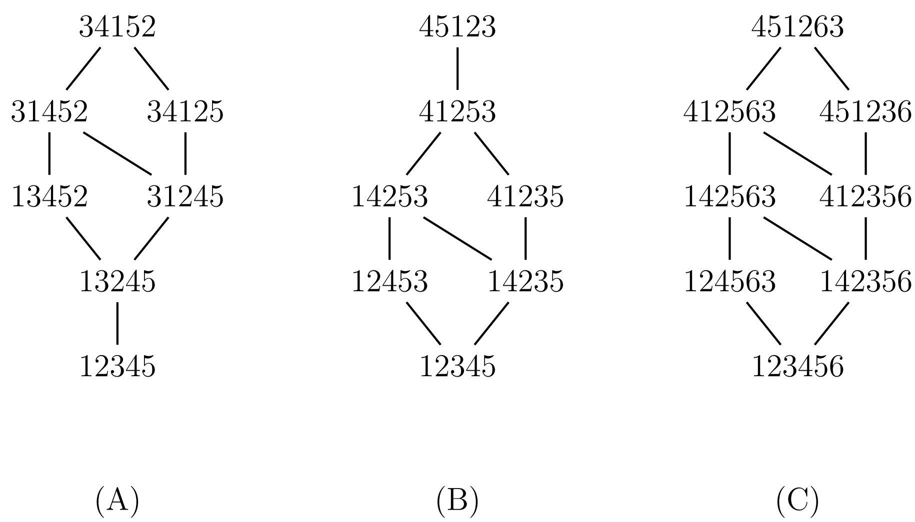


Figure 2.8: $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1, 1, q_3, 1)$, $q_3 \geq 2$.

(A) $Bl(W_I) = (1, 1, 2)$ and $Bl(W_J) = (1, 1, 2)$;

(B) $Bl(W_I) = (1, 1, 2)$ and $Bl(W_J) = (1, 2, 1)$;

(C) $Bl(W_I) = (1, 1, 2)$ and $Bl(W_J) = (2, 1, 1)$;

(D) $Bl(W_I) = (1, 1, 3)$ and $Bl(W_J) = (1, 2, 2)$;

(E) $Bl(W_I) = (1, 1, 3)$ and $Bl(W_J) = (2, 1, 2)$;

(F) $Bl(W_I) = (1, 1, 3)$ and $Bl(W_J) = (2, 2, 1)$;

(G) $Bl(W_I) = (1, 1, 4)$ and $Bl(W_J) = (2, 2, 2)$.

The Hasse diagrams of these possible posets, that are denoted by (A) to (G), are depicted in Figure **??**.

2.4.8 $Bl(W_I) = (1, p_2, 1)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

We will assume that $p_2 \geq 2$. In this case, by the appropriate reduction arguments as in the previous cases we see that it is safe to assume $q_1, q_2, q_3 \leq 2$, so, we have the following distinct possibilities:

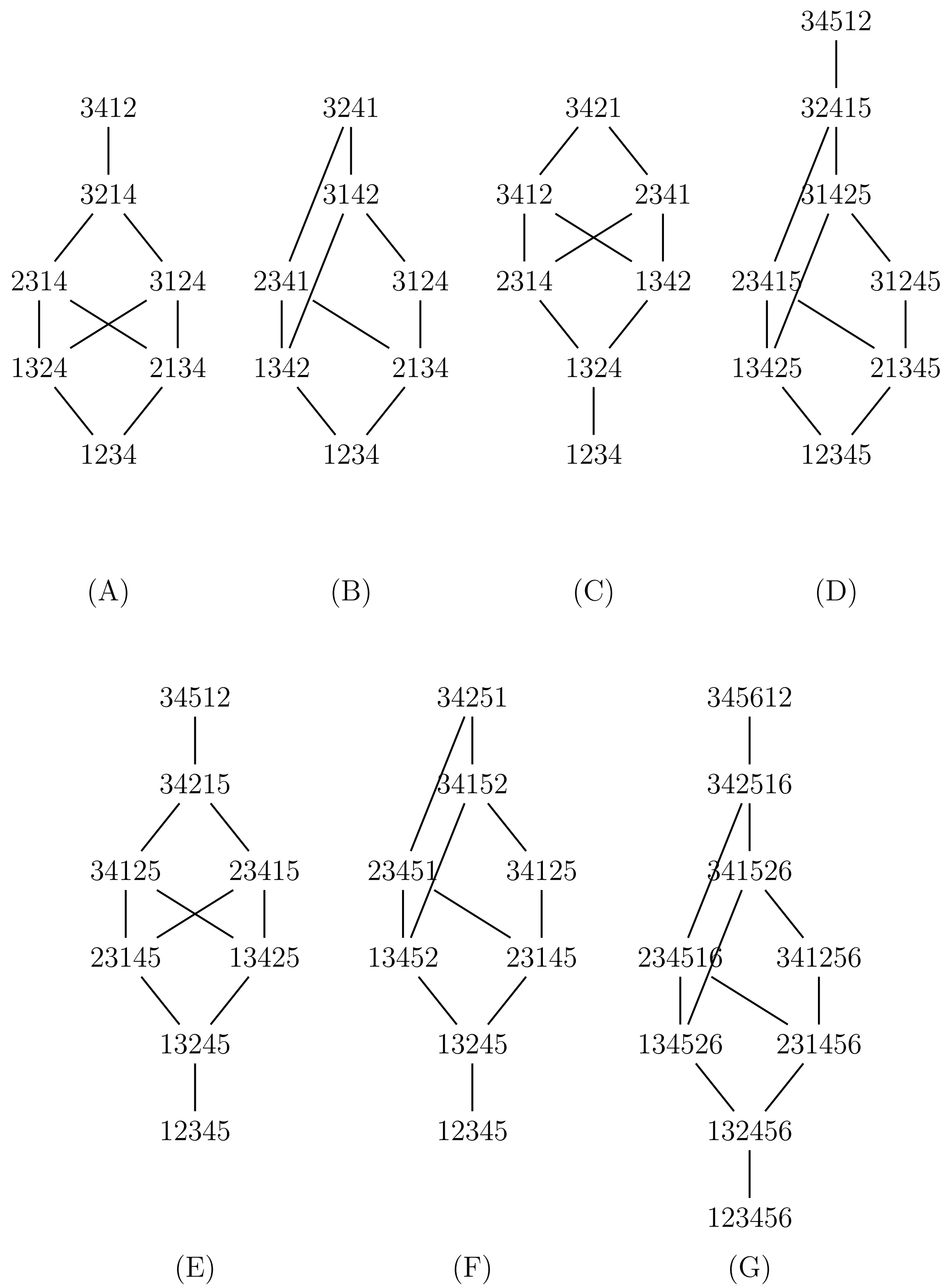


Figure 2.9: $Bl(W_I) = (1, 1, p_3)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

(A) $Bl(W_I) = (1, 2, 1)$ and $Bl(W_J) = (1, 1, 2)$;

(B) $Bl(W_I) = (1, 2, 1)$ and $Bl(W_J) = (1, 2, 1)$;

(C) $Bl(W_I) = (1, 3, 1)$ and $Bl(W_J) = (1, 2, 2)$;

(D) $Bl(W_I) = (1, 3, 1)$ and $Bl(W_J) = (2, 1, 2)$;

(E) $Bl(W_I) = (1, 4, 1)$ and $Bl(W_J) = (2, 2, 2)$.

The Hasse diagrams of the possible posets, that are denoted by (A), (B), (C), (D), and (E), are depicted in Figure **??**.

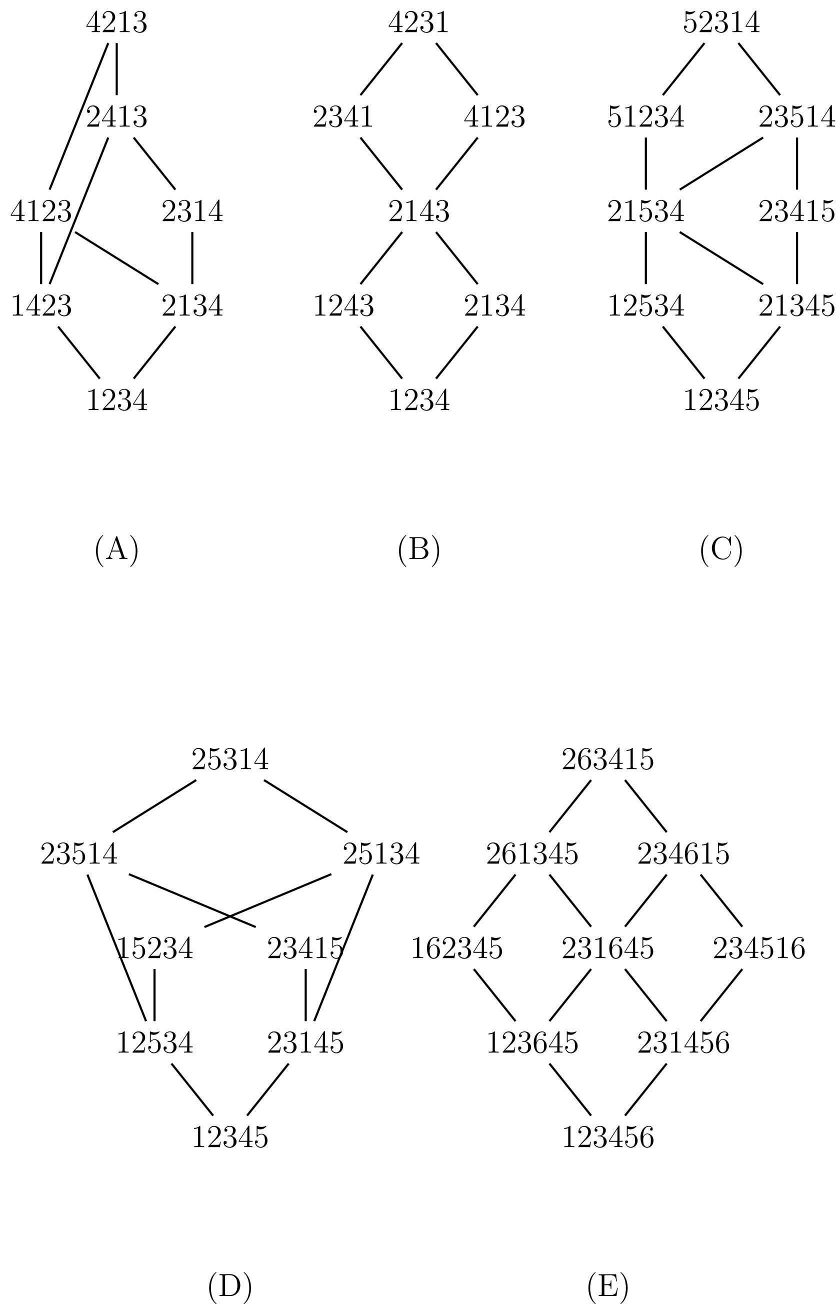


Figure 2.10: $Bl(W_I) = (1, p_2, 1)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

Chapter 3

Using Bruhat order on matrices for Type A and D

According to Magyar, Weyman, and Zelevinsky [?], Bongartz [?] has an alternative matrix interpretation for the poset on the double cosets when the Dynkin diagram corresponding to R_m is of type A_n, D_n, E_6, E_7, or E_8. The Bruhat order on those matrices are:

$$M \leq M' \iff \sum_{k=1}^{i}\sum_{l=1}^{j} m_{kl} \geq \sum_{k=1}^{i}\sum_{l=1}^{j} m'_{kl}.$$

If we use Bongartz's matrix interpretation on all the cases of complexity 1, we will have an equivalent poset structure to our result for each case. It can be obtained in the following way. For example, if we work with the case $Bl(W_I) = (3, p_2), p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3), q_1, q_2, q_3 \geq 2$. Then, the G-orbits of $G/P_I \times G/P_J$ are parametrized by 2×3 nonnegative integer matrices with row sums $3, p_2$ and column sums q_1, q_2, q_3.

Because the first row sum is 3, we have limited all the possible entries for the first row to only few cases where they add up equal to 3. Each case will determine the Bruhat order of the matrix. As the results, we have the diagram of the posets equivalent to the diagram that we have.

As before, G is a connected reductive complex algebraic group and X is a double flag variety $G/P_I \times G/P_J$.

3.1 Type A, complexity 0

In this section, we will show another proof of Theorem **??**, using Bongartz's matrix interpretation.

Theorem **??**: Let G denote SL_n and let X be a double flag variety $G/P_I \times G/P_J$. If $c_G(X) = 0$, then the inclusion poset of G-orbit closures in X is a finite lattice.

We will use the same table **??** that lists all the complexity 0 of SL_n.

3.1.1 $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (q_1, q_2)$.

Here is the diagram of the poset when $p_1 \geq q_1$ and $p_2 \geq q_1$. If $p_1 < q_1$ or $p_2 < q_1$, some of the matrices will be omitted. Figure **??**. Note: the poset is a chain and is not bounded.

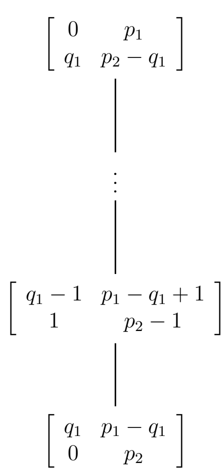


Figure 3.1: $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (q_1, q_2)$.

3.1.2 $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (1, q_2, q_3)$.

The full diagram is when $p_1 \geq q_2 + 1$ and $p_2 \geq q_2 + 1$. If $p_1 \leq q_2$ or $p_2 \leq q_2$, some of the matrices will be omitted. Figure **??**. Note: the poset is a lattice and is not bounded.

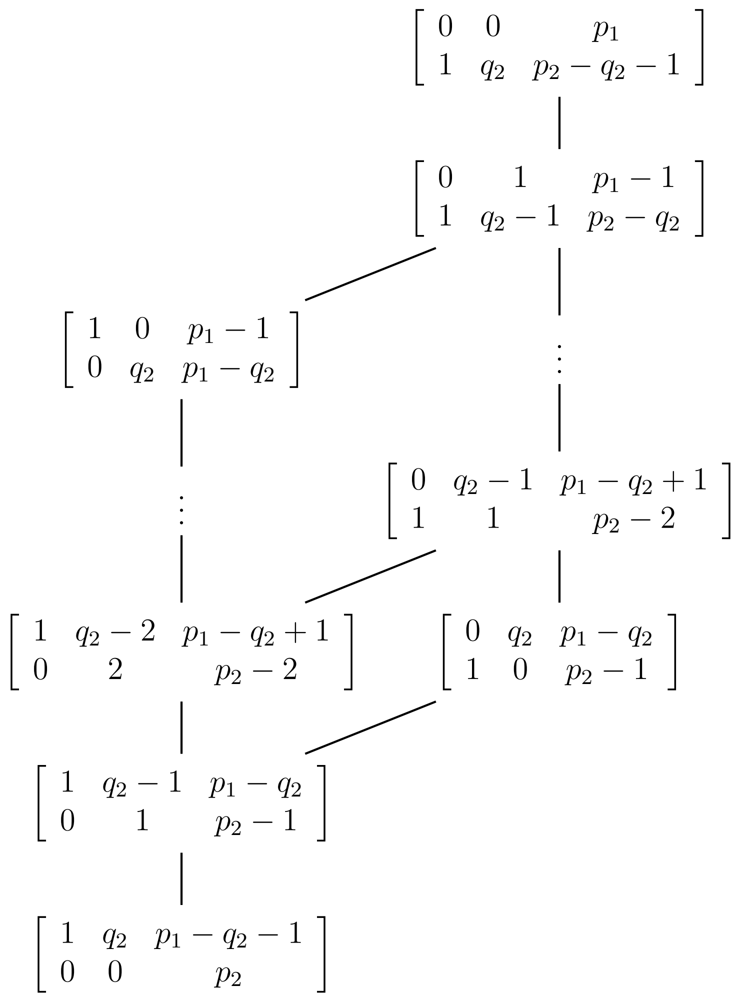


Figure 3.2: $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (q_1, 1, q_3)$.

3.1.3 $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (q_1, 1, q_3)$.

The full diagram is when $p_1 \geq q_1 + 1$ and $p_2 \geq q_1 + 1$. If $p_1 \leq q_1$ or $p_2 \leq q_1$, then some of the matrices will be omitted. Figure ??. Note: the poset is a lattice and is not bounded.

$$\begin{bmatrix} 0 & 0 & p_1 \\ q_1 & 1 & p_2 - q_1 - 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & p_1 - 1 \\ q_1 & 0 & p_2 - q_1 \end{bmatrix}$$

$$\vdots$$

$$\vdots$$

$$\begin{bmatrix} q_1 - 1 & 0 & p_1 - q_1 + 1 \\ 1 & 1 & p_2 - 2 \end{bmatrix}$$

$$\begin{bmatrix} q_1 - 1 & 1 & p_1 - q_1 \\ 1 & 0 & p_2 - 1 \end{bmatrix}$$

$$\begin{bmatrix} q_1 & 0 & p_1 - q_1 \\ 0 & 1 & p_2 - 1 \end{bmatrix}$$

$$\begin{bmatrix} q_1 & 1 & p_1 - q_1 - 1 \\ 0 & 0 & p_2 \end{bmatrix}$$

Figure 3.3: $Bl(W_I) = (p_1, p_2)$ and $Bl(W_J) = (q_1, 1, q_3)$.

3.1.4 $Bl(W_I) = (2, p_2)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

Here is the diagram of the poset when $q_1, q_2, q_3 \geq 2$. If some value $q_i < 2$, some of the matrices will be omitted. Figure **??**. Note: the poset is a lattice and is bounded.

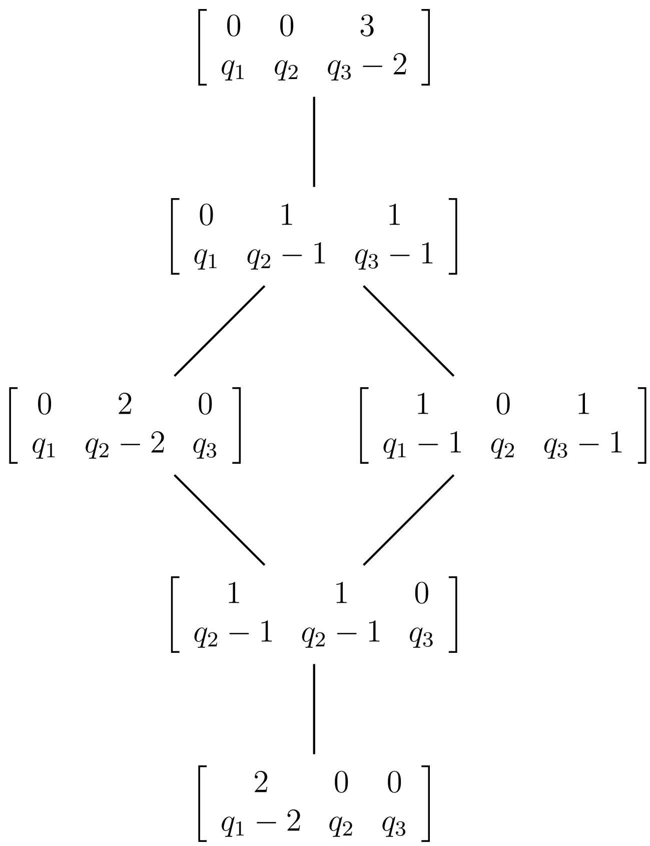


Figure 3.4: $Bl(W_I) = (2, p_2)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

3.1.5 $Bl(W_I) = (1, p_2)$ and $Bl(W_J) = (q_1, q_2, \ldots, q_s)$.

We move 1 along the first row from the left to the right. Figure **??**. Note: the poset is a chain and is not bounded.

$$\begin{bmatrix} 0 & 0 & \ldots & 1 \\ q_1 & q_2 & \ldots & q_s - 1 \end{bmatrix}$$

$$\Big|$$

$$\vdots$$

$$\Big|$$

$$\begin{bmatrix} 0 & 1 & \ldots & 0 \\ q_1 & q_2 - 1 & \ldots & q_s \end{bmatrix}$$

$$\Big|$$

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ q_1 - 1 & q_2 & \ldots & q_s \end{bmatrix}$$

Figure 3.5: $Bl(W_I) = (1, p_2)$ and $Bl(W_J) = (q_1, q_2, \ldots, q_s)$.

3.2 Type A_n, complexity 1

Theorem **??**: If $G = SL_n$ and $c_G(X) = 1$, then the inclusion poset of G-orbit closures in X is a finite poset.

We use the same table **??** as of previous chapter.

3.2.1 $Bl(W_I) = (3, p_2)$, $p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3)$, $q_1, q_2, q_3 \geq 2$.

Here is the diagram of the poset when $q_1, q_2, q_3 \geq 3$. If some value $q_i < 3$, some of the matrices will be omitted. Figure **??**. Note: the poset is a lattice and is bounded.

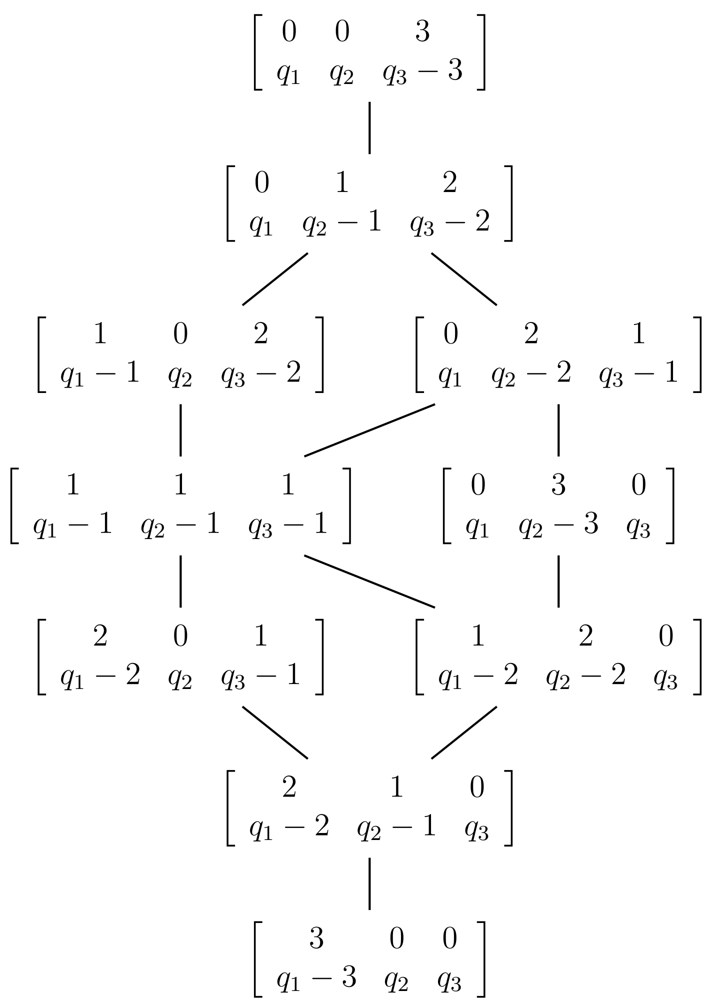


Figure 3.6: $Bl(W_I) = (3, p_2)$, $p_2 \geq 3$ and $Bl(W_J) = (q_1, q_2, q_3)$, $q_1, q_2, q_3 \geq 2$.

3.2.2 $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 3$ and $Bl(W_J) = (2, 2, q_3)$, $q_3 \geq 2$.

Here is the diagram of the poset when $p_1, p_2 \geq 4$. If some value $p_i < 4$, some of the matrices will be omitted. Figure ??. Note: the poset is a lattice and is bounded.

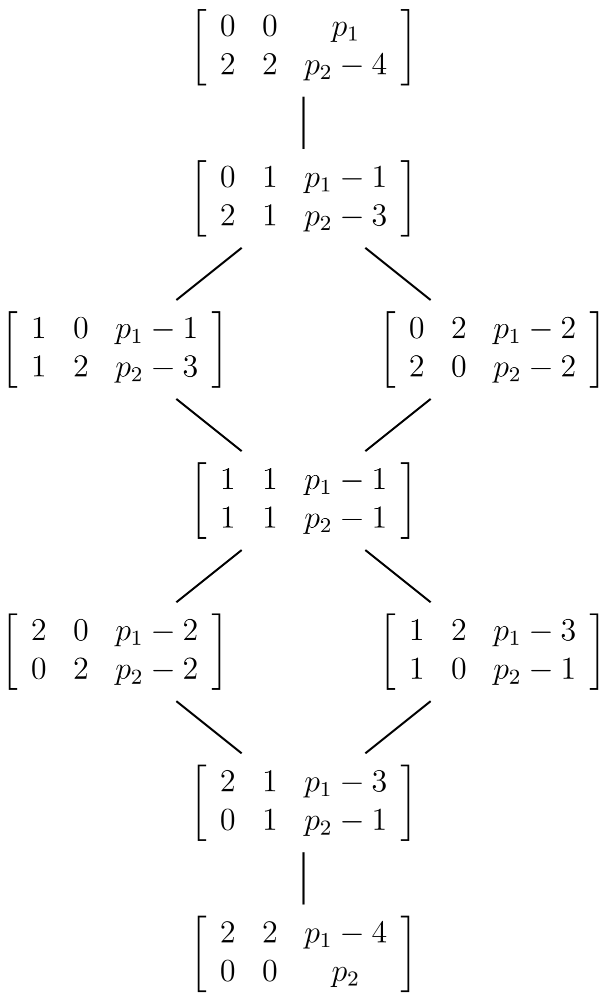


Figure 3.7: $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 3$ and $Bl(W_J) = (2, 2, q_3)$, $q_3 \geq 2$.

3.2.3 $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 3$ and $Bl(W_J) = (2, q_2, 2)$, $q_2 \geq 2$.

Here is the diagram of the poset when $p_1, p_2 \geq 4$. If some value $p_i < 4$, some of the matrices will be omitted. Figure ??. Note: the poset is a lattice and is bounded.

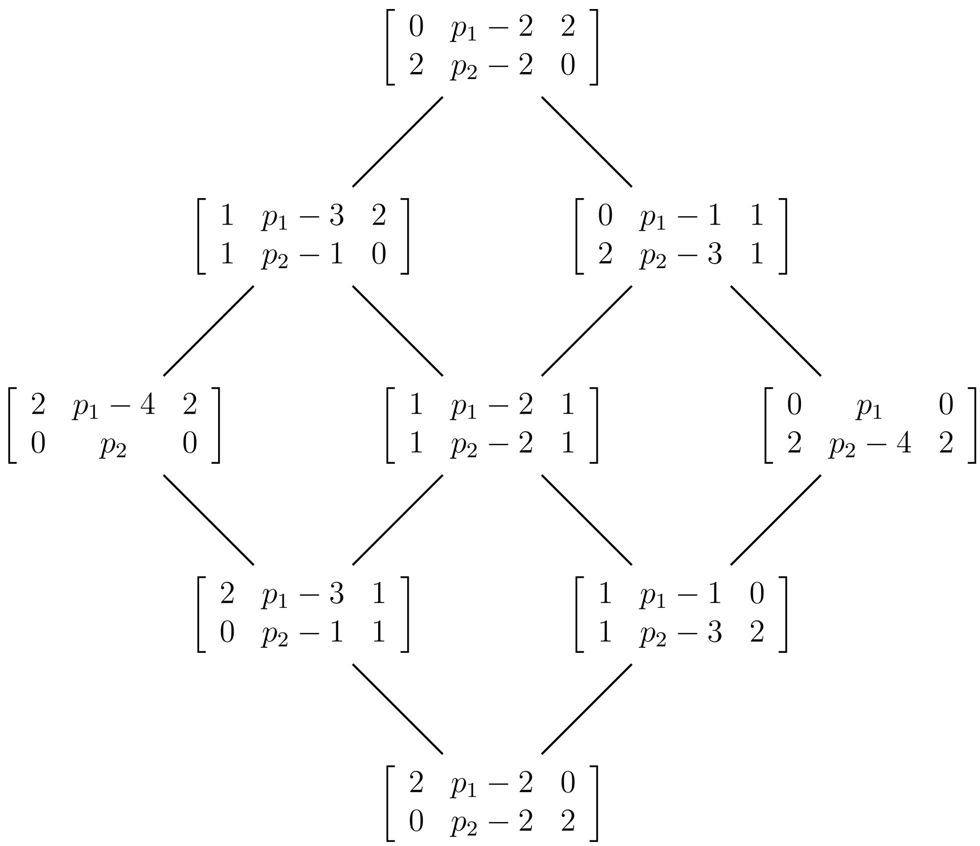


Figure 3.8: $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 3$ and $Bl(W_J) = (2, q_2, 2)$, $q_2 \geq 2$.

3.2.4 $Bl(W_I) = (2, p_2)$ and $Bl(W_J) = (q_1, q_2, q_3, q_4)$.

Here is the diagram of the poset when $q_1, q_2, q_3, q_4 \geq 2$. If some value $q_i < 2$, some of the matrices will be omitted. Figure **??**. Note: the poset is a lattice and is bounded.

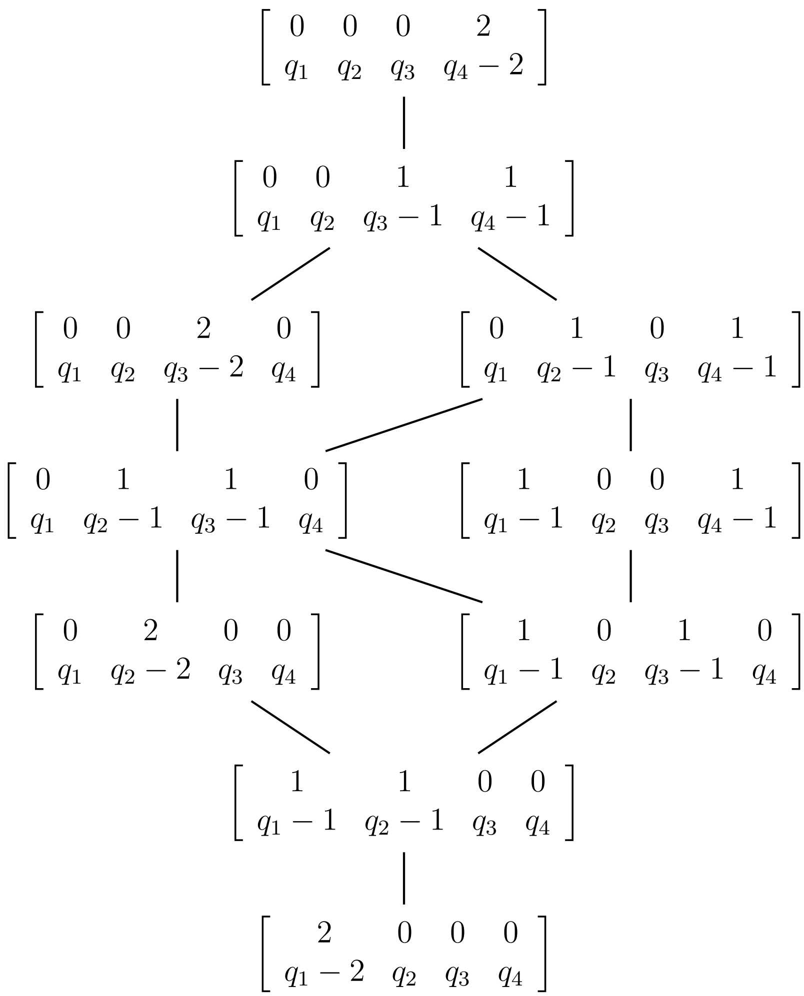


Figure 3.9: $Bl(W_I) = (2, p_2)$ and $Bl(W_J) = (q_1, q_2, q_3, q_4)$.

3.2.5 $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1, 1, 1, q_4)$.

Here is the diagram of the poset when $p_1, p_2 \geq 3$. If some value $p_i < 3$, some of the matrices will be omitted. Figure ??. Note: the poset is a lattice and is bounded.

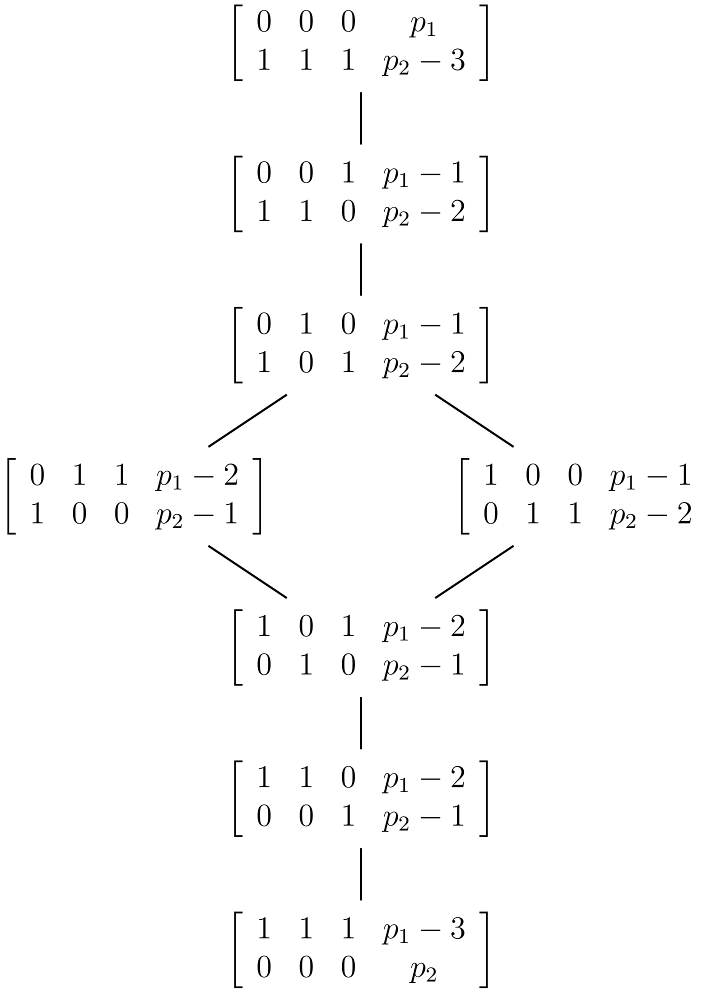


Figure 3.10: $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1, 1, 1, q_4)$.

3.2.6 $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1, 1, q_3, 1)$.

Here is the diagram of the poset when $p_1, p_2 \geq 3$. If some value $p_i < 3$, some of the matrices will be omitted. Figure ??. Note: the poset is a lattice and is bounded.

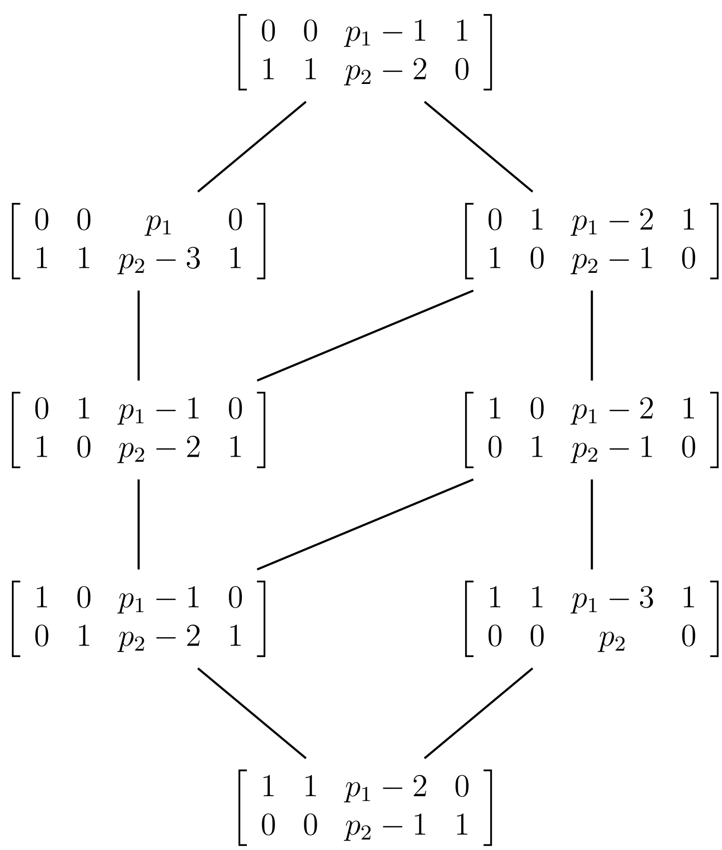


Figure 3.11: $Bl(W_I) = (p_1, p_2)$, $p_1, p_2 \geq 2$ and $Bl(W_J) = (1, 1, q_3, 1)$.

3.2.7 $Bl(W_I) = (1, 1, p_3)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

Figure **??**. The poset is bounded but is NOT a lattice in general.

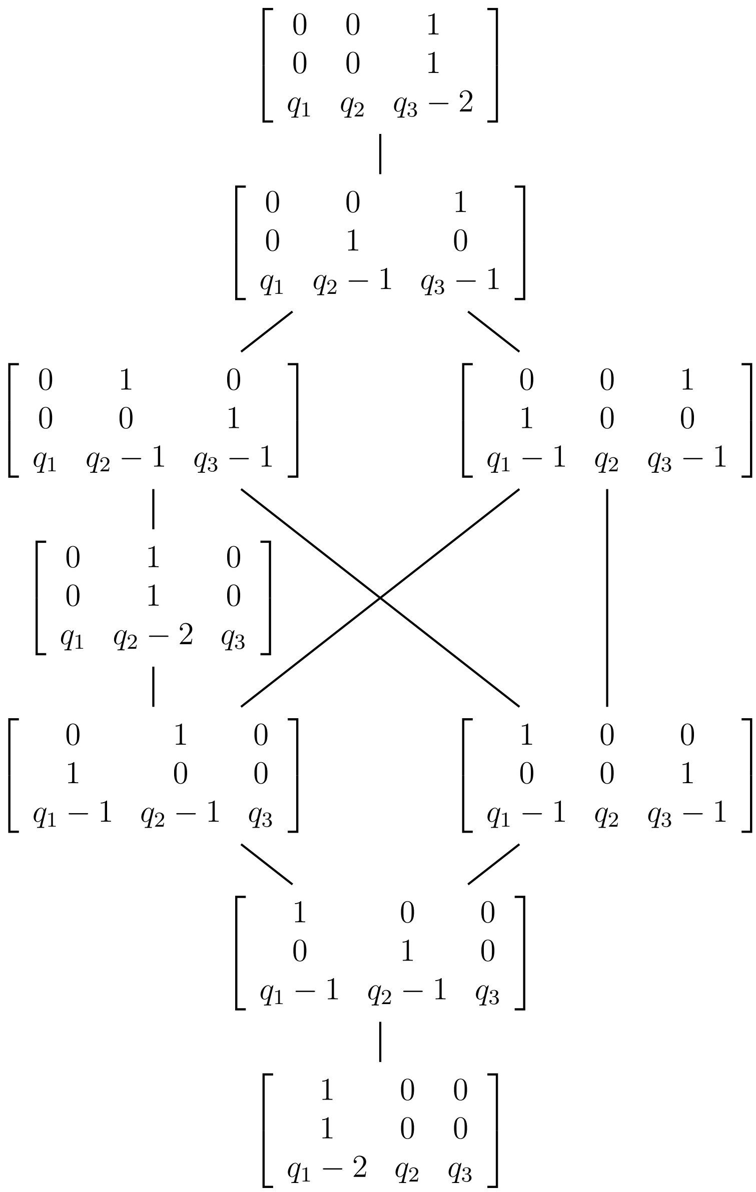


Figure 3.12: $Bl(W_I) = (1, 1, p_3)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

3.2.8 $Bl(W_I) = (1, p_2, 1)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

Here is the diagram of the poset when $q_1, q_2, q_3 \geq 2$. If some value $q_i < 2$, some of the matrices will be omitted. Figure ??. Note: If $q_2 = 1$, the poset is NOT a lattice. Otherwise, the poset is a lattice and is bounded.

$$\begin{bmatrix} 0 & 0 & 1 \\ q_1 - 1 & q_2 & q_3 - 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ q_1 & q_2 - 1 & q_3 - 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ q_1 - 1 & q_2 - 1 & q_3 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ q_1 & q_2 & q_3 - 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ q_1 & q_2 - 2 & q_3 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ q_1 - 2 & q_2 & q_3 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ q_1 & q_2 - 1 & q_3 - 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ q_1 - 1 & q_2 - 1 & q_3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ q_1 - 1 & q_2 & q_3 - 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 3.13: $Bl(W_I) = (1, p_2, 1)$ and $Bl(W_J) = (q_1, q_2, q_3)$.

3.3 Type D_n, complexity 0

Theorem 3.3.1. *Let G denote SO_{2n} and let X be a double flag variety $G/P_I \times G/P_J$. If $c_G(X) = 0$, then the inclusion poset of G-orbit closures in X is a lattice.*

Refer to Ponomavera's table **??**.

Number of blocks	$Bl(W_I)$	$Bl(W_J)$
2, 2	(p, p)	(p, p)
2, 3	(p, p) (p, p)	$(q_1, q_2, q_1),\ q_1 \leq 3$ $(q, 2, q)$
2, 4	(p, p) $(4, 4)$	$(1, q, q, 1)$ $(2, 2, 2, 2)$
2, 5	(p, p)	$(1, 1, q, 1, 1)$
3, 3	$(1, p, 1)$ $(p, 1, p)$	(q_1, q_2, q_1) $(p, 1, p)$
3, 4	$(1, p, 1)$	(q_1, q_2, q_2, q_1)

Table 3.1: The list of all complexity 0 double flag varieties for SO_{2n}.

3.3.1 $Bl(W_I) = (p, p)$ and $Bl(W_J) = (p, p)$.

Figure **??**. Note: the poset is a chain and is not bounded.

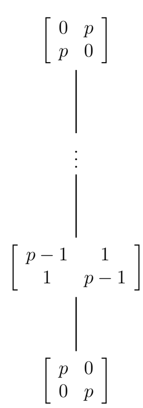


Figure 3.14: $Bl(W_I) = (p, p$ and $Bl(W_J) = (p, p)$.

3.3.2 $Bl(W_I) = (p, p)$ and $Bl(W_J) = (q_1, q_2, q_1)$, $q_1 \leq 3$.

$p + p = q_1 + q_2 + q_1$ implies $p = q_1 + \frac{1}{2}q_2$. So q_2 is an even number. Figure **??**. Note: the poset is a lattice and is bounded.

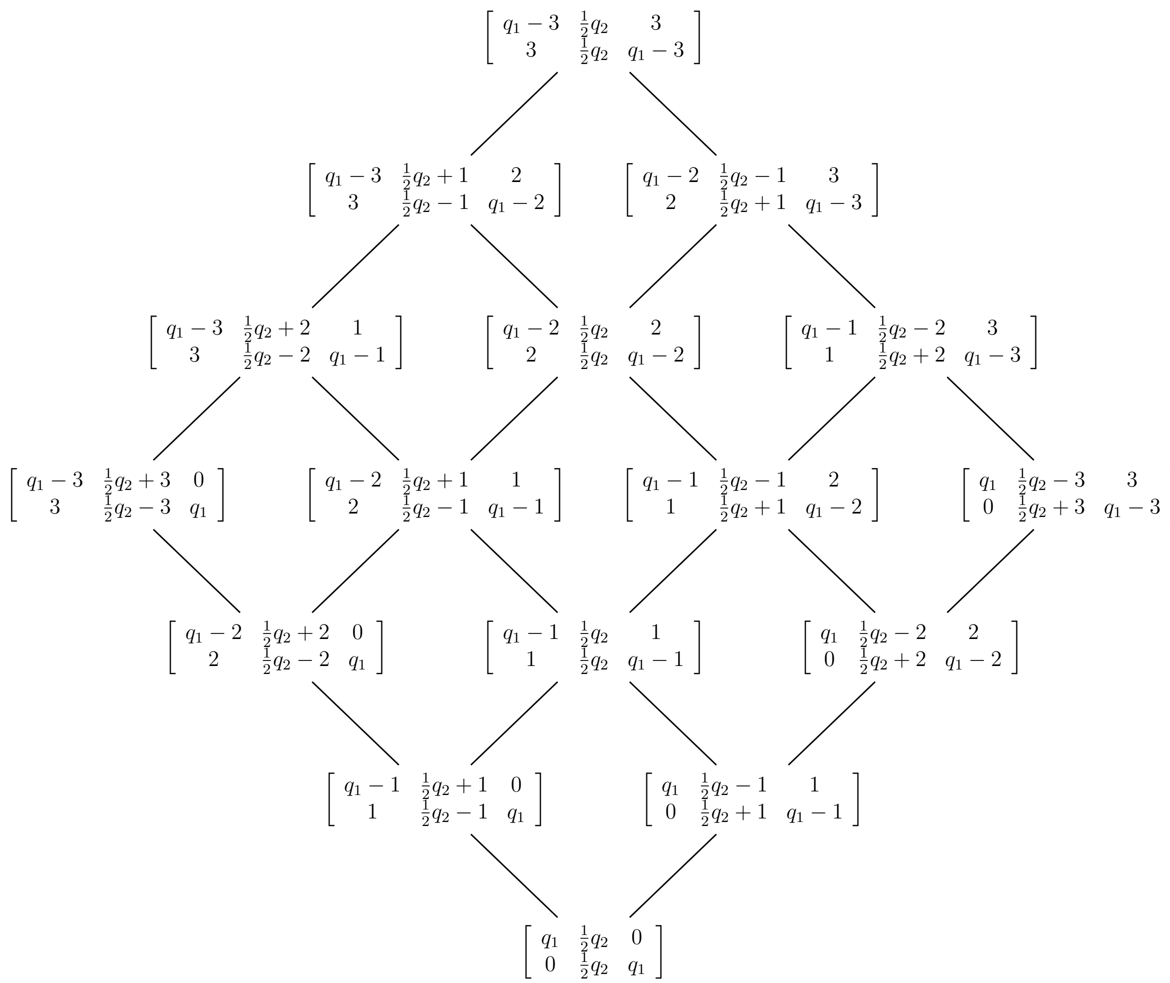


Figure 3.15: $Bl(W_I) = (p, p)$ and $Bl(W_J) = (q_1, q_2, q_1)$, $q_1 \leq 3$.

3.3.3 $Bl(W_I) = (p, p)$ and $Bl(W_J) = (q, 2, q)$.

$p + p = q + 2 + q$ implies $p = q + 1$. Figure **??**. Note: the poset is a lattice but is NOT bounded.

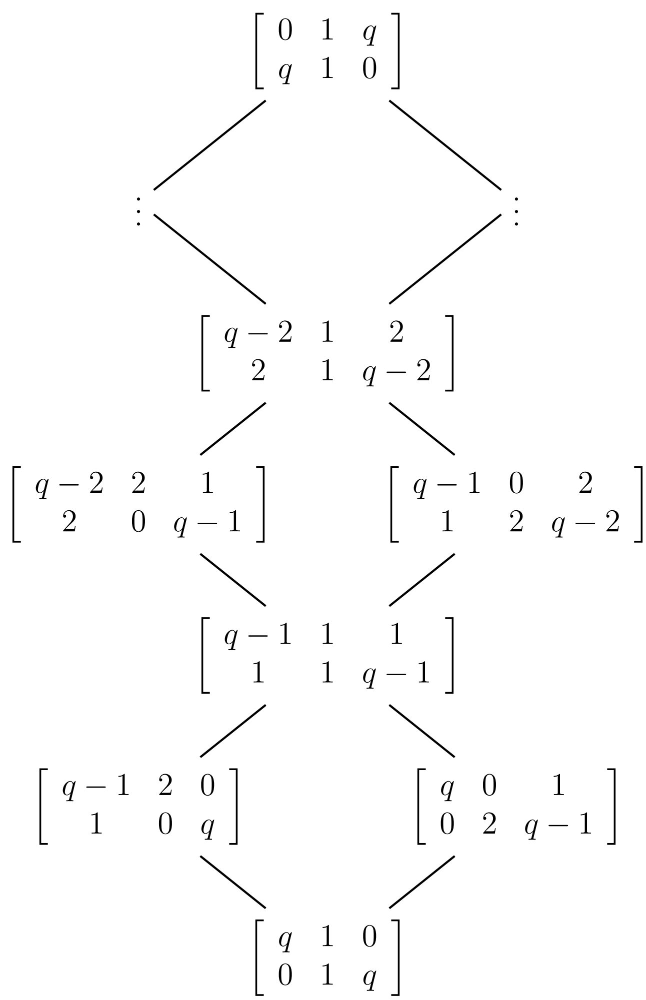


Figure 3.16: $Bl(W_I) = (p, p)$ and $Bl(W_J) = (q, 2, q)$.

3.3.4 $Bl(W_I) = (p, p)$ and $Bl(W_J) = (1, q, q, 1)$.

$p + p = 1 + q + q + 1$ implies $p = q + 1$. Figure **??**. Note: the poset is a lattice but is NOT bounded.

Figure 3.17: $Bl(W_I) = (p, p)$ and $Bl(W_J) = (1, q, q, 1)$.

3.3.5 $Bl(W_I) = (4, 4)$ and $Bl(W_J) = (2, 2, 2, 2)$.

Note: the poset is a lattice and is bounded. Figure ??.

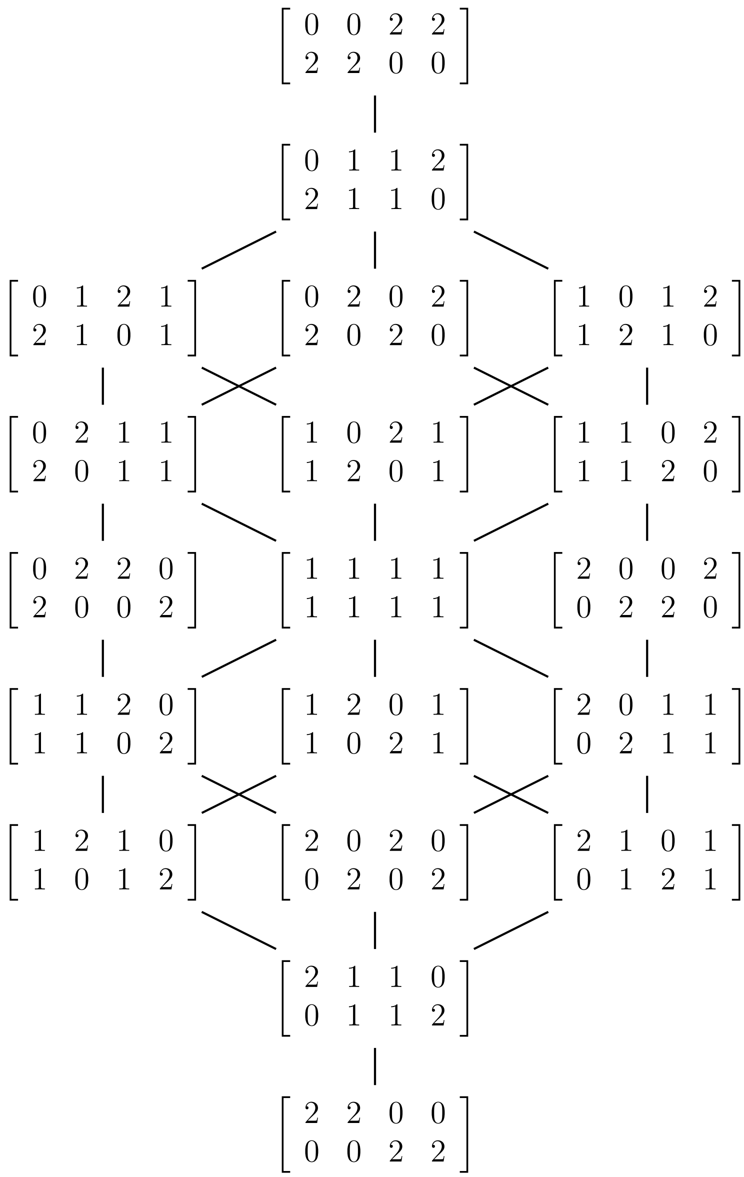


Figure 3.18: $Bl(W_I) = (4, 4)$ and $Bl(W_J) = (2, 2, 2, 2)$.

3.3.6 $Bl(W_I) = (p, p)$ and $Bl(W_J) = (1, 1, q, 1, 1)$.

$p + p = 1 + 1 + q + 1 + 1$ implies $q = 2p - 4$. So q is an even number. Figure ??.

Note: the poset is a lattice and is bounded.

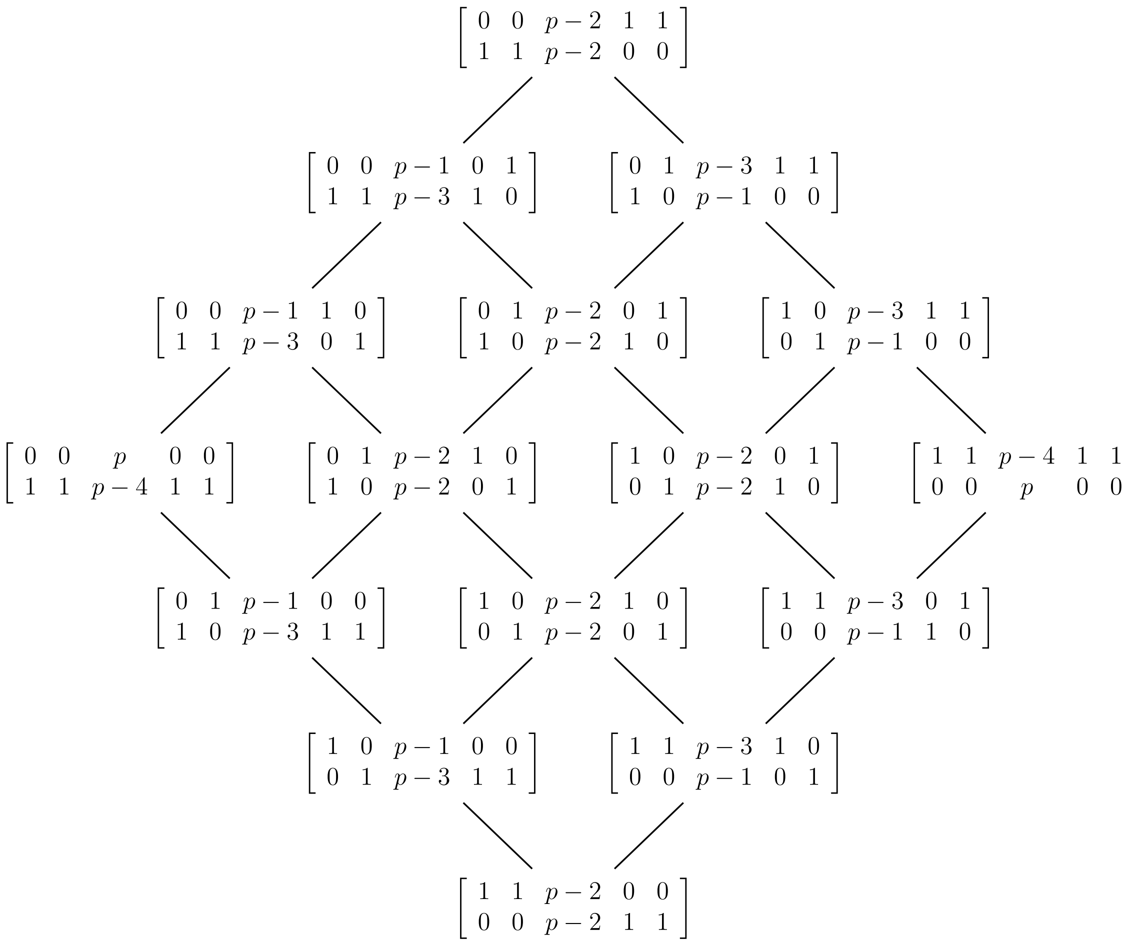


Figure 3.19: $Bl(W_I) = (p, p)$ and $Bl(W_J) = (1, 1, q, 1, 1)$.

3.3.7 $Bl(W_I) = (1, p, 1)$ and $Bl(W_J) = (q_1, q_2, q_1)$.

The requirement is that p is an even number. Here is the diagram of the poset when $q_1, q_2 \geq 2$. If some value $q_i < 2$, some of the matrices will be omitted. Figure **??**. Note: If $q_2 = 1$, the poset is NOT a lattice. Otherwise, the poset is a lattice and is bounded.

$$\begin{bmatrix} 0 & 0 & 1 \\ q_1 - 1 & q_2 & q_1 - 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\diagup \qquad \diagdown$$

$$\begin{bmatrix} 0 & 0 & 1 \\ q_1 & q_2 - 1 & q_1 - 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ q_1 - 1 & q_2 - 1 & q_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\diagup \qquad \diagdown \qquad \diagup \qquad \diagdown$$

$$\begin{bmatrix} 0 & 0 & 1 \\ q_1 & q_2 & q_1 - 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ q_1 & q_2 - 2 & q_1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ q_1 - 2 & q_2 & q_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\diagdown \qquad \diagup \qquad \diagdown \qquad \diagup$$

$$\begin{bmatrix} 0 & 1 & 0 \\ q_1 & q_2 - 1 & q_1 - 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ q_1 - 1 & q_2 - 1 & q_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\diagdown \qquad \diagup$$

$$\begin{bmatrix} 1 & 0 & 0 \\ q_1 - 1 & q_2 & q_1 - 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 3.20: $Bl(W_I) = (1, p, 1)$ and $Bl(W_J) = (q_1, q_2, q_1)$.

3.3.8 $Bl(W_I) = (1, p, 1)$ and $Bl(W_J) = (q_1, q_2, q_2, q_1)$.

The requirement is that p is an even number. Figure **??**. Note: the poset is a lattice and is bounded.

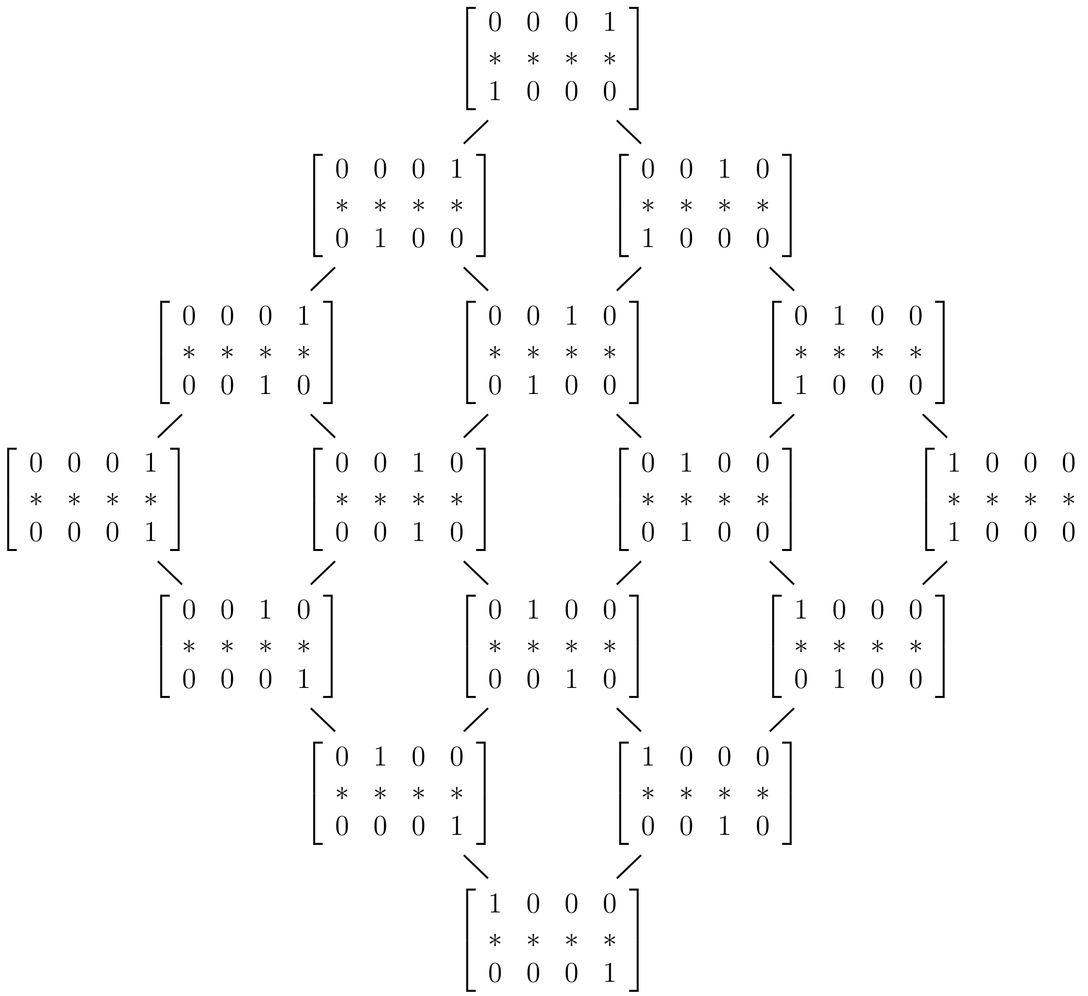


Figure 3.21: $Bl(W_I) = (1, p, 1)$ and $Bl(W_J) = (q_1, q_2, q_2, q_1)$.

3.4 Type D_n, complexity 1

Theorem 3.4.1. *Let G denote SO_{2n}, and let X be a double flag variety $G/P_I \times G/P_J$. If $c_G(X) = 1$, then the inclusion poset of G-orbit closures in X is a finite poset.*

Refer to Ponomavera's table **??**.

Number of blocks	$Bl(W_I)$	$Bl(W_J)$
$2,3$	$(6,6)$	$(4,4,4)$
$2,4$	$(4,4)$	$(2,2,2,2)$
	$(5,5)$	$(2,3,3,2)$
	$(5,5)$	$(3,2,2,3)$
$2,5$	$(4,4)$	$(1,2,2,2,1)$
	$(4,4)$	$(2,1,2,1,2)$
$2,6$	$(4,4)$	$(1,1,2,2,1,1)$
$3,3$	$(2,2,2)$	$(2,2,2)$
	$(2,p,2),\ p>1$	$(q,1,q)$
$3,4$	$(2,2,2)$	$(1,2,2,1)$
$3,5$	$(1,p,1)$	(q_1,q_2,q_3,q_2,q_1)
	$(2,1,2)$	$(1,1,1,1,1)$
$3,6$	$(1,p,1)$	$(q_1,q_2,q_3,q_3,q_2,q_1)$
$4,4$	$(1,2,2,1)$	$(1,2,2,1)$

Table 3.2: The list of all complexity 1 double flag varieties for SO_{2n}.

3.4.1 $Bl(W_I) = (6,6)$ and $Bl(W_J) = (4,4,4)$.

Note: the poset is a lattice and is bounded. Figure **??**.

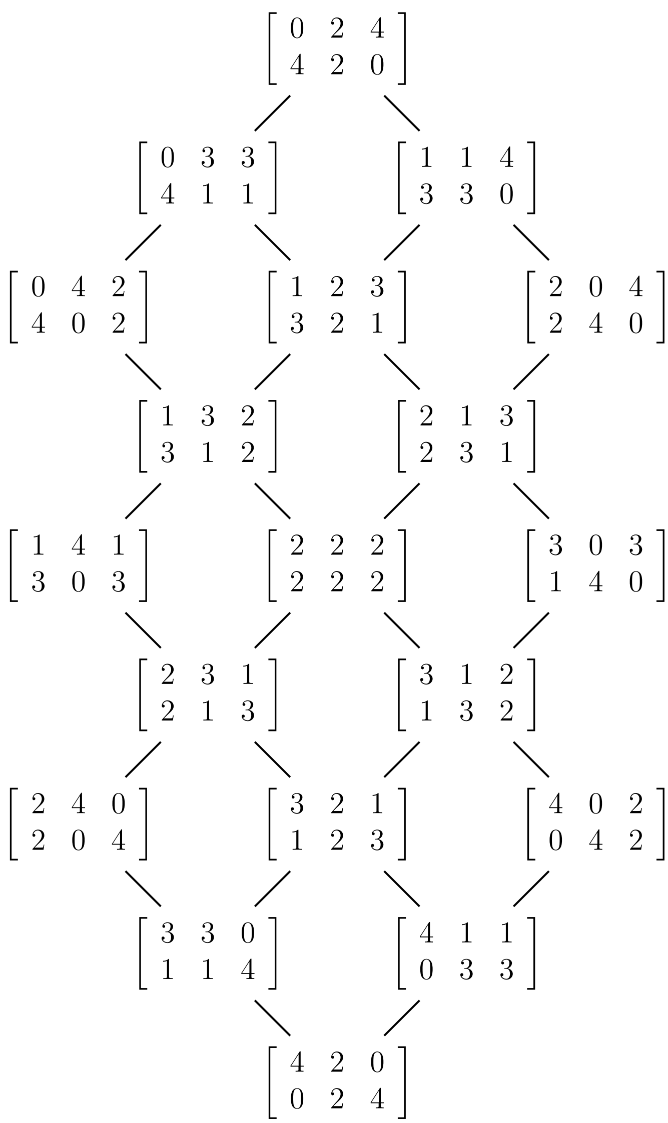


Figure 3.22: $Bl(W_I) = (6,6)$ and $Bl(W_J) = (4,4,4)$.

3.4.2 $Bl(W_I) = (4, 4)$ and $Bl(W_J) = (2, 2, 2, 2)$.

Note: the poset is a lattice and is bounded. Figure **??**.

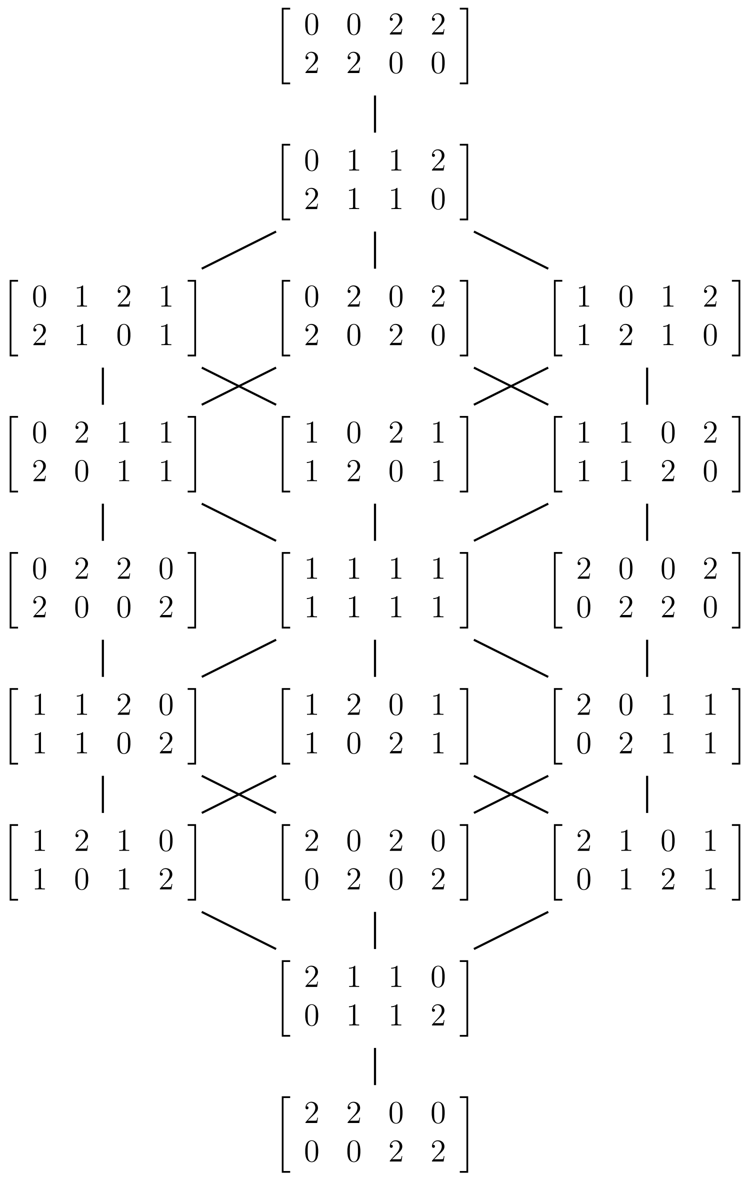


Figure 3.23: $Bl(W_I) = (4, 4)$ and $Bl(W_J) = (2, 2, 2, 2)$.

3.4.3 $Bl(W_I) = (5,5)$ and $Bl(W_J) = (2,3,3,2)$.

Note: the poset is a lattice and is bounded. Figure **??**.

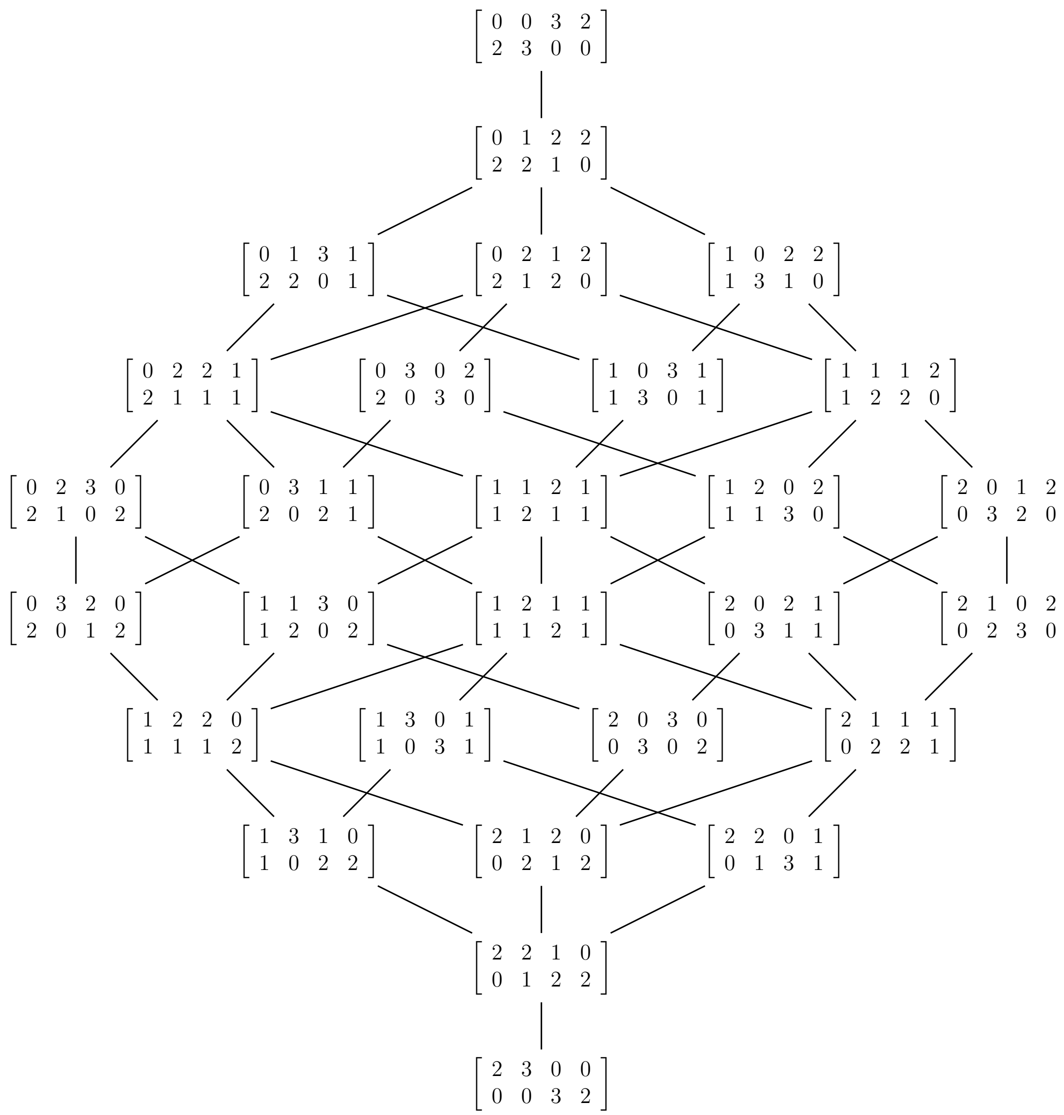


Figure 3.24: $Bl(W_I) = (5,5)$ and $Bl(W_J) = (2,3,3,2)$.

3.4.4 $Bl(W_I) = (5, 5)$ and $Bl(W_J) = (3, 2, 2, 3)$.

Note: the poset is a lattice and is bounded. Figure **??**.

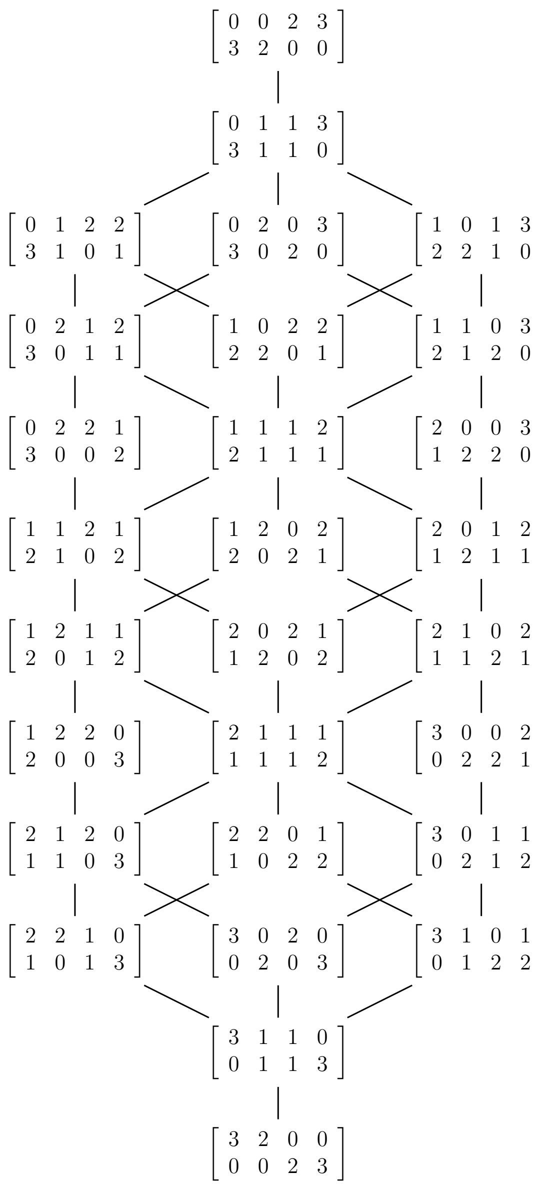


Figure 3.25: $Bl(W_I) = (5, 5)$ and $Bl(W_J) = (3, 2, 2, 3)$.

3.4.5 $Bl(W_I) = (4,4)$ and $Bl(W_J) = (1,2,2,2,1)$.

Note: the poset is a lattice and is bounded. Figure **??**.

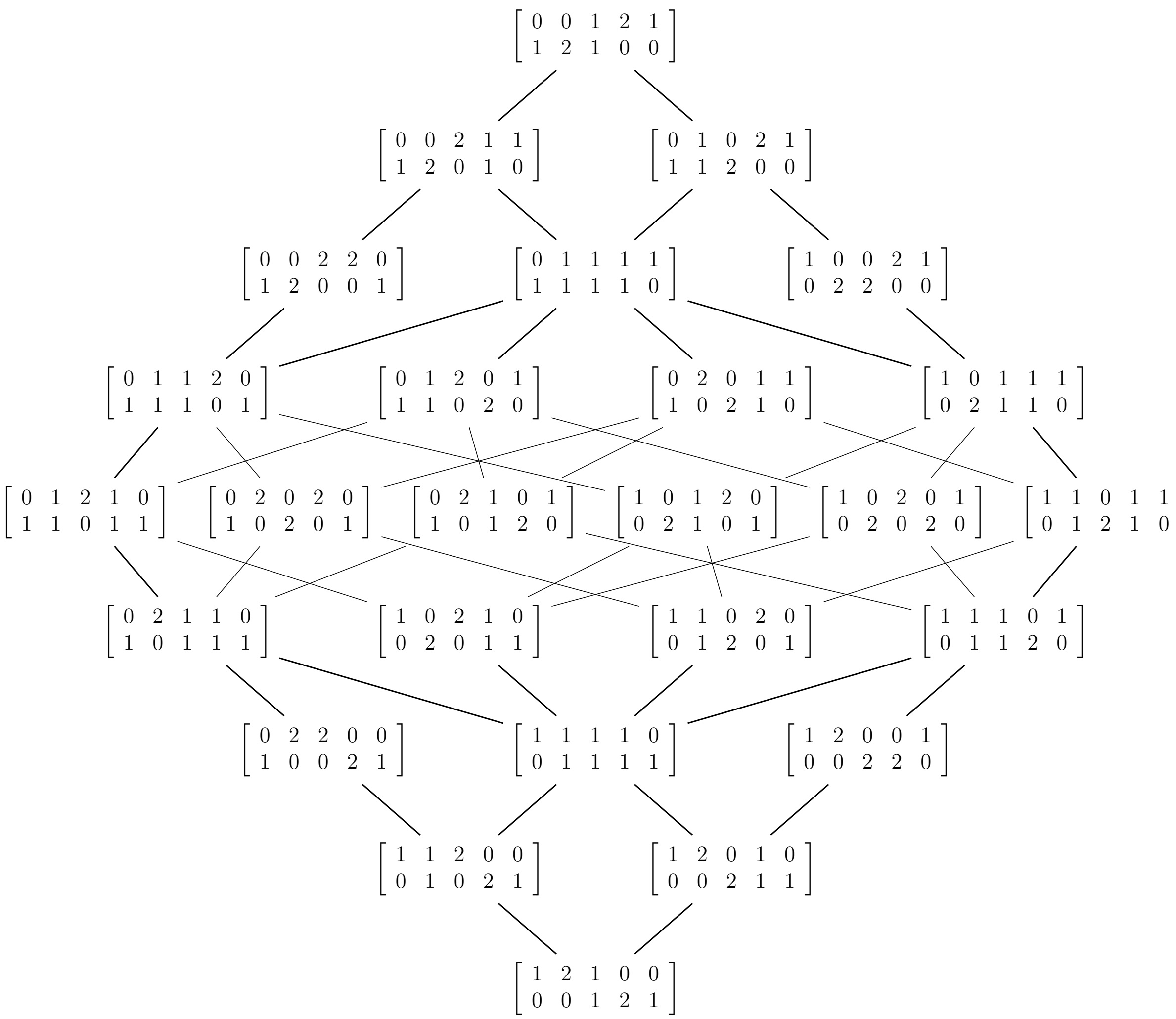


Figure 3.26: $Bl(W_I) = (4,4)$ and $Bl(W_J) = (1,2,2,2,1)$.

3.4.6 $Bl(W_I) = (2, 2, 2)$ and $Bl(W_J) = (2, 2, 2)$

Note: the poset is bounded but is NOT a lattice. Figure **??**.

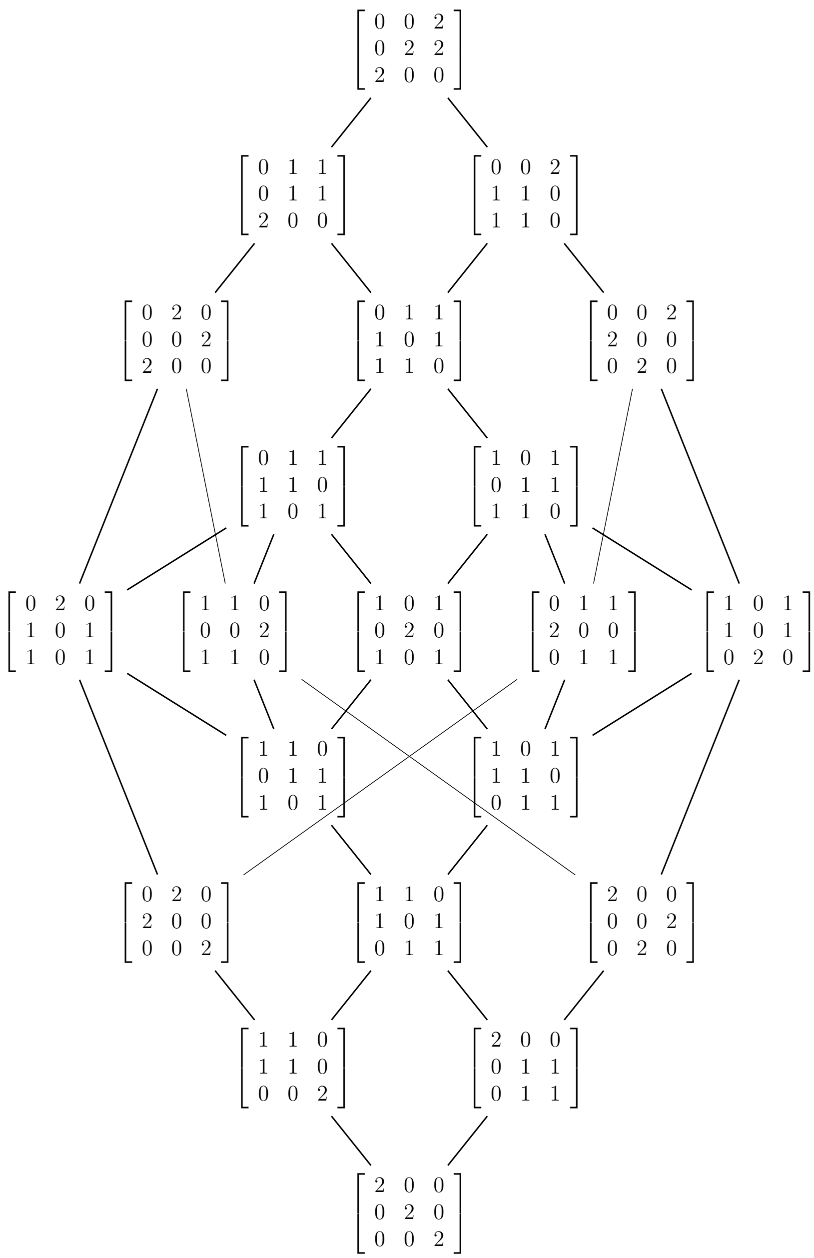


Figure 3.27: $Bl(W_I) = (2, 2, 2)$ and $Bl(W_J) = (2, 2, 2)$.

Chapter 4

Summary

By reviewing all the cases that we have, these are the main results for type A_n and type D_n.

Theorem **??**: If $G = SL_n$ and $c_G(X) = 0$, then the inclusion poset of G-orbit closures in X is a graded lattice.

Theorem **??**: If $G = SL_n$ and $c_G(X) = 1$, then the inclusion poset of G-orbit closures in X is a finite poset.

Theorem **??**: If $G = SO_{2n}$ and $c_G(X) = 0$, then the inclusion poset of G-orbit closures in X is a graded lattice.

Theorem **??**: If $G = SO_{2n}$ and $c_G(X) = 1$, then the inclusion poset of G-orbit closures in X is a finite poset.

Remark: All the posets of type A and type D with complexity 1 is a finite poset.

Future work: We will extend our work to type B (SO_{2n+1}) and type C (SP_{2n}). Note that Bongartz's matrix interpretation does not apply to these types. However, we can still apply other characterizations of Bruhat order for types B, C.

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Biography

The author was born in Hanoi, Vietnam in 1987. He graduated from the State University of New York at Plattsburgh with a Bachelor's degree in a double major Finance and Mathematics in 2010; and graduated from the Western New England University with a Master's degree in Mathematics for Teachers in 2012. The author started the Ph.D. program at the Tulane University mathematics department in August 2014, eventually completing the program in May 2020.